JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 72, 29–33 (1979)

Regular Multiparameter Eigenvalue Problems with Several Parameters in the Boundary Conditions

PATRICK J. BROWNE* AND B. D. SLEEMAN

Department of Mathematics, University of Dundee, Dundee, Scotland DD1 4HN

Submitted by Ky Fan

This paper extends the work of J. Walter on regular eigenvalue problems with eigenvalue parameter in the boundary condition to the multiparameter setting. The main results include several completeness and expansion theorems.

1. INTRODUCTION

Walter [5] has recently studied a modified Sturm–Liouville problem consisting of the usual Sturm–Liouville differential equation subjected to boundary conditions containing the eigenparameter of the problem. He regards such a problem as "self-adjoint" if it can be considered as the eigenvalue problem of a self-adjoint operator in some appropriate Hilbert space. This approach leads readily to an eigenfunction expansion theorem and a Parseval equality associated with the problem. Fulton [3] also studies this problem but uses the function theoretic methods developed by Titchmarsh. Both [3] and [5] contain extensive bibliographies of contributions on such problems and their applications.

This note discusses the multiparameter analog of Walter's work. We shall consider a linked system of such modified Sturm–Liouville equations, the linking being via the spectral parameters in the usual multiparameter way. An eigenfunction expansion theorem and Parseval equality will follow from a suitable interpretation of the problem in appropriately defined Hilbert spaces.

2. THE PROBLEM

Consider the system of ordinary differential equations

$$(p_r(x_r)\, y_r'(x_r))' + q_r(x_r)\, y_r(x_r) + \sum_{s=1}^{k} \lambda_s a_{rs}(x_r)\, y_r(x_r) = 0, \qquad 1 \leqslant r \leqslant k. \tag{1}$$

* Research supported in part by National Research Council of Canada Grant A9073.

0022-247X/79/110029–05$02.00/0

Here the coefficient functions p_r, q_r, a_{rs} are real valued and defined on a finite interval $[a_r, b_r] \subseteq \mathbb{R}$. We assume q_r, a_{rs} to be continuous while p_r will be taken as positive and once continuously differentiable. The parameters λ_s, $1 \leqslant s \leqslant k$, are complex numbers.

We come now to the boundary conditions to be imposed on the differential equations. Given points $\boldsymbol{\alpha}^r$, $\boldsymbol{\beta}^r$, $\boldsymbol{\gamma}^r$, $\boldsymbol{\delta}^r \in \mathbb{R}^2$ with $\boldsymbol{\beta}^r \neq 0$, $\boldsymbol{\delta}^r \neq 0$ and given a function y_r defined on (a_r, b_r) we define

$$(y_r)_{\boldsymbol{\alpha}^r} = \lim_{x_r \to a_r+} (\alpha_1{}^r y_r(x_r) - \alpha_2{}^r p_r(x_r)\, y_r'(x_r)),$$

$$(y_r)_{\boldsymbol{\beta}^r} = \lim_{x_r \to a_r+} (\beta_1{}^r y_r(x_r) - \beta_2{}^r p_r(x_r)\, y_r'(x_r)),$$

$$(y_r)_{\boldsymbol{\gamma}^r} = \lim_{x_r \to b_r+} (\gamma_1{}^r y_r(x_r) - \gamma_2{}^r p_r(x_r)\, y_r'(x_r)),$$

$$(y_r)_{\boldsymbol{\delta}^r} = \lim_{x_r \to b_r+} (\delta_1{}^r y_r(x_r) - \delta_2{}^r p_r(x_r)\, y_r'(x_r)),$$

whenever these limits exist, $1 \leqslant r \leqslant k$. We further assume that for each r

$$-p_r(a_r)\,(\alpha_1{}^r\beta_2{}^r - \alpha_2{}^r\beta_1{}^r) = p_r(b_r)\,(\gamma_1{}^r\delta_2{}^r - \gamma_2{}^r\delta_1{}^r) = 1. \tag{2}$$

The boundary value problem can now be formulated. It consists of the linked system of differential equations (1) together with the boundary conditions

$$\begin{aligned} -(y_r)_{\beta^r} &= \sum_{s=1}^{k} \lambda_s \theta_{rs} (y_r)_{\boldsymbol{\alpha}^r}, \\ -(y_r)_{\boldsymbol{\delta}^r} &= \sum_{s=1}^{k} \lambda_s \phi_{rs} (y_r)_{\boldsymbol{\gamma}^r}, \end{aligned} \tag{3}$$

where θ_{rs}, ϕ_{rs}, $1 \leqslant r$, $s \leqslant k$, are real numbers which subsequently will be required to satisfy certain conditions.

