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Invariant of the hypergeometric group associated to the quantum cohomology of the projective space

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Abstract

We present a simple method to calculate the Stokes matrix for the quantum cohomology of the projective spaces $\mathbf{CP}^{k-1}$ in terms of certain hypergeometric group. We present also an algebraic variety whose fibre integrals are solutions to the given hypergeometric equation.

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1. Generalized hypergeometric function

We begin with a short review on the motivation of our problem making reference to the works [5,11] where one can find precise definitions of the notions below.

At first, we consider a k-dimensional Frobenius manifold F with flat coordinates $(t_1, \ldots, t_k) \in F$ where the coordinate t_i corresponds to coefficients of the basis Δ_i of the quantum cohomology $H^*(\mathbf{CP}^{k-1})$. On $H^*(\mathbf{CP}^{k-1})$ one can define so called quantum multiplication

$$\Delta_\alpha \bullet \Delta_\beta = C^\gamma_{\alpha,\beta} \Delta_\gamma,$$

or

$$\frac{\partial}{\partial t_\alpha} \cdot \frac{\partial}{\partial t_\beta} = C^\gamma_{\alpha,\beta} \frac{\partial}{\partial t_\gamma},$$

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on the level of vector fields on F. The Frobenius manifold is furnished with the Frobenius algebra on the tangent space T_tF depending analytically on $t \in F$, $T_tF = (A_t, \langle\, , \rangle_t)$ where A_t is a commutative associative $\mathbf{C}$ algebra and $\langle\, , \rangle_t : A_t \times A_t \to \mathbf{C}$ a symmetric non-degenerate bilinear form. The bilinear form $\langle\, , \rangle_t$ defines a metric on F and the Levi-Civita connexion ∇ for this metric can be considered. Dubrovin introduces a deformed flat connexion $\widetilde{\nabla}$ on F by the formula $\widetilde{\nabla}_u v := \nabla_u v + xu \cdot v$ with $x \in \mathbf{C}$ the deformation parameter. Further he extends $\widetilde{\nabla}$ to $F \times \mathbf{C}$. Especially we have $\widetilde{\nabla}_{\partial/\partial x} = \frac{\partial}{\partial x} - E(t) - \frac{\mu}{x}$, where $E(t)$ corresponds to the multiplication by the Euler vector field $E(t) = \sum_{1 \leqslant j \neq 2 \leqslant k-1} (2-j) t_j \frac{\partial}{\partial t_j} + k t_2 \frac{\partial}{\partial t_2}$.

After [5,11] the quantum cohomology $\vec{u}(x) = (u_1(x), \ldots, u_k(x))$ for the projective space $\mathbf{CP}^{k-1}$ at a semisimple point $(0, t_2, 0, \ldots, 0)$ (i.e. the algebra $(A_t, \langle\, , \rangle_t)$ is semisimple there) satisfies the following system of differential equation:

$$\partial_x \vec{u}(x) = \left(k\mathcal{C}_2(t) + \frac{\mu}{x}\right)\vec{u}(x), \tag{1.1}$$

where

$$\mathcal{C}_2(0, t_2, 0, \ldots, 0) = \left(C^{\gamma}_{2,\beta}\right)_{1 \leqslant \beta, \gamma \leqslant k} = \begin{pmatrix} 0 & 0 & \ldots & 0 & \mathrm{e}^{t_2} \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

The matrix μ denotes a diagonal matrix with rational entries:

$$\mu = \mathrm{diag}\left\{-\frac{k-1}{2}, -\frac{k-3}{2}, \ldots, \frac{k-3}{2}, \frac{k-1}{2}\right\}.$$

The last component $u^k(z)$ (after the change of variables $z := kx\,\mathrm{e}^{t_2/k}$) of the above system for the quantum cohomology satisfies a differential equation as follows [11]:

$$\left[(\vartheta_z)^k - z^k\right] z^{(-k+1)/2} u^k(z) = 0, \tag{1.2}$$

with $\vartheta_z = z\frac{\partial}{\partial z}$. After the Fourier–Laplace transformation

$$\tilde{u}(\lambda) = \int \mathrm{e}^{\lambda z} z^{(-k+1)/2} u^k(z)\, \mathrm{d}z,$$

we obtain an equation as follows:

$$\left[(\vartheta_\lambda + 1)^k - \left(\frac{\partial}{\partial \lambda}\right)^k\right]\tilde{u}(\lambda) = 0.$$

Here the notation ϑ_λ stands for $\lambda\frac{\partial}{\partial\lambda}$. After multiplying λ^k from the left, we obtain

$$\left[\lambda^k(\vartheta_\lambda + 1)^k - \vartheta_\lambda(\vartheta_\lambda - 1)(\vartheta_\lambda - 2)\cdots\left(\vartheta_\lambda - (k-1)\right)\right]\tilde{u}(\lambda) = 0.$$

The equation for $\lambda\tilde{u}(\lambda)$, the Fourier–Laplace transform of $\frac{\partial}{\partial z} z^{(-k+1)/2} u^k(z)$ should be

$$\left[\lambda^k(\vartheta_\lambda)^k - (\vartheta_\lambda - 1)(\vartheta_\lambda - 2)\cdots(\vartheta_\lambda - k)\right]\left(\lambda\tilde{u}(\lambda)\right) = 0. \tag{1.3}$$

It is evident that the Stokes matrix for $\frac{\partial}{\partial z} z^{(-1+k)/2} u^k(z)$ is identical with that of the original solution $u^k(z)$.

Before proceeding further, we remind the following theorem that gives connexion between the Stokes matrix of the system (1.1) with the monodromy of Eq. (1.3). Let us consider the Fourier–Laplace transform of the system (1.1):

$$(\vartheta_\lambda + \mathrm{id}_k)\vec{\tilde{u}}(\lambda) = \bigl(k\mathcal{C}_2(t)\partial_\lambda - \mu\bigr)\vec{\tilde{u}}(\lambda). \tag{1.1$'$}$$

In a slightly more general setting, let us observe a system with regular singularities:

$$(\Lambda - \lambda \cdot \mathrm{id}_k)\partial_\lambda \vec{\tilde{u}}(\lambda) = \bigl(\mathrm{id}_k + A_1(\lambda)\bigr)\vec{\tilde{u}}(\lambda) \tag{1.1$''$}$$

with $\Lambda \in \mathrm{GL}(k, \mathbf{C})$ whose eigenvalues $(\lambda_1, \ldots, \lambda_k)$ are all distinct, $A_1(\lambda) \in \mathrm{End}(\mathbf{C}^k) \otimes \mathcal{O}_{\mathbf{C}}$ with $A_1(0) = \mathrm{diag}(\rho_1, \ldots, \rho_k)$ where none of the ρ_j's is an integer. We call solutions to a scalar differential equation deduced from $(1.1'')$ *component solutions*. Thus solutions to (1.3) are component solutions to $(1.1')$.

