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A TOPOLOGICAL CHARACTERIZATION OF COMPLETE DISTRIBUTIVE LATTICES

Lucian BEZNEA

Department of Mathematics, Increst, Bd. Pacii 220, 79622 Bucharest, Romania

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An *ordered compact space* is a compact topological space X, endowed with a partially ordered relation, whose graph is a closed set of $X \times X$ (cf. [4]). An important subclass of these spaces is that of *Priestley spaces*, characterized by the following property: for every $x, y \in X$ with $x \not\leq y$ there is an increasing clopen set A (i.e. A is closed and open and such that $a \in A$, $a \leq z$ implies that $z \in A$) which separates x from y, i.e., $x \in A$ and $y \notin A$. It is known (cf. [5, 6]) that there is a dual equivalence between the category **Ld01** of distributive lattices with least and greatest element and the category **P** of Priestley spaces.

In this paper we shall prove that a lattice $L \in$ **Ld01** is complete if and only if the associated Priestley space X verifies the condition: (E0) $D \subseteq X$, D is increasing and open implies $\bar{D}^*$ is increasing clopen (where A^* denotes the least increasing set which includes A).

This result generalizes a well-known characterization of complete Boolean algebras in terms of associated Stone spaces (see [2, Ch. III, Section 4, Lemma 1], for instance).

We shall also prove that an ordered compact space that fulfils (E0) is necessarily a Priestley space.

1. Preliminaries

The results concerning ordered topological spaces are taken from [4] and those about Priestley spaces are from [1].

A compact topological space X endowed with an order relation is called an *ordered compact space* provided the graph of the order relation is closed in the product space $X \times X$. For every $A \subseteq X$ we set:

$$A^* = \{x \in X \mid \text{there exists } y \in A \text{ such that } y \leq x\},$$

$$_*A = \{x \in X \mid \text{there exists } y \in A \text{ such that } x \leq y\}.$$

A subset $A \subseteq X$ of a partially ordered set $(X, \leq)$ is said to be *increasing* (*decreasing*) if $x \in A$, $y \in X$ and $x \leq y$ $(y \leq x)$ imply $y \in A$.

Clear for every $A \subseteq X$: A is increasing iff $A = A^*$ so A^* is the increasing closer of A.

Lemma 1 [3, Lemma 2]. *Let X be an ordered compact space. If $A \subseteq X$ is closed then $_*A$ and A^* are also closed.*

Lemma 2 [3, Theorem 1]. *Let X be an ordered compact space and two closed sets F_0, $F_1 \subseteq X$ such that $F_0 \cap F_1 = \emptyset$, F_0 is increasing and F_1 is decreasing. Then there* *exist two open sets* A_0, $A_1 \subseteq X$ *such that* $A_0 \cap A_1 = \emptyset$, $F_0 \subseteq A_0$, $F_1 \subseteq A_1$, A_0 *is increasing and* A_1 *is decreasing.*

A topological space endowed with a partially ordered $\leqslant$ is called a *totally disconnected ordered topological space* provided $x \not\leqslant y$ imply there exist an increasing clopen set A such that $x \in A$ and $y \notin A$. If, moreover, the space is compact, then it is said to be a *Priestley space.*

It is easily seen from the definition that every Priestley space is an ordered compact space (the complement of the graph of the order relation is clearly an open set), therefore the above mentioned properties of ordered compact spaces apply in particular to Priestley spaces. For every ordered compact space X set:

$$Q(X) = \{A \subseteq X \mid A \text{ increasing clopen}\},$$
$$Q'(X) = \{A \subseteq X \mid A \text{ decreasing clopen}\},$$

and notice that $A \in Q(X)$ iff $X \setminus A \in Q'(X)$.

2. Characterization of complete distributive lattices

In the sequel, unless otherwise mentioned, X will be a fixed Priestley space.

Lemma 3. *Let* $A \in Q(X)(A \in Q'(X))$ *and* D *an increasing (a decreasing) and closed subset of* A *and* $x \in A \setminus D$. *Then there exists* $A' \in Q(X)$ $(A' \in Q'(X))$ *such that* $D \subseteq A' \subseteq A$ *and* $x \notin A'$.

Proof. Let $y \in D$. As D is increasing, it follows that $y \not\leqslant x$, therefore there exist $W_y \in Q(X)$ such that $y \in W_y$ and $x \notin W_y$. On the other hand D is closed, hence compact, so that there exist $y_1, \ldots, y_n \in D$ such that $D \subseteq \bigcup_{i=1}^n W_{y_i}$. Let $W = \bigcup_{i=1}^n W_{y_i}$. Then: $W \in Q(X)$, $x \notin W$ and $D \subseteq W$. Taking $A' = W \cap A$ it follows obviously that $D \subseteq A' \subseteq A$, $x \notin A'$ and $A' \in Q(X)$. The case $A \in Q'(X)$ is treated similarly.

Corollary 1. *If* $D \subseteq X$ *is a decreasing and closed set, then* $D = \bigcap \{B \mid B \in Q'(X), D \subseteq B\}$.

Proof. $\subseteq$. Obvious.

