Research problems from the 20th British Combinatorial Conference

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The Research Problems section presents unsolved problems in discrete mathematics. In special issues, these typically are problems collected by the guest editors. In regular issues, they generally consist of problems submitted on an individual basis.

Older problems are acceptable if they are not widely known and the exposition features a new partial result. Concise definitions and commentary (such as motivation or known partial results) should be provided to make the problems accessible and interesting to a broad cross-section of the readership. Problems are solicited from all readers. Ideally, they should be presented in the style below, occupy at most one journal page, and be sent to Douglas B. West.

Most of the problems in this issue were presented at the problem session of the 20th British Combinatorial Conference at the University of Durham, 10–15 July 2005. Some submitted after the session were added. Problem BCC20.11 on the parity of the numbers of complete bipartite and tripartite subgraphs of a graph, posed by Ron Shaw, has been solved by Balazs Montagh and has been withdrawn from the list: see Shaw’s paper in this volume for further details. Two other problems were removed for editorial reasons. Progress on other problems is reported in editorial notes. The problems were collected by Peter J. Cameron.

Comments and questions of a technical nature about a particular problem should be sent to the correspondent for that problem. Other comments and information about partial or full solutions should be sent to Professor Cameron (for potential later updates).

PROBLEM 910. (BCC20.2) A more general “cycle-plus-triangles” problem

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The famous “cycle-plus-triangles” problem was posed by Erdős and resolved positively by Fleischner and Stiebitz \cite{1}, see \cite{2}. In that problem, a cycle of length $3m$ is augmented by $m$ disjoint triangles on the same vertex set. The theorem is that the chromatic number of the resulting graph is $3$ (in fact, Fleischner and Stiebitz proved the stronger result that the graph is 3-choosable). Here we pose a generalization of this Erdős question.

\textbf{Question.} Let $G$ be an $n$-vertex graph that decomposes into a spanning cycle plus some vertex-disjoint triangles and edges. Is it always true that $G$ has chromatic number $3$ if $n$ is not congruent to $1$ mod $3$?
Comment. The restriction on \( n \) is necessary. Otherwise, there are counterexamples with exactly two single-edge subgraphs in the decomposition: the simplest is the complete graph on 4 vertices, and Sachs found a counterexample on 7 vertices (see [2]). Indeed, counterexamples exist for all \( n \) congruent to 1 mod 3. On the other hand, if there are no triangles in the decomposition, then the maximum degree is 3, and the answer follows immediately from Brooks’ theorem. The problem is to determine what happens between these extremal cases, and the conjecture is that the chromatic number never rises above 3.

When there is only one loose edge, Häggkvist and Johansson (personal communication) proved that the answer is positive, using the same method as Fleischner and Stiebitz. For graphs on at most 12 vertices, the proposer has verified the conjecture using exhaustive search by computer.

References


PROBLEM 911. (BCC20.3) Choosability with separation

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Let \( G = (V, E) \) be a graph. A \( k \)-list assignment for \( G \) consists of a list \( L(v) \subseteq \mathbb{Z} \) of size \( k \), for each \( v \in V \). The choosability \( ch(G) \) of \( G \) is the smallest \( k \) such that, for any \( k \)-list assignment, a proper colouring \( c : V \to \mathbb{Z} \) can be found with \( c(v) \in L(v) \) for \( v \in V \).

The choosability with separation \( s \) of \( G \), denoted \( ch_s(G) \), is the smallest \( k \) such that, for any \( k \)-list assignment, a colouring \( c \) can be found with \( c(v) \in L(v) \) for \( v \in V \), and \( |c(v) - c(w)| \geq s \) for \( vw \in E \); hence \( ch_1(G) = ch(G) \).

Problem. For \( s \geq 2 \), find an upper bound for \( ch_s(G) \) in terms of \( ch(G) \).

Conjecture. For all \( s \geq 2 \) and every graph \( G \), \( ch_s(G) \leq (2s - 1)(ch(G) - 1) + 1 \).

Comment. The best general upper bound known to the proposer is superexponential, based on a probabilistic result of Alon [1] which relates \( ch(G) \) to the average degree of \( G \). However, the correct bound surely ought to be linear. The conjectured value is attained, for any \( s \geq 2 \) and \( ch(G) = k \geq 2 \), by the complete bipartite graph \( G = K_{k-1,m} \), if \( m \) is sufficiently large.