Theorem 1.1 [1,5]. *Under the assumption that the eigenvalues of the matrix $A_1(0)$ are distinct, the Stokes matrix S for the component solutions of* (1.1) *expresses the symmetric Gram matrix G of the component solutions of* $(1.1')$ *as follows*:

$${}^tS + S = 2G.$$

As for the definition of the Stokes matrix S for the system (1.4) we refer to [5,11]. The main theorem of this article is the following:

Theorem 1.2. *The i, j component S_{ij}, $1 \leqslant i, j \leqslant k$, of the Stokes matrix to the system* (1.1) *has the following expression*:

$$S = \begin{cases} (-1)^{i-j}\,{}_kC_{i-j}, & i \geqslant j, \\ 0, & i < j. \end{cases}$$

This theorem has already been shown by D. Guzzetti [11] by means of a detailed study of braid group actions etc on the set of solutions to (1.2). We present here another approach to understand the structure of the Stokes matrix.

Remark 1. In this article we observe the convention of the matrix multiplication as follows:

$$A \cdot x = (a_{ij})_{0 \leqslant i,j \leqslant k-1}(x_i)_{0 \leqslant i \leqslant k-1} = \left\langle \sum_{i=0}^{k-1} a_{ij} x_i \right\rangle_{0 \leqslant j \leqslant k-1}.$$

The matrix operates on the vector from left, in contrast to the convention used in [5,11].

On the other hand it has been known since [3] that a collection of coherent sheaves $\mathcal{O}(-i)$, $0 \leqslant i \leqslant k-1$, on $\mathbf{CP}^{k-1}$ satisfies the following relation

$$\mathrm{Hom}\bigl(\mathcal{O}(-i), \mathcal{O}(-j)\bigr) = S^{i-j}\bigl(\mathbf{C}^k\bigr), \quad 0 \leqslant i, j \leqslant k-1,$$
$$\mathrm{Ext}^\ell\bigl(\mathcal{O}(-i), \mathcal{O}(-j)\bigr) = 0, \quad 0 \leqslant i, j \leqslant k-1,\ \ell > 0.$$

These relation entails immediately the equality

$$\chi\big(\mathcal{O}(-i),\mathcal{O}(-j)\big) := \sum_{\ell=0}(-1)^{\ell}\operatorname{Ext}^{\ell}\big(\mathcal{O}(-i),\mathcal{O}(-j)\big) = \begin{cases} {}_{k+i-j-1}C_{i-j}, & i \geqslant j,\\ 0, & i < j.\end{cases}$$

We consider action of the braid group $\beta_i \in \mathbf{B}_k$, $1 \leqslant i \leqslant k-1$, that corresponds to the braid action between i-th basis and $(i+1)$-st basis of the space on which act a matrix. In our situation, β_i represents the braid action between $\mathcal{O}(1-i)$ and $\mathcal{O}(-i)$. In literature on coherent sheaves on algebraic varieties, this procedure is called mutation (e.g. [9]). Let us denote by β an element of the braid group $\mathbf{B}_k$

$$\beta = \beta_1(\beta_2\beta_1)\cdots(\beta_{k-1}\cdots\beta_2\beta_1).$$

We introduce a matrix of reordering $J = \delta_{i,k-1-i}$, $0 \leqslant i \leqslant k-1$. In this situation our Stokes matrix from Theorem 1.2 is connected with the matrix $\boldsymbol{\chi} := (\chi(\mathcal{O}(-i),\mathcal{O}(-j))$, $0 \leqslant i, j \leqslant k-1$, in the following way,

$${}^{t}S = J\beta\chi\beta J.$$

Eventually it turns out that $\chi = S^{-1}$. This general fact on the braid group is explained in [16], §2.4.

As our Stokes matrix is determined up to the change of basis, including effects by braid group actions, the Theorem 1.2 is a confirmation of an hypothesis [6] that the matrix for certain exceptional collection of coherent sheaves on a good Fano variety Y must coincide with the Stokes matrix for the quantum cohomology of Y.

Our strategy to prove Theorem 1.2 consists in the study of system $(1.1')$, instead of (1.1) itself.

Further we consider so called the Kummer covering (naming after N. Katz) of the projective space $\mathbf{CP}^1$ by $\zeta = \lambda^k$ to deduce an hypergeometric equation:

$$\left[\zeta(\vartheta_\zeta)^k - \left(\vartheta_\zeta - \frac{1}{k}\right)\left(\vartheta_\zeta - \frac{2}{k}\right)\cdots(\vartheta_\zeta - 1)\right]v(\zeta) = 0, \tag{1.4}$$

for $v(\lambda^k) = \lambda\tilde{u}(\lambda)$. We remind of us here a famous theorem due to A.H.M. Levelt that allows us to express the (global) monodromy group of the solution to (1.4) in quite a simple way. For the hypergeometric equation in general,

$$\left[\prod_{\ell=1}^{k}(\vartheta_\zeta - \alpha_\ell) - \zeta\prod_{\ell=1}^{k}(\vartheta_\zeta - \beta_\ell)\right]v(\zeta) = 0, \tag{1.5}$$

we define two vectors $(A_1,\ldots,A_k)$ and $(B_1,\ldots,B_k)$ in the following way:

$$\prod_{\ell=1}^{k}\big(t - \mathrm{e}^{2\pi\alpha_\ell \mathrm{i}}\big) = t^k + A_1t^{k-1} + A_2t^{k-2} + \cdots + A_k,$$

$$\prod_{\ell=1}^{k}\big(t - \mathrm{e}^{2\pi\beta_\ell \mathrm{i}}\big) = t^k + B_1t^{k-1} + B_2t^{k-2} + \cdots + B_k.$$

Definition 1.3. A linear map $L \in \mathrm{GL}(k, \mathbf{C})$ is called pseudo-reflexion if it satisfies the condition $\mathrm{rank}(\mathrm{id}_k - L) = 1$. A pseudo-reflexion R satisfying an additional condition $R^2 = \mathrm{id}_k$ is called a reflexion.