An eigenvalue $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ and eigenfunction $y(x) = y_1(x_1) \cdots y_k(x_k)$ are then a k-tuple of complex numbers and a decomposable function so that Eqs. (1) and Conditions (3) are satisfied. Condition (2) does not permit ordinary Sturm–Liouville conditions to be applied at some of the end points a_r, b_r. We shall incorporate this possibility later.

3. A Hilbert Space Formulation

We define a measure ρ_r on $[a_r, b_r]$ by setting

$$\rho_r(\{a_r\}) = p_r(a_r), \qquad \rho_r(\{b_r\}) = p_r(b_r),$$

$$\rho_r(M) = \int_M 1\,dx_r \qquad \text{for } M \subseteq (a_r, b_r).$$

H_r will denote $L^2([a_r, b_r]; \rho_r)$ and in this space we define operators V_{rs}, $1 \leqslant s \leqslant k$, by

$$\begin{aligned}(V_{rs}f_r)(x_r) &= a_{rs}(x_r) f_r(x_r), \qquad && x_r \in (a_r, b_r),\\ &= \theta_{rs} f_r(a_r), && x_r = a_r,\\ &= \phi_{rs} f_r(b_r), && x_r = b_r.\end{aligned}$$

Each of these operators is Hermitian and may be considered as a multiplication operator $V_{rs}f_r = A_{rs}f_r$, where A_{rs} is given by the above formulas. We require the fundamental positivity hypothesis:

There is a constant $c > 0$ so that

$$\forall \mathbf{x} = (x_1, \ldots, x_k) \in \mathop{\mathsf{X}}_{r=1}^{k} [a_r, b_r] = [\mathbf{a}, \mathbf{b}],$$
$$\det\{A_{rs}(x_r)\} = |A|(\mathbf{x}) \geqslant c. \tag{4}$$

This condition will be satisfied, for example, in either of the following two situations:

(i) Let $\det a_{rs}(x_r)$ be strictly positive and bounded away from zero for all $\mathbf{x}$, and assume that there are points x_r', $x_r'' \in [a_r, b_r]$ so that $\theta_{rs} = a_{rs}(x_r')$, $\phi_{rs} = a_{rs}(x_r'')$, $1 \leqslant r$, $s \leqslant k$.

(ii) Let $\det a_{rs}(x_r)$ be strictly positive and bounded away from zero for all $\mathbf{x}$ and assume that any subdeterminant formed by deleting row and column i, row and column j,... has the same property as a function of the remaining variables. Then for $\theta_{rs} = \xi_r \delta_{rs}$, $\phi_{rs} = \eta_r \delta_{rs}$, where $\xi_r, \eta_r > 0$, condition (4) is satisfied.

If we let $\rho = \rho_1 \times \cdots \times \rho_k$ on $[\mathbf{a}, \mathbf{b}]$ it follows that on $L^2([\mathbf{a}, \mathbf{b}]; \rho)$ the multiplication operator Δ_0 defined by $(\Delta_0 f)(\mathbf{x}) = |A|(\mathbf{x}) f(\mathbf{x})$ is Hermitian and strictly positive definite.

Operators T_r: $\mathscr{D}(T_r) \subseteq H_r \to H_r$ are defined as follows:

(i) $\mathscr{D}(T_r) = \{u_r \in H_r \mid u_r, p_r u_r'$ are absolutely continuous on (a_r, b_r), $(p_r u_r')' + q_r u_r \in L^2(a_r, b_r)$, $u_r(a_r) = (u_r)_{\alpha r}$, $u_r(b_r) = (u_r)_{\gamma r}\}$,

(ii) for $u_r \in \mathscr{D}(T_r)$,

$$\begin{aligned}(T_r u_r)(x_r) &= (p_r(x_r)\, u_r'(x_r))' + q_r(x_r)\, u_r(x_r), \qquad && x_r \in (a_r, b_r),\\ &= -(u_r)_{\beta r}, && x_r = a_r,\\ &= -(u_r)_{\delta r}, && x_r = b_r.\end{aligned}$$

We now appeal to Theorem 1 of Walter [5] and claim that in H_r, T_r is a self-adjoint operator with compact resolvent. We should point out that Walter has only considered modified boundary conditions at one end of the interval but as he remarks at the opening of Section 2 this is a matter of notational convenience.

Accordingly we have

(a) the operators $V_{rs}: H_r \to H_r$ are Hermitian,

(b) the operator Δ_0 in $L^2([\mathbf{a}, \mathbf{b}]; \rho)$ is strictly positive definite,

(c) $T_r: \mathscr{D}(T_r) \subseteq H_r \to H_r$ is self-adjoint and has compact resolvent.