The conjecture has been proved for graphs with \( ch(G) = 2 \), using the structural description of such graphs in [2]. This proof method is unlikely to work in general since, for \( k \geq 3 \), no structural characterisation of graphs \( G \) with \( ch(G) \leq k \) is known (and it is unlikely that any useful characterisation exists). The conjecture and some related results appear in [3].

References


PROBLEM 912. (BCC20.4) Arc-dominated digraphs

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A digraph $D$ of minimum out-degree $d$ is $d$-arc-dominated if for every arc there is one vertex of degree $d$ dominating its endpoints. That is, for all $xy \in E(D)$, there exists $z \in V(D)$ with out-degree $d$ such that $zx, zy \in E(D)$. We consider only digraphs that are oriented graphs, meaning orientations of simple graphs.

A conjecture attributed to Bermond and Thomassen [1, p. 555] states that, for any $k \geq 1$, a digraph of minimum out-degree at least $2k - 1$ contains $k$ (vertex)-disjoint directed cycles.

**Problem 1.** Characterize the $d$-arc-dominated oriented graphs.

**Conjecture 2.** The Bermond–Thomassen Conjecture holds for $d$-arc-dominated digraphs. That is, if $d \geq 2k - 1$ and $D$ is a $d$-arc-dominated digraph, then $D$ contains $k$ disjoint cycles.

**Comment.** A conjecture in the preliminary version of these problems, asserting that any arc-dominated digraph contains a directed triangle, has been refuted by Stéphan Thomassé.

**Reference**


**PROBLEM 913.** (BCC20.5) Mediated digraphs

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A digraph $D$ is mediated if every two nonadjacent vertices $x$ and $y$ have a common successor (a vertex $z$ such that $xz, yz \in E(D)$). Tournaments and symmetric digraphs of diameter 2 (we allow 2-cycles) are mediated. Mediated digraphs are models of quantum nonlocality in quantum mechanics; this connection is discussed in [1].

Let $\Delta^-(D) = \max\{d^-(x): x \in V(D)\}$, and let $\mu(n)$ be the minimum of $\Delta^-(D)$ over mediated digraphs with $n$ vertices. The figure below shows that $\mu(6) \leq 2$. For $n \geq 4$, it is easy to show that $\mu(n) > 1$. The value of $\mu(n)$ is of interest in the quantum mechanical context.

![Mediated Digraph](image)

**Question 1.** Is it true that $\mu(n + 1) \geq \mu(n)$ for all $n$?

**Conjecture 2.** There is a constant $c$ such that $\mu(n) \leq f(n) + c$, where $f(n) = \left\lceil \frac{1}{2}\left(\sqrt{4n - 3} - 1\right)\right\rceil$.

**Comment.** It is known that $\mu(n) \leq f(n) + 1$ for $n \leq 133$. In general, $f(n) \leq \mu(n) \leq f(n)(1 + o(1))$ (see [1]). The lower bound comes from an easy counting argument. Equality holds when $n = q^2 + q + 1$ if and only if there is a projective plane of order $q$.

**Reference**

PROBLEM 914. (BCC20.6) Separability of the combinatorial Laplacian

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For a graph $G$, the (unit trace) combinatorial Laplacian $L(G)$ is the matrix defined by

$$L(G) = \frac{1}{\text{tr}(D(G))}(D(G) - A(G)),$$

where $A(G)$ is the adjacency matrix, $D(G)$ is the diagonal matrix of vertex degrees, and tr$(D(G))$ is degree-sum (the trace of $D(G)$).

When the number of vertices factors as $p$ times $q$, we view the vertices of $G$ as the ordered pairs $(i, j)$ such that $1 \leq i \leq p$ and $1 \leq j \leq q$. The partial transpose of $G$ with respect to this labelling, denoted by $G^T$, is the graph defined as follows: $V(G^T) = V(G)$, and the unordered pair $\{(i, j), (k, l)\}$ is an edge of $G^T$ if and only if $\{(i, l), (k, j)\}$ is an edge of $G$.

A density matrix is an Hermitian, positive-semidefinite, unit trace matrix. For example, $L(G)$ is a density matrix.

The matrix $L(G)$ is separable with respect to $p$ and $q$ if there exist $p \times p$ density matrices $M_1, \ldots, M_k$ and $q \times q$ density matrices $N_1, \ldots, N_k$ (not necessarily distinct) such that

$$L(G) = \sum_{i=1}^{k} \omega_i M_i \otimes N_i,$$

where $\omega_i \in (0, 1]$ and $\sum_{i=1}^{k} \omega_i = 1$.