Proposition 1.4 [4,13]. *For the solutions to* (1.5)*, the monodromy action on them at the points* $\zeta = 0, \infty, 1$ *has the following expressions*:

$$
\begin{aligned}
h_0 &= \begin{pmatrix} 0 & 0 & \dots & 0 & -A_k \\ 1 & 0 & \dots & 0 & -A_{k-1} \\ 0 & 1 & \dots & 0 & -A_{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -A_1 \end{pmatrix}, \\
(h_\infty)^{-1} &= \begin{pmatrix} 0 & 0 & \cdots & 0 & -B_k \\ 1 & 0 & \dots & 0 & -B_{k-1} \\ 0 & 1 & \dots & 0 & -B_{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -B_1 \end{pmatrix},
\end{aligned}
\tag{1.6}
$$

whereas $h_1 = (h_0 h_\infty)^{-1}$ *is a pseudo-reflexion.*

It is worthy to notice that the above proposition does not precise for which bases of solution to (1.5) the monodromy is calculated. As a corollary to the Proposition 1.4, however, we see that the monodromy action on the solutions to our equation (1.4) can be written down with respect to a certain basis as follows:

$$
\begin{aligned}
h_0 &= \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \\
h_\infty &= \begin{pmatrix} {}_kC_1 & 1 & 0 & \dots & 0 & 0 \\ -{}_kC_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{k-1}{}_kC_{k-2} & 0 & 0 & \dots & 1 & 0 \\ (-1)^k{}_kC_{k-1} & 0 & 0 & \dots & 0 & 1 \\ -(-1)^k & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{1.7}
$$

In other words,

$$
\det(t - h_0) = t^k - 1, \qquad \det(t - h_\infty) = (t-1)^k. \tag{1.8}
$$

Furthermore we have,

$$
h_1 = \begin{pmatrix} (-1)^{k-1} & 0 & 0 & \dots & 0 & 0 \\ (-1)^{k-2}{}_kC_{k-1} & 1 & 0 & \dots & 0 & 0 \\ (-1)^{k-3}{}_kC_{k-2} & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ {}_kC_1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{1.9}
$$

In the next section we will see that the theory of Levelt supplies us with necessary data to calculate further the Stokes matrix of the solutions to (1.1)

2. Invariants of the hypergeometric group

Let us begin with a detailed description of the generators of the hypergeometric group defined for the solutions to Eq. (1.3).

Proposition 2.1 (cf. [8], I, 8.5). *The generators of the hypergeometric group H of Eq.* (1.3) *are expressed in terms of the matrices introduced in the Proposition* 1.4 *as follows*:

$$M_0 = h_0^k = 1, \quad M_1 = h_1 = (h_0 h_\infty)^{-1}, \quad M_\infty = h_\infty^k,$$
$$M_{\omega^i} = h_\infty^{-i} h_1 h_\infty^i \quad (i = 1, 2, \dots, k-1), \tag{2.1}$$

where M_t denotes the monodromy action around the point $t \in \mathbf{CP}^1_\lambda$. The generators around singular points $\omega^i = \mathrm{e}^{2\pi\sqrt{-1}i/k}$ naturally satisfy the Riemann–Fuchs relation:

$$M_\infty M_{\omega^{k-1}} M_{\omega^{k-2}} \cdots M_\omega M_1 = \mathrm{id}_k. \tag{2.2}$$

Proof. Let us think of a k-leaf covering $\widetilde{\mathbf{CP}}^1_\lambda$ of $\mathbf{CP}^1_\zeta$ that corresponds to the Kummer covering $\zeta^k = \lambda$. In lifting up the path around $\zeta = 1$ the first leaf of $\widetilde{\mathbf{CP}}^1_\lambda$, the monodromy h_1 is sent to the conjugation with a path around $\lambda = \infty$. That is to say we have $M_\omega = h_\infty^{-1} h_1 h_\infty$. For other leaves the argument is similar. □

Let us denote by X^K a $k \times k$ matrix that satisfies the relation

$$\bar{g} X^{K\,t} g = X^K, \tag{2.3}$$

for every element g of a group $K \subset \mathrm{GL}(k, \mathbf{C})$. From the definition, the set of all X^K for a group K represents a $\mathbf{C}$ vector space in general. We will call a matrix of this space the quadratic invariant of the group K.

In the special case in which we are interested, the following statement holds.

Lemma 2.2. *For the hypergeometric group H generated by the pseudo-reflexions as in* (2.1), *for every X^H there exists a non-zero $k \times k$ matrix $\widetilde{X}^H$ such that $X^H = \lambda \widetilde{X}^H$ for some $\lambda \in \mathbf{C} \setminus \{0\}$.*

Proof. The relation

$$h_1 X\, {}^t h_1 = X \tag{2.4}$$

gives rise to equations on x_{0j} and x_{j0}. That is to say, the first row of (2.4) corresponds to

$$(-1)^i {}_kC_i x_{00} - (-1)^{k-1} x_{0i} = x_{0i}, \quad 1 \leqslant i \leqslant k-1,$$

while

$$(-1)^i {}_kC_i x_{00} - (-1)^{k-1} x_{i0} = x_{i0}, \quad 1 \leqslant i \leqslant k-1.$$

Thus we obtained $2(k-1)$ linearly independent equations. Further by concrete calculus one can easily see that

$$M_{\omega^\ell} = \mathrm{id}_k + T_\ell,$$

where

$$T_\ell = \begin{pmatrix} t_0^{(\ell)}\tau_0 & t_1^{(\ell)}\tau_0 & \dots & t_\ell^{(\ell)}\tau_0 & 0 & \dots & 0 \\ t_0^{(\ell)}\tau_1 & t_1^{(\ell)}\tau_1 & \dots & t_\ell^{(\ell)}\tau_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_0^{(\ell)}\tau_{k-1} & t_1^{(\ell)}\tau_{k-1} & \dots & t_\ell^{(\ell)}\tau_{k-1} & 0 & \dots & 0 \end{pmatrix},$$

with $(k-\ell)$-zero columns from the right. The remaining columns are generated from T_1 after simple linear recurrent relations by an inductive way. The relation $M_\omega X^t M_\omega = X$ gives rise to new equations

$$\bigl(1 + t_1^{(1)}\tau_1\bigr)^2 x_{11} + \text{linear functions in } (x_{0i}, x_{i0}) = x_{11},$$

with $1 + t_1^{(1)}\tau_1 = -1 + ({}_kC_1)^2 \neq 1$ and

$$\bigl(1 + t_1^{(1)}\tau_1\bigr) x_{1i} + \text{linear functions in } (x_{0i}, x_{i0}, x_{11}) = x_{1i},$$
$$\bigl(1 + t_1^{(1)}\tau_1\bigr) x_{i1} + \text{linear functions in } (x_{0i}, x_{i0}) = x_{i1}.$$