$\supset$. Let $\mathscr{F} = \{B \mid B \in Q'(X), D \subseteq B\}$ and suppose that $\bigcap \mathscr{F} \not\subseteq D$. Let $x \in \bigcap \mathscr{F} \setminus D$. Lemma 3 applied for $A = X$ implies the existence of $A' \in \mathscr{F}$, $x \notin A'$. It follows that $x \notin \bigcap \mathscr{F}$, a contradiction. Therefore $D = \bigcap \mathscr{F}$. □

Corollary 2. *Let* $D \subseteq X$. *Then the following conditions are equivalent:*

(1) *D is an increasing and open set.*

(2) $D = \bigcup_{i \in I} A_i$, $A_i \in Q(X)$ *for any* $i \in I$.

Proof. (2) $\Rightarrow$ (1). Obvious.

(1)$\Rightarrow$(2). The set $D' = X\setminus D$ is decreasing and closed. Then:

$$X\setminus D = D' = \bigcap \{B \mid B \in Q'(X), X - D \subseteq B\},$$

by Corollary 1 applied to D'. It follows successively:

$$D = \bigcup \{X\setminus B \mid B \in Q'(X), X\setminus D \subseteq B\},$$
$$D = \bigcup \{X\setminus B \mid X\setminus B \in Q(X), X\setminus B \subseteq D\},$$
$$D = \bigcup \{A \mid A \in Q(X), A \subseteq D\}.$$

Theorem 1. *The following conditions are equivalent for a Priestley space X:*

(C) *$Q(X)$ is a complete lattice.*

(E0) *$D \subseteq X$, D is increasing and open implies that $\bar{D}^*$ is increasing clopen* (i.e., $\bar{D}^* \in Q(X)$).

Proof. (E0)$\Rightarrow$(C). It suffices to prove that the lattice $Q(X)$ is upper complete. Let $(A_i)_{i\in I} \subseteq Q(X)$ and $D = \bigcup_{i\in I} A_i$. D being increasing and open, the hypothesis (E0) implies that $\bar{D}^* \in Q(X)$. Obviously: $\bigcup_{i\in I} A_i \subseteq \bar{D}^*$. Besides $\bigcup_{i\in I} A_i \subseteq E \in Q(X)$ implies $D \subseteq E$, then $\bar{D} \subseteq E$, therefore $\bar{D}^* \subseteq E$. Hence there exists $\bigvee_{i\in I} A_i = \bar{D}^*$.

(C)$\Rightarrow$(E0). Let D an increasing and open subset of X. It follows from Corollary 2 that $D = \bigcup_{i\in I} A_i$, $A_i \in Q(X)$ for any $i \in I$. Set $A = \bigvee_{i\in I} A_i \in Q(X)$.

Obviously $D \subseteq A$, therefore $\bar{D}^* \subseteq A$.

$\bar{D}$ is closed, hence compact, so that $\bar{D}^*$ is increasing and closed by Lemma 1. Let us prove that $\bar{D}^* = A$. Suppose $\bar{D}^* \neq A$. It follows that there exists $x \in A\setminus \bar{D}^*$ and hence there exists $A' \in Q(X)$, $\bar{D}^* \subseteq A' \subseteq A$ and $x \notin A'$ by Lemma 3. Finally: $\bigcup_{i\in I} A_i \subseteq A' \subsetneq A$, which contradicts $A = \bigvee_{i\in I} A_i$. Therefore $\bar{D}^* = A$, hence $\bar{D}^* \in Q(X)$ which completes the proof. □

Remark that a totally disconnected space is a Priestley space with respect to identity taken as partial order, while property (E0) reduces to (E), therefore in this case Theorem 1 reduces to:

Theorem 0 [2, Ch. III, Section 4, Lemma 1]. *The following conditions are equivalent for a totally disconnected compact space X:*

(C) *The Boolean algebra of the clopen subsets of X is complete.*

(E) *X is extremal, i.e., $D \subseteq X$, D is open implies that $\bar{D}$ is clopen.*

Also, taking into account the dual equivalence between the category **Ld01** of distributive lattices with least and greatest element and the category **P** of Priestley spaces, Theorem 1 can be reformulated as follows:

Theorem 1'. *A distributive lattice with least and greatest element is complete if and only if the associated Priestley space fulfils* (E0).

In category language we obtain:

Theorem 1″. *The category of complete distributive lattices is dually equivalent to the category of those Priestley spaces which fulfil* (E0).

Finally let us prove:

Theorem 2. *An ordered compact space that fulfils* (E0) *is necessarily a Priestley space.*

Proof. Let X be an ordered compact space fulfilling (E0) and let us show that: $x \not\leqslant y$ implies there exist $A \in Q(X)$ such that $x \in A$ and $y \notin A$. So let $x \not\leqslant y$. Then the sets:

$$x^* = \{x\}^* \quad \text{and} \quad {}_*y = {}_*\{y\}$$

are closed by Lemma 1. Obviously $x^* \cap {}_*y = \emptyset$. Now Lemma 2 implies there exist an increasing and open set D and a decreasing and open set F such that $x^* \subseteq D$, ${}_*y \subseteq F$ and $D \cap F = \emptyset$.

Therefore $x \in D$ and $y \in F$, while (E0) implies $\bar{D}^* \in Q(X)$. But $\bar{D} \cap F = \emptyset$, hence $y \notin \bar{D}^*$ for otherwise there is an element $z \in \bar{D}$, $z \leqslant y$ and F being decreasing we would obtain $z \in F$, in contradiction with $\bar{D} \cap F = \emptyset$. Taking $A = \bar{D}^*$ we get $A \in Q(X)$, $x \in A$ and $y \notin A$. Thus X is a Priestley space.

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