Conjecture. If $G$ is a simple graph with $pq$ vertices, then $L(G)$ is separable with respect to $p$ and $q$ if and only if $\Delta(G) = \Delta(G^T)$.

Comment. This conjecture is interesting in the context of quantum information theory (see [1,2,3]).

References

PROBLEM 915. (BCC20.7) Hamiltonicity and orthogonality

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An $n$-vertex digraph $D$ supports the $n \times n$ complex matrix $M$ provided that $(i, j) \in E(D)$ if and only if $M_{i,j} \neq 0$. An $n \times n$ complex matrix $U$ is unitary if $U^{-1} = U^\dagger$, where $U^\dagger$ denotes the adjoint of $U$. 
Conjecture. If a digraph $D$ supports a unitary matrix, then each component of $D$ has a spanning cycle.

Comment. Undirected graphs are naturally included in the conjecture.

Reference

PROBLEM 916. (BCC20.8) Positive and negative eigenvalues

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Let $s$ and $t$ be the numbers of positive and negative eigenvalues of the adjacency matrix of a graph. Torgašev [2] (see also [1]) proved that $s$ is bounded when $t$ is fixed.

Question. What is the best bound on $s$ as a function of $t$? In particular, is there a polynomial (or even a quadratic) bound?

References

PROBLEM 917. (BCC20.9) Parity of clique numbers of Paley graphs

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Let $q$ be a prime power congruent to 1 mod 4. The Paley graph $P_q$ is defined as follows: $V(G)$ is the finite field $\mathbb{F}_q$, and $xy \in E(G)$ if and only if $x - y$ is a nonzero square in $\mathbb{F}_q$.

Shearer [2] computed the clique numbers $\omega(P_p)$ for prime $p$ with $p \leq 7000$. He found that about four-fifths of them are odd.

Problem. Explain this observation!

Comment. The proportion of $\omega(P_p)$ which are odd does not appear to decrease as $p$ increases.

It is thought that some Paley graphs might satisfy $\omega(P_q) \leq (2 - \varepsilon) \log_2 q$ for some $\varepsilon > 0$. Since $P_q$ is self-complementary, $\omega(P_q) = \pi(P_q)$, so such a result would improve the lower bound on diagonal Ramsey numbers. This is very hard, but progress on the parity question may throw some light on it.

If $q$ is a square, then $\omega(P_q) = \sqrt{q}$. A conjecture of Baker et al. [1] would then imply that the second largest maximal cliques in $P_q$ have odd order.

References
PROBLEM 918. (BCC20.10) A determinant problem

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Let $p$ be a prime congruent to 3 mod 4, and let $k = (p + 1)/2$. Let $A$ be the $k \times k$ matrix with $(i, j)$ entry $((j - i)/p)$, where $(x/p)$ is the Legendre symbol (equal to $+1$ if $x$ is a nonzero square mod $p$, to $-1$ if $x$ is a nonsquare mod $p$, and to 0 if $p$ divides $x$).

**Conjecture.** $\det(A) = 1$.

**Comment.** This conjecture is true for primes $p < 1000$. See [1] for background and related results.

**Reference**

PROBLEM 919. (BCC20.12) Two conjectures on cylinders

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A cylinder is the union of $p$ parallel lines in the affine space $AG(3, p)$, where $p$ is prime. We seek characterisations of cylinders.

We say that a direction (a parallel class of lines) is *not determined* by a point set $S$ if every line in the class contains at most one point of $S$.

**Conjecture 1 (Strong cylinder conjecture).** Let $S$ be a set of $p^2$ points in $AG(3, p)$. If $|S \cap P|$ is divisible by $p$ for every plane $P$, then $S$ is a cylinder.

**Conjecture 2 (Weak cylinder conjecture).** Let $S$ be a set of $p^2$ points in $AG(3, p)$. If there are at least $p$ directions that are not determined by $S$, then $S$ is a cylinder.

**Comment.** If $S$ is a set of $p^2$ points and at least $p$ directions are not determined by $S$, then $|S \cap P|$ is divisible by $p$ for every plane $P$ (see [1, Theorem 2.2]). Thus, Conjecture 1 implies Conjecture 2.

**Reference**

PROBLEM 920. (BCC20.13) A problem on covering arrays

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We are given an $N \times k$ array $A$ on $v + 1$ symbols, one of which is the “do not care” symbol $\ast$, and a parameter $t$ referred to as the *strength*. For each $t$-element subset $T$ of $\{1, \ldots, k\}$, let $R_A(T)$ denote the set of distinct rows omitting $\ast$ in the $N \times t$ subarray of $A$ consisting of the columns are indexed by $T$. Clearly $|R_A(T)| \leq N$ and $|R_A(T)| \leq v^t$. 
**Question.** When does an \((N - 1) \times k\) array \(A'\) exist such that \(RA(T) \subseteq RA'(T)\) for all \(T\)?