Thus we get $2k-3$ new linearly independent equations. In general for (ℓ, ℓ) term, we get from the relation $M_{\omega^\ell} X\,{}^t M_{\omega^\ell} = X$, $1 \leqslant \ell \leqslant k-1$,

$$\bigl(1 + t_\ell^{(\ell)}\tau_\ell\bigr)^2 x_{\ell\ell} + \text{linear functions in } (x_{\nu i}, x_{i\nu}, 0 \leqslant \nu \leqslant \ell-1) = x_{\ell\ell},$$

with $1 + t_\ell^{(\ell)}\tau_\ell = -1 + ({}_kC_\ell)^2 \neq 1$. For $x_{i\ell}$

$$\bigl(1 + t_\ell^{(\ell)}\tau_\ell\bigr) x_{i\ell} + \text{linear functions in } (x_{\nu i}, x_{i\nu}, 0 \leqslant \nu \leqslant \ell-1, x_{\ell\ell}) = x_{i\ell}.$$

In this way we get a set of $2(k-1) + \sum_{\ell=1}^{k-1}(2(k-\ell)-1) = k^2 - 1$ independent linear equations with respect to the elements of X. $\square$

The quadratic invariant X^{H_0} for $H_0 = \{h_0, h_\infty\}$ is invariant with respect to H. After Lemma 2.2, **C** vector space of quadratic invariants X^H is one-dimensional. Thus every X^{H_0} is also X^H. Hence we can calculate the quadratic invariant X^H after the following relations,

$$\bar{h}_0 X^H\, {}^t h_0 = X^H, \qquad \bar{h}_\infty X^H\, {}^t h_\infty = X^H. \tag{2.5}$$

From [4] we know that the inverse to $X^{H_0} = X^H$, if it exists, must be a Toeplitz matrix i.e.:

$$\bigl(X^{H_0}\bigr)^{-1} = \begin{pmatrix} x_0 & x_1 & x_2 & \dots & x_{k-1} \\ x_{-1} & x_0 & x_1 & \dots & x_{k-2} \\ x_{-2} & x_{-1} & x_0 & \dots & x_{k-3} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ x_{-(k-1)} & x_{-(k-2)} & x_{-(k-3)} & \dots & x_0 \end{pmatrix}.$$

Making use of this circumstances, it is possible to show that the system of equations that arises from the relations

$$ {}^{t}h_{\infty}\big(X^{H_0}\big)^{-1}\bar{h}_{\infty} = \big(X^{H_0}\big)^{-1}, \qquad {}^{t}h_{0}\big(X^{H_0}\big)^{-1}\bar{h}_{0} = \big(X^{H_0}\big)^{-1}, $$

for $(X^{H_0})^{-1}$ consists of $2(k-1)$ equations.

$$ x_{k-1-i} = x_{-i-1}, \tag{2.6$'$} $$

$$ \begin{aligned} &(-1)^{k+1}x_{k-1-i} + (-1)^{k}{}_{k}C_{k-1}x_{k-2-i} + \cdots +_{k} C_3 x_{2-i} -_{k} C_2 x_{1-i} + kx_{-i} \\ &\quad = x_{-1-i}. \end{aligned} \tag{2.6$''$} $$

This calculates the matrix X^H for the case k-odd.

As for the case k-even, our matrix X^H has the following form

$$ X^H = \begin{pmatrix} 0 & y_1 & y_2 & \dots & y_{k-1} \\ y_{-1} & 0 & y_1 & \dots & y_{k-2} \\ y_{-2} & y_{-1} & 0 & \dots & y_{k-3} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ y_{-(k-1)} & y_{-(k-2)} & y_{-(k-3)} & \dots & 0 \end{pmatrix}, $$

where $y_{-(k-1)}, \dots, y_{k-1}$ satisfy $2(k-1)$ equations for some constant y_0,

$$ y_i + y_{-i} = 0, \quad y_i - y_{-i} = 2(-1)^i{}_kC_i\, y_0, \quad 1 \leqslant i \leqslant k-1, \tag{2.6$'''$} $$

which are derived from (2.5). Thus the matrix X^H for the case k-even is obtained.

We remember here a classical theorem on the pseudo-reflexions.

Theorem 2.3 (cf. Bourbaki Groupe et Algèbre de Lie, Chapitre V, §6, Exercise 3). *Let E be a vector space with basis $(e_1, \dots, e_d)$, and their dual basis $(f_1, \dots, f_d) \in E^*$. Let us set $a_{ij} = f_i(e_j)$. The pseudo-reflexion s_i with respect to the basis f_i is defined as*

$$ s_i(e_j) = e_j - f_i(e_j)e_i = e_j - a_{ij}e_i. $$

Set

$$ V = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{d1} \\ 0 & a_{22} & a_{32} & \dots & a_{d2} \\ 0 & 0 & a_{33} & \dots & a_{d3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{dd} \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ a_{12} & 0 & 0 & \dots & 0 \\ a_{13} & a_{23} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1d} & a_{2d} & \dots & a_{d-1,d} & 0 \end{pmatrix}. \tag{2.7} $$

Under these notations, the composition of all possible reflexions $s_d s_{d-1}\cdots s_1$ (*a Coxeter element*) *with respect to the basis $(e_1, \dots, e_d)$ is expressed as follows*:

$$ s_d s_{d-1} \cdots s_1 = (\mathrm{id}_d - V)(\mathrm{id}_d + U)^{-1}. \tag{2.8} $$

Proof. For $1 \leqslant i, k \leqslant d$ we define

$$ y_i = s_{i-1}\cdots s_1(e_i). $$

It is possible to see that

$$e_i = y_i + \sum_{k<i\leqslant d} a_{ki} y_k, \qquad s_d \cdots s_1(e_i) = y_i - \sum_{i\leqslant k\leqslant d} a_{ki} y_k.$$

The statement follows immediately from these relations. □

To establish a relationship between the invariant X^H and the Gram matrix necessary for calculus of the Stokes matrix, we investigate a situation where the generators of the hypergeometric group have special forms. Namely consider a hypergeometric group Γ of rank k generated by pseudo-reflexions $R_0, \ldots, R_{k-1}$ where

$$R_j = \mathrm{id}_k - Q_j, \tag{2.9}$$

with

$$Q_j = \begin{pmatrix} 0 & \ldots & 0 & t_{j0} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & t_{j1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & t_{j2} & 0 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & \ldots & 0 & t_{j,k-1} & 0 & \ldots & 0 \end{pmatrix}, \quad 0 \leqslant j \leqslant k-1, \tag{2.10}$$

all zero components except for the j-th column. Let us define the Gram matrix G associated to the above collection of pseudo-reflexions:

$$G = \begin{pmatrix} t_{00} & t_{10} & \cdots & t_{k-1,0} \\ t_{01} & t_{11} & \cdots & t_{k-1,1} \\ t_{02} & t_{12} & \cdots & t_{k-1,2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{0,k-1} & t_{1,k-1} & \cdots & t_{k-1,k-1} \end{pmatrix}. \tag{2.11}$$