In short we have the necessary ingredients for an abstract multiparameter eigenvalue problem of the type considered by Browne [2] and Sleeman [4]. An eigenvalue $\boldsymbol{\lambda}$ and eigenfunction $\psi(\mathbf{x}) = \psi_1(x_1) \cdots \psi_k(x_k)$ of this problem are then a point in $\mathbb{R}^k$ and a nonzero function in $L^2([\mathbf{a}, \mathbf{b}]; \rho)$, respecively, so that

$$T_r\psi_r + \sum_{s=1}^{k} \lambda_s V_{rs}\psi_r = 0, \qquad 1 \leqslant r \leqslant k.$$

It is now easily checked that over (a_r, b_r), ψ_r is a solution of the original second-order differential equation and that at the end points a_r, b_r, boundary conditions (3) are satisfied. From [2, Theorem 10; 4, Chap. 4] we can now immediately deduce

THEOREM. *The multiparameter eigenvalue problem* (1), (3) *has a countable infinity of eigenvalues* $\boldsymbol{\lambda}^i \in \mathbb{R}^k$ *each of finite multiplicity and having no finite point of accumulation. There is a corresponding set of decomposable eigenfunctions* $\psi^i(\mathbf{x})$ *forming an orthonormal basis for* $L^2([\mathbf{a}, \mathbf{b}]; \rho)$ *renormed according to*

$$|||f|||^2 = \int_{[\mathbf{a},\mathbf{b}]} |A|(\mathbf{x})\, |f(\mathbf{x})|^2\, d\rho(\mathbf{x}). \tag{5}$$

COROLLARIES. (i) *Every* $u \in L^2([\mathbf{a}, \mathbf{b}]; \rho)$ *has an expansion*

$$u = \sum_{i=1}^{\infty} \psi^i \int_{[\mathbf{a},\mathbf{b}]} |A|(\mathbf{x})\, u(\mathbf{x})\, \psi^i(\mathbf{x})\, d\rho(\mathbf{x}),$$

the summation converging with respect to the norm (5)

$$|||u|||^2 = \sum_{i=1}^{\infty} \left| \int_{[\mathbf{a},\mathbf{b}]} |A|(\mathbf{x})\, u(\mathbf{x})\, \psi^i(\mathbf{x})\, d\rho(\mathbf{x}) \right|^2.$$

(ii) *Let* $v \in L^2([\mathbf{a}, \mathbf{b}]; \rho)$ *be such that* $v(\mathbf{x}) = 0$ *if there exists r such that* $x_r = a_r$ *or* $x_r = b_r$ *and let* $\hat{\psi}^i$ *denote the restriction of* ψ^i *to the open interval* $(\mathbf{a}, \mathbf{b})$. *Then*

$$v = \sum_{i=1}^{\infty} \hat{\psi}^i \int_{(\mathbf{a},\mathbf{b})} |A|(\mathbf{x})\, v(\mathbf{x})\, \psi^i(\mathbf{x})\, d\mathbf{x},$$

the summation converging in $L^2((\mathbf{a}, \mathbf{b}); \det(a_{rs}(a_{rs}(x_r))))$.

(iii) $0 = \sum_{i=1}^{\infty} \hat{\psi}^i \prod_{r=1}^{k} (\psi_r{}^i)_{a^r}$, *the summation converging as in* (ii). *Further*

$$\prod_{r=1}^{k} \frac{1}{p_r(a_r)} = \Phi \sum_{i=1}^{\infty} \prod_{r=1}^{k} (\psi_r{}^i)_{a^r},$$

where $\Phi = \det(\theta_{rs}) > 0$.

Proofs. The theorem is an easy consequence of the cited references, and Corollaries (i), (ii) are merely restatements of it. For Corollary (iii) we apply the theorem to the function

$$u(\mathbf{x}) = 1, \qquad \mathbf{x} = (a_1, \dots, a_k),$$
$$= 0, \qquad \text{otherwise.}$$

Many other statements similar to Corollary (iii) can be generated by applications of the theorem to functions concentrated at one or more of the corners of $[\mathbf{a}, \mathbf{b}]$.

Our results here parallel Theorems 1 and 2 of Walter [5].

Should the boundary conditions at one end of one of the intervals $[a_r, b_r]$ take the normal Sturm–Liouville form, for example,

$$\beta_1{}^r u_r(a_r) - \beta_2{}^r u_r'(a_r) = 0,$$

we should then define the new measure ρ_r by using Lebesgue measure over $[a_r, b_r)$ and placing an atom of measure $p_r(b_r)$ at b_r as before. The theory then continues mutatis mutandis as in the case discussed in detail.

References

1. F. V. Atkinson, "Multiparameter Eigenvalue Problems: Matrices and Compact Operators," Academic Press, New York/London, 1972.
2. P. J. Browne, Abstract multiparameter theory, I, *J. Math. Anal. Appl.* **60** (1977), 259–273.
3. C. T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A.* **77** (1977), 293–308.
4. B. D. Sleeman, "Multiparameter Spectral Theory in Hilbert Space," Pitman, London, 1978.
5. J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary condition, *Math. Z.* **133** (1973), 301–312.