**Comment.** A necessary condition is that \(|RA(T)| \leq N - 1\) for all \(T\). A rather strong sufficient condition is that \(A\) has two rows that never have nontrivial entries in the same column. Collapsing the two rows while retaining the nontrivial elements yields the desired \(A'\) (note that \(RA(T)\) is a set of row vectors, not a set of row indices). For example,

\[
\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & * & * & * \\
* & * & a_{24} & a_{25} & a_{26} & \\
\end{array}
\]

can be replaced by

\[
\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & a_{24} & a_{25} & a_{26} \\
\end{array}
\]

See [1] for further information on covering arrays.

**Reference**


**PROBLEM 921. (BCC20.14) Balanced arrays of triples**

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We use the notation \(SBIBD(v, k, \lambda)\) for a symmetric balanced incomplete block design with \(v\) varieties (or “points” or “treatments”) and \(v\) blocks, and with \(k\) entries per block. The parameter \(\lambda\) is irrelevant to the present problem, which concerns \(SBIBDs\) with \(v = ck \pm 1\) for some \(c > 1\).

Consider the following 7-cyclically generated 15 \(\times\) 15 array:

\[
\begin{array}{ccccccccccccccccccc}
\end{array}
\]

If each triple in this array is replaced by a 1, and each dash by a 0, then we obtain the incidence matrix of an \(SBIBD(15, 7, 3)\). For each choice of distinct positions \(r\) and \(s\) in \(\{1, 2, 3\}\) and distinct elements \(i\) and \(j\) in \(\{1, \ldots, 7\}\), there are two entries having \(i\) in position \(r\) and \(j\) in position \(s\), and there are three entries having \(i\) in both positions.

**Question.** Let \(A\) be the incidence matrix of an \(SBIBD(v, k, \lambda)\) with \(v = ck + \varepsilon\), where \(\varepsilon \in \{+1, -1\}\). Under what conditions does there exist a \(v \times v\) array, with a triple in each position where \(A\) is 1 and empty entries where \(A\) is 0,
such that for distinct $i, j \in \{1, \ldots, k\}$ and $r, s \in \{1, 2, 3\}$, there are $c$ entries in the array whose $r$th position is $i$ and $s$th position is $j$, and there are $c + e$ entries whose $r$th position and $s$th position are both $i$? That is, restricting to any two coordinate positions yields a $k \times k$ incidence matrix equal to $cJ + eI$, where $J$ is the all-1 matrix, and incidence is common occurrence.

**Comment.** The $15 \times 15$ array above appears in [1], along with a similar $13 \times 13$ array for $(v, k, \lambda) = (13, 4, 1)$. However, no similar array is known for any other parameter-set $(v, k, \lambda)$ with $v = ck \pm 1$, and no array is known containing 4-tuples satisfying the analogous conditions. Finding such arrays for further parameter-sets is an important problem in the construction of balanced hyper-Graeco-Latin designs. Attention is particularly drawn to those SBIBDs with $(v, k) = (2p + 1, p)$, where $p \equiv 3\text{ mod } 4$, which include the Paley designs.

**Reference**


**PROBLEM 922. (BCC20.16) Index of connection and Möbius number**

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There is a well-known relation between the three polyhedral groups $\text{Alt}(4), \text{Sym}(4)$, and $\text{Alt}(5)$ (the groups of rotations of the tetrahedron, octahedron, and dodecahedron), and the three exceptional root lattices $E_6$, $E_7$, and $E_8$. (See the “McKay correspondence” [3] and various results in singularity theory [1]).

For a finite group $G$, the Möbius number $\mu(G)$ is the value of $\mu(1, G)$, where $\mu$ is the Möbius function of the subgroup lattice of $G$ with 1 denoting the trivial subgroup (see [4], for example). Let $n(G) = |G|/\mu(G)$.

The index of connection of a root lattice $L$ is its index in the dual lattice, or the determinant of the Coxeter matrix, see [2, p. 40]. The Coxeter matrix is $2I - A$, where $A$ is the adjacency matrix of the Coxeter–Dynkin diagram of the lattice. These diagrams are shown below for the exceptional root lattices $E_6$, $E_7$, and $E_8$, respectively.