We shall treat the cases where G is either symmetric or anti-symmetric. Let us introduce an upper triangle matrix S satisfying

$$G = S + {}^tS \quad (\text{resp. } G = S - {}^tS),$$

for a symmetric (anti-symmetric) matrix G. In the anti-symmetric case, we shall use a convention so that the diagonal part of S is a scalar multiplication on the unit matrix. It is easy to see that for the symmetric (resp. anti-symmetric) G the diagonal element $t_{jj} = 2$ (resp. $t_{jj} = 0$).

Proposition 2.4. *For an hypergeometric group Γ defined over* $\mathbf{R}$, *the following statements hold.*

(1) *Suppose that the space of quadratic invariant matrices X^Γ is* 1*-dimensional. Then X^Γ coincides with the Gram matrix G* (2.11) *up to scalar multiplication.*
(2) *The composition of all generators $R_0, \ldots, R_{k-1}$ gives us the Seifert form*:

$$R_{k-1} \cdots R_0 = \mp {}^tS \cdot S^{-1}, \tag{2.12}$$

where to the minus sign corresponds symmetric G and to the plus sign anti-symmetric G.

Proof. (1) It is enough to prove that the Gram matrix is a quadratic invariant. We calculate

$$R_j G^t R_j = (\mathrm{id}_k - Q_j) G (\mathrm{id}_k - {}^t Q_j).$$

It is easy to compute

$$Q_j G = (t_{aj} t_{jb})_{0 \leqslant a,b \leqslant k-1}, \qquad G^t Q_j = (t_{ja} t_{jb})_{0 \leqslant a,b \leqslant k-1},$$
$$Q_j G^t Q_j = t_{jj} G^t Q.$$

It yields the following equality,

$$G^t Q_j + Q_j G - Q_j G^t Q_j = \big(t_{jb}\big((1 - t_{jj}) t_{ja} + t_{aj}\big)\big)_{0 \leqslant a,b \leqslant k-1},$$

that vanishes for G symmetric with $t_{jj} = 2$ and for G anti-symmetric with $t_{jj} = 0$.

(2) It is possible to apply directly our situation to that of Theorem 2.3. In the symmetric case, $t_{ii} = 2$ and

$$V = \begin{pmatrix} 2 & t_{10} & t_{20} & \dots & t_{k-1,0} \\ 0 & 2 & t_{21} & \dots & t_{k-1,1} \\ 0 & 0 & 2 & \dots & t_{k-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ t_{10} & 0 & 0 & \dots & 0 \\ t_{20} & t_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{k-1,0} & t_{k-1,1} & \dots & t_{k-1,k-2} & 0 \end{pmatrix},$$

in accordance with the notation (2.7). The formula (2.8) means (2.12) with minus sign. In the anti-symmetric case $t_{ii} = 0$, $0 \leqslant i \leqslant k-1$, and (2.7) yields (2.11) with plus sign. □

Corollary 2.5. *We can determine the Stokes matrix S by the following relation*

$$S = (\mathrm{id}_k - R_{k-1} \cdots R_0)^{-1} G, \tag{2.13}$$

with the aid of the Gram matrix and pseudo-reflexions.

In some sense, a converse to Proposition 2.4 holds. To show this, we remember a definition and a proposition from [14].

Definition 2.6. The fundamental set $(u_0(\lambda), \dots, u_{k-1}(\lambda))$ of the system $(1.1'')$ is a set of its component solutions satisfying the following asymptotic expansion:

$$u_j(\lambda) = (\lambda - \lambda_j)^{\rho_j} \sum_{r=0}^{\infty} g_r^{(j)} (\lambda - \lambda_j)^r,$$

where $(\lambda_0, \dots, \lambda_{k-1})$ are eigenvalues of the matrix Λ. The exponents ρ_j are diagonal elements of the matrix $A_1(0)$.

After [14], the fundamental set to the system $(1.1'')$ is uniquely determined.

Proposition 2.7 [14]. *Every generator of an hypergeometric group Γ over* $\mathbf{R}$ *defined for the system of type* $(1.1'')$ (*without logarithmic solution*) *is a product of pseudo-reflexions of the following form expressed with respect to its fundamental set*:

$$M_j = \mathrm{id}_k - \begin{pmatrix} 0 & \dots & 0 & s_{j0} & 0 & \dots & 0 \\ 0 & \dots & 0 & s_{j1} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \cdots & 0 & s_{j,k-1} & 0 & \dots & 0 \end{pmatrix}, \tag{2.14}$$

where $s_{jj} = 2$ *or* 0.

We get the following corollary to the above Proposition 2.7.

Corollary 2.8. *Assume that the hypergeometric group Γ is generated by pseudo-reflexions $T_0, \dots, T_{k-1}$ such that* $\mathrm{rank}(T_i - \mathrm{id}_k) = 1$ *for* $0 \leqslant i \leqslant k-1$. *Then it is possible to choose a suitable set of pseudo-reflexions generators R_j like* (2.9), (2.10), *up to constant multiplication on Q_j, so that they determine the quadratic invariant Gram matrix like* (2.11).

Proof. Proposition 2.7 implies that every generator T_i is a product of pseudo-reflexions M_j with s_{jk} possibly different from t_{jk}. From the condition on the quadratic invariant X^{Γ} and Proposition 2.4, s_{jk} must coincide with t_{jk}. That is to say Γ must be generated by $M_0, \dots, M_{k-1}$ with $s_{ja}/t_{ja} = s_{jb}/t_{jb}$ for all $a, b, j \in \{0, \dots, k-1\}$. This means that Γ has as its generators the pseudo-reflexions $R_0, \dots, R_{k-1}$ of (2.12) up to constant multiplication on Q_j. $\square$

Proof of Theorem 1.2. First we remark that solutions to (1.3) have no logarithmic asymptotic behaviour around any of their singular points except for the infinity.

In the case with k odd for X^{H_0}, there exists $a \neq 0$ such that the vector $\vec{v}_0 :=^t (1 + (-1)^{k-1}, -k, {}_kC_2, \dots, (-1)^{k-2}{}_kC_{k-2}, (-1)^{k-1}{}_kC_{k-1}) \in \mathbf{R}^k$ satisfies the relation:

$$X^{H_0}\vec{v}_0 =^t (a, 0, 0, \dots, 0).$$

Actually this fact can be proven almost without calculation in the following way. First we introduce a series of vectors

$$\vec{w}_\ell = (x_{-\ell}, x_{-\ell+1}, \dots, x_{k-1-\ell}), \quad \ell = 0, 1, \dots, k-1.$$

Then Eq. $(2.6'')$ can be rewritten in terms of $\vec{w}_\ell$:

$$\vec{w}_\ell \cdot \vec{v}_0 = \sum_{i=0}^{k-1} (-1)^i {}_kC_i \cdot x_{i-\ell} = 0 \quad \text{for } 1 \leqslant \ell \leqslant k-1.$$

On the other hand, the vector $\vec{w}_0$ is linearly independent of the vectors $\vec{w}_1, \dots, \vec{w}_{k-1}$ by virtue of the construction of the matrix X. Therefore $\vec{w}_0 \cdot \vec{v}_0 \neq 0$ as $\vec{v}_0 \neq 0$. This means the existence of the non-zero constant a as above.

This relation with Corollary 2.8 gives immediately the expression below for the pseudo-reflexions

$$R_j = \mathrm{id}_k - \begin{pmatrix} 0 & \dots & 0 & (-1)^{j+k-1}{}_kC_j \cdot r & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & -(-1)^{k-1}k \cdot r & 0 & \dots & 0 \\ 0 & \dots & 0 & (1+(-1)^{k-1}) \cdot r & 0 & \dots & 0 \\ 0 & \dots & 0 & -k \cdot r & 0 & \dots & 0 \\ 0 & \dots & 0 & {}_kC_2 \cdot r & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & (-1)^{k-1-j}{}_kC_{k-j-1} \cdot r & 0 & \dots & 0 \end{pmatrix}, \quad 0 \leqslant j \leqslant k-1, \tag{2.15}$$

whose Gram matrix is equal to

$$\begin{aligned} G_{ij} &= (-1)^{i-j+k-1}{}_kC_{i-j} \cdot r, & i > j, \\ G_{ii} &= (1+(-1)^{k-1}) \cdot r, & i = j, \\ G_{ij} &= (-1)^{i-j}{}_kC_{j-i} \cdot r, & j > i \end{aligned} \tag{2.16}$$

with some constant r. As for the case k-even, Eqs. $(2.6''')$ and Corollary 2.8 gives us the expression (2.15) for the pseudo-reflexion generators.

Taking into account Theorem 1.1 for the symmetric Gram matrix, we obtain the desired statement for the case k-odd, as it is required from Proposition 2.7 $G_{ii} = 2 = 2r$.

For the case k-even, we remember a statement on the Stokes matrix from [1] (Proposition 1.2) which claims that if the matrix μ of (1.1) has integer eigenvalues, the equality $\det(S + {}^tS) = 0$ must hold. Corollary 2.5 gives us the relation

$$S = \left(\mathrm{id}_k + (\mathrm{id}_k - V)(\mathrm{id}_k + U)^{-1}\right)^{-1} G = (\mathrm{id}_k + U)G^{-1}G = \mathrm{id}_k + U,$$

with

$$U_{ij} = (-1)^{i-j+k-1}{}_kC_{i-j} \cdot r, \quad i > j.$$

We shall choose the constant $r = 1$ so that $S + {}^tS = 2\,\mathrm{id}_k + U + {}^tU$ possesses an eigenvector $(1, -1, \dots, 1, -1)$ with zero eigenvalue. □

Remark 2. The Gram matrix (2.16) that has been calculated for the fundamental set (Definition 2.6) of Eq. (1.3) gives directly a suitable Stokes matrix we expected. For other Fano varieties, however, the Gram matrix calculated with respect to the fundamental set does not necessarily give a desirable form, as it is seen from the case of odd dimensional quadrics. This situation makes us to be careful in the choice of the base of solutions for which we calculate the Gram matrix.

3. Geometric interpretation of the hypergeometric equation

In this section we show that Eq. (1.4) arises from the differential operator that annihilates the fibre integral associated to the family of variety defined as a complete intersection

$$X_s := \left\{(x_0, \dots, x_k) \in \mathbf{C}^{k+1};\ f_1(x) + s = 0,\ f_2(x) + 1 = 0\right\}. \tag{3.1}$$

where

$$f_1(x) = x_0 x_1 \cdots x_k, \qquad f_2(x) = x_0 + x_1 + \cdots + x_k.$$

This result has been already announced by [7,8] and [2]. Our main theorem of this section is the following

Theorem 3.1. *Let us assume that* $\Re(f_1(x) + s)|_\Gamma < 0$, $\Re(f_2(x) + s)|_\Gamma < 0$, *out of a compact set for a Leray coboundary cycle* $\Gamma \in H^{k+1}(\mathbf{C}^{k+1} \setminus X_s)$ *avoiding the hypersurfaces* $f_1(x) + s = 0$ *and* $f_2(x) + 1 = 0$. *For such a cycle we consider the following residue integral*:

$$I^{(v_1,v_2)}_{x^{\mathbf{i}},\Gamma}(s) = \int_\Gamma x^{\mathbf{i}+\mathbf{1}}\big(f_1(x)+s\big)^{-v_1}\big(f_2(x)+1\big)^{-v_2}\frac{\mathrm{d}x}{x^{\mathbf{1}}}, \tag{3.2}$$

for the monomial $x^{\mathbf{i}} := x_0^{i_0}\cdots x_k^{i_k}$, $x^{\mathbf{1}} := x_0\cdots x_k$. *Then the integral* $I^{(1,1)}_{x^0,\Gamma}(s)$ *satisfies the following hypergeometric differential equation*

$$\left[\vartheta_s^k - k^k s\left(\vartheta_s + \frac{1}{k}\right)\left(\vartheta_s + \frac{2}{k}\right)\cdots\left(\vartheta_s + \frac{k}{k}\right)\right] I^{(1,1)}_{1,\Gamma}(s) = 0 \tag{3.3}$$

which has unique holomorphic solution at $s = 0$,

$$I_0(s) = \sum_{m \geqslant 0} \frac{(km)!}{(m!)^k} s^m. \tag{3.4}$$

We shall put $\zeta = 1/(k^k s)$, to get (1.4) from (3.3). Our calculus is essentially based on the Cayley trick method developed in [15].