![Coxeter–Dynkin diagrams for exceptional root lattices](image)

**Question.** Is it just coincidence that, ignoring signs, the values of $n(G)$ for the three polyhedral groups are equal to the indices of connection of the corresponding root lattices? The numbers are 3, 2, 1, respectively.

**References**


**PROBLEM 923. (BCC20.17) Indecomposable permutations and transitive groups**

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A permutation $\pi$ of $\{1, \ldots, n\}$ is indecomposable if there is no index $k$ with $0 < k < n$ such that $\pi$ carries the set $\{1, \ldots, k\}$ to itself. Let $c_n$ be the number of indecomposable permutations of $\{1, \ldots, n\}$. See Comtet [1].

Dixon [2] calculated the number of ordered pairs of permutations of $\{1, \ldots, n\}$ that generate a transitive subgroup of the symmetric group $S_n$. This number turns out to be $(n-1)! \cdot c_{n+1}$.

**Problem.** Find a bijective proof of this fact. That is, find a bijection from the set of pairs of permutations generating a transitive subgroup of $S_n$ to the set of pairs $(\pi, \sigma)$, where $\pi \in S_{n-1}$, $\sigma \in S_{n+1}$, and $\sigma$ is indecomposable.

**References**


**PROBLEM 924. (BCC20.18) Binary X-rays of permutations and score sequences of tournaments**

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Let $\mathcal{S}_n$ be the set of all permutations of $\{1, 2, \ldots, n\}$ and let $P_\pi$ be the permutation matrix corresponding to $\pi \in \mathcal{S}_n$. For $1 \leq k \leq 2n-1$, the $k$th antidiagonal sum of $P_\pi$, written $x_k(\pi)$ is $\sum_{i+j=k} [P_\pi]_{i,j}$. Let $x(\pi) = x_1(\pi) \ldots x_{2n-1}(\pi)$; the vector $x(\pi)$ is the (antidiagonal) X-ray of $\pi$. For example, the following table contains the X-rays of all permutations in $\mathcal{S}_3$:

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>123</th>
<th>231</th>
<th>312</th>
<th>312</th>
<th>213</th>
<th>321</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(\pi)$</td>
<td>10101</td>
<td>01110</td>
<td>01110</td>
<td>10020</td>
<td>02001</td>
<td>00300</td>
</tr>
</tbody>
</table>

An X-ray $x(\pi)$ is binary if $x_i(\pi) \in \{0, 1\}$ for $1 \leq i \leq 2n-1$. Permutations with binary X-rays correspond to placements of $n$ nonattacking queens on an $n \times n$ chessboard under a modified rule in which two queens do not attack each other if they are in the same northwest–southeast diagonal.

A tournament is an orientation of a complete graph. Its score sequence is the vector whose entries are the outdegrees of the vertices, arranged in nondecreasing order. The following conjecture is from [1].

**Conjecture.** The number of distinct binary X-rays of permutations in the symmetric group $S_n$ equals the number of distinct score sequences of tournaments on $n$ vertices.

**Comment.** In [1], it was proved for each $n$ that the number of score sequences is an upper bound on the number of binary X-rays. The conjecture has been verified for $n \leq 9$ using Matlab.

**Reference**


**PROBLEM 925. (BCC20.19) A number-theoretic problem**

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**Conjecture 1.** $\lfloor \frac{3^m}{2^m} \rfloor = \lfloor \frac{(3^m - 1)}{(2^m - 1)} \rfloor$ for every positive integer $m$. 
Comment. Equality has been checked for $m \leq 10,000$ using Maple. A theorem of Mahler [1] shows that equality holds for sufficiently large $m$. S. Severini and S. Velani noted a relation between Conjecture 1 and Waring’s Problem. Let $g(k)$ denote the least $m$ such that every positive integer can be expressed as the sum of $m$ $k$th powers. The function $g$ can be explicitly determined if, for every positive integer $n$,

$$\left\{ \left( \frac{3}{2} \right)^n \right\} \leq 1 - \left( \frac{3}{4} \right)^n,$$

where $\{x\}$ denotes $x - \lfloor x \rfloor$. Inequality (1) is true for $n < 471600000$ and believed to be true for all $n$, which would imply Conjecture 1. Now, Conjecture 1 actually requires that

$$\left\{ \left( \frac{3}{2} \right)^n \right\} < 1 - \left( \frac{3^n - 2^n}{4^n - 2^n} \right),$$

which is somewhat weaker than (1). Nevertheless, they are probably of the same order of difficulty.

Reference