Proof of Theorem 3.1. Let us consider the Mellin transform of the fibre integral (3.2)

$$M^{(v_1,v_2)}_{\mathbf{i},\Gamma}(z) := \int_\Pi s^z I^{(v_1,v_2)}_{x^{\mathbf{i}}}(s)\frac{\mathrm{d}s}{s}. \tag{3.5}$$

For the Mellin transform (3.5), we have the following

$$\begin{aligned} M^{(v_1,v_2)}_{\mathbf{i},\Gamma}(z) = g(z)\prod_{\ell=0}^{k-1}\Gamma(z+i_\ell+1-v_2)\,\Gamma\left(-\sum_{\ell=0}^{k-1}(i_\ell+1) - kz + v_1 + kv_2\right) \\ \times\Gamma(-z+v_2)\Gamma(z), \end{aligned} \tag{3.5$'$}$$

with $g(z)$ a rational function in $\mathrm{e}^{\pi \mathrm{i} z}$. The formula (3.5′) shall be proven below. In substituting $\mathbf{i} = 0$, $v_1 = v_2 = 1$, we see that

$$I^{(1,1)}_{x^0,\Gamma}(s) = \int_{\check\Pi} s^{-z} g(z)\frac{\Gamma(z)^k}{\Gamma(kz)}\,\mathrm{d}z,$$

where $\check\Pi$ denotes the path $(-\mathrm{i}\infty, +\mathrm{i}\infty)$ avoiding the poles of $\Gamma(z) = 0, -1, -2, \ldots$. From this integral representation, Eq. (3.3) immediately follows in taking account the fact that the factor $g(z)$ plays no role in establishment of the differential equation. □

Proof of (3.5′). In making use of the Cayley trick, we transform the integral (3.5) into the following form

$$M^{(v_1,v_2)}_{\mathbf{i},\Gamma}(z) = \int\limits_{\Pi\times\mathbf{R}^2_+\times\Gamma} x^{\mathbf{i}+\mathbf{1}}\, \mathrm{e}^{y_1(f_1(x)+s)+y_2(f_2(x)+1)} y_1^{v_1} y_2^{v_2} s^z \frac{\mathrm{d}x}{x^{\mathbf{1}}}\frac{\mathrm{d}y}{y^{\mathbf{1}}}\frac{\mathrm{d}s}{s^{\mathbf{1}}}, \tag{3.6}$$

with $\mathbf{R}_+$ the positive real axis in $\mathbf{C}_{y_p}$ for $p = 1$ or 2. Here we introduce new variables $T_0, \ldots, T_{k+2}$,

$$\begin{aligned} &T_i = y_1 x_i, \quad 0 \leqslant i \leqslant k-1, \\ &T_k = y_1 s, \quad T_{k+1} = y_2 x_0 x_1 \cdots x_{k-1}, \quad T_{k+2} = y_2 \end{aligned} \tag{3.7}$$

in such a way that the phase function of the right-hand side of (3.6) becomes

$$y_1(f_1(x)+s) + y_2(f_2(x)+1) = T_0 + T_1 + \cdots + T_{k+2}.$$

If we set

$$\begin{aligned} &\mathrm{Log}\, T :=^t (\log T_0, \ldots, \log T_{k+2}), \\ &\Xi :=^t (x_0, \ldots, x_{k-1}, s, y_1, y_2), \\ &\mathrm{Log}\, \Xi :=^t (\log x_0, \ldots, \log x_{k-1}, \log s, \log y_1, \log y_2). \end{aligned}$$

Then the above relationship (3.7) can be written down as

$$\mathrm{Log}\, T = \mathsf{L} \cdot \mathrm{Log}\, \Xi, \tag{3.8}$$

where

$$\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 1 \end{bmatrix}.$$

This yields immediately

$$\mathrm{Log}\, \Xi = \mathsf{L}^{-1} \cdot \mathrm{Log}\, T,$$

with

$$\mathsf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & \ldots & 1 & -k & -1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & \ldots & -1 & k & 1 & 0 \end{bmatrix}.$$

If we set

$$(i_0, \dots, i_{k-1}, z, v_1, v_2) \cdot \mathsf{L}^{-1} = \big(\mathcal{L}_0(\mathbf{i}, z, v_1, v_2), \dots, \mathcal{L}_{k+2}(\mathbf{i}, z, v_1, v_2)\big). \tag{3.9}$$

then we can see that

$$\begin{aligned} M^{(v_1,v_2)}_{\mathbf{i},\Gamma}(z) &= \int\limits_{\Pi\times\mathbf{R}^2_+\times\Gamma} x^{\mathbf{i}+\mathbf{1}}\, \mathrm{e}^{T_0+\cdots+T_{k+2}} y_1^{v_1} y_2^{v_2} s^z \frac{\mathrm{d}x}{x^{\mathbf{1}}}\frac{\mathrm{d}y}{y^{\mathbf{1}}}\frac{\mathrm{d}s}{s^{\mathbf{1}}} \\ &= \int\limits_{\mathsf{L}_*(\Pi\times\mathbf{R}^2_+\times\Gamma)} \mathrm{e}^{T_0+\cdots+T_{k+2}} \prod_{0\leqslant i\leqslant k+2} T_i^{\mathcal{L}_i(\mathbf{i},z,v_1,v_2)} \bigwedge_{0\leqslant i\leqslant k+2} \frac{\mathrm{d}T_i}{T_i}. \end{aligned}$$

Here $\mathsf{L}_*(\Pi\times\mathbf{R}^2_+\times\Gamma)$ denotes a $(k+3)$-chain in $T_0\cdots T_{k+2}\neq 0$ that obtained as a image of $\Pi\times\mathbf{R}^2_+\times\Gamma$ under the transformation induced by L. In view of the choice of the cycle Γ, we can apply the formula to calculate Γ function to our situation:

$$\int\limits_C \mathrm{e}^{-T} T^{\sigma}\frac{\mathrm{d}T}{T} = \big(1-\mathrm{e}^{2\pi\mathrm{i}\sigma}\big)\Gamma(\sigma),$$

for the unique nontrivial cycle C turning around $T=0$ that begins and returns to $\Re T\to +\infty$. Here one can consider the natural action $\lambda : C_a \to \lambda(C_a)$ defined by the relation,

$$\int\limits_{\lambda(C_a)} \mathrm{e}^{-T_a} T_a^{\sigma_a}\frac{\mathrm{d}T_a}{T_a} = \int\limits_{(C_a)} \mathrm{e}^{-T_a}\big(\mathrm{e}^{2\pi\sqrt{-1}}T_a\big)^{\sigma_a}\frac{\mathrm{d}T_a}{T_a}.$$

In terms of this action $\mathsf{L}_*(\Pi\times\mathbf{R}^2_+\times\Gamma)$ is shown to be homologous to a chain

$$\sum_{(j_0^{(\rho)},\dots,j_{k+2}^{(\rho)})\in[1,\Delta]^{k+3}} m_{j_0^{(\rho)},\dots,j_{k+2}^{(\rho)}} \prod_{a=0}^{1} \lambda^{j_a^{(\rho)}}(\mathbf{R}_+) \prod_{a'=2}^{k+2} \lambda^{j_{a'}^{(\rho)}}(C_{a'}),$$

with $m_{j_0^{(\rho)},\dots,j_{k+2}^{(\rho)}}\in\mathbf{Z}$. This explains the appearance of the factor $g(z)$ in front of the Γ function factors in $(3.5')$.

The direct calculation of (3.9) shows that

$$\begin{aligned} &\mathcal{L}_\ell(\mathbf{i}, z, v_1, v_2) = z + i_\ell + 1 - v_2, \quad 0\leqslant \ell\leqslant k-1, \\ &\mathcal{L}_k(\mathbf{i}, z, v_1, v_2) = -\sum_{\ell=0}^{k-1}(i_\ell+1) + v_1 + k(v_2 - z), \\ &\mathcal{L}_{k+1}(\mathbf{i}, z, v_1, v_2) = -z + v_2, \quad \mathcal{L}_{k+2}(\mathbf{i}, z, v_1, v_2) = z. \end{aligned}$$

This shows the formula $(3.5')$. □

In combining Theorems 1.2, 3.1, we can state that we found out a deformation of an algebraic variety $X_\lambda = \{(\lambda/k)^k(x_0x_1\cdots x_k)+1=0, x_0+x_1+\cdots+x_k=1=0\}$ such that its variation gives rise to Eq. (1.3). It means that we establish a connexion between an exceptional collection of $\mathbf{CP}^{k-1}$ and a set of vanishing cycles for its mirror counter part X_λ. Thus our theorems give an affirmative answer to the hypothesis stating the existence of such relationship between two mirror symmetric varieties (so called Bondal–Kontsevich hypothesis) in a special case. See [8] and [12] in this respect for the detail.

It is known from the theory of period integrals associated to the complete intersections [10] that the integrals $I^{(v_1,v_2)}_{x^{\mathbf{i}},\Gamma}((k/\lambda)^k)$ for $\Gamma \in H_{k+1}(\mathbf{C}^{k+1} \setminus X_\lambda, \mathbf{Z})$ has singularities only at the discriminant locus of X_λ where the cycle Γ becomes singular (or vanishes). On the other hand, in §2 we found a set of solutions called fundamental such that $u_j(\lambda)$ has an singular point $\lambda = \mathrm{e}^{2\pi\sqrt{-1}j/k}$. Two solutions to an hypergeometric differential equation (1.3) with the same assigned asymptotic behaviours at all possible singular points must coincide. In combination of this argument with the Picard–Lefschetz theorem, we obtain the following.

Corollary 3.2. *There exists a set of cycles $\gamma_j \in H_{k-1}(X_\lambda, \mathbf{Z})$, $0 \leqslant j \leqslant k-1$, such that for their Leray's coboundary $\Gamma_j \in H_{k+1}(\mathbf{C}^{k+1} \setminus X_\lambda, \mathbf{Z})$ we have the identity,*

$$I^{(1,1)}_{x^0,\Gamma_j}\left(\left(\frac{k}{\lambda}\right)^k\right) = u_j(\lambda), \quad 0 \leqslant j \leqslant k-1,$$

with $u_j(\lambda)$ the fundamental solution to (1.3) *in the sense of Definition* 2.6. *Consequently the Gram matrix G of* (2.16) *is equal to the intersection matrix $(\langle\gamma_i, \gamma_j\rangle)_{0\leqslant i,j\leqslant k-1}$ after proper choice of constant $r = 1$.*

References

[1] W. Balser, W.B. Jurkat, D.A. Lutz, On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities, I, II, SIAM J. Math. Anal. 12 (5) (1981) 691–721;
W. Balser, W.B. Jurkat, D.A. Lutz, SIAM J. Math. Anal. 19 (2) (1988) 398–443.

[2] S. Barannikov, Semi-infinite Hodge structures and mirror symmetry for projective spaces, math.AG/0010157.

[3] A.A. Beilinson, Coherent sheaves on $\mathbf{CP}^n$ and problems of linear algebra, Funct. Anal. Appl. 13 (2) (1978) 68–69.

[4] F. Beukers, G. Heckman, Monodromy for the hypergeometric function ${}_nF_{n-1}$, Invent. Math. 95 (1989) 325–354.

[5] B. Dubrovin, Painlevé transcendents in two dimensional topological field theory, The Painlevé Property, in: CRM Ser. Math. Phys., Springer, 1999, pp. 287–412; math.AG/9803107.

[6] B. Dubrovin, Geometry and analytic theory of Frobenius manifolds, in: Proceedings ICM Berlin vol. II, 1998, pp. 315–326; math.AG/9807034.

[7] A. Givental', Equivariant Gromov–Witten invariants, Intern. Math. Res. Notices 13 (1996) 613–663.

[8] V.V. Golyshev, Riemann–Roch variations, Izv. Math. 65 (5) (2001) 853–887.

[9] A.L. Gorodentsev, S.A. Kuleshov, Helix theory, MPIM preprint series 97, 2001.

[10] G.-M. Greuel, Der Gauss–Manin Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten, Math. Ann. 214 (3) (1975) 235–266.

[11] D. Guzzetti, Stokes matrices and monodromy of the quantum cohomology of projective spaces, Comm. Math. Phys. 207 (2) (1999) 341–383.

[12] M. Kontsevich, Homological algebra of mirror symmetry, in: Proceedings of ICM (Zürich, 1994) 1994, pp. 120–139; alg-geom/9411018.

[13] A.H.M. Levelt, Hypergeometric functions, Indag. Math. 23 (1961) 361–403.

[14] K. Okubo, On the Group of Fuchsian Equations, Tokyo Metropolitan University, 1987;

K. Okubo, Connection problems for systems of linear differential equations, in: Lecture Notes in Math., vol. 243, Springer, 1971, pp. 238–248.

[15] S. Tanabé, Transformée de Mellin des intégrales – fibres associées à l'intersection complète non-dégénérée, Math. AG/0405399.

[16] E. Zaslow, Solitons and Helices: The Search for a Math-Phys Bridge, Comm. Math. Phys. 175 (1996) 337–375.