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Four-genera of quasipositive knots

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Abstract

By using a result of Rudolph concerning the four-genera of classical knots, we give an infinite family of knots which have arbitrary large gaps between the four-genera and the topological four-genera. © 1998 Published by Elsevier Science B.V.

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1. Introduction

A *link* is a closed oriented 1-manifold smoothly embedded in the 3-sphere S^3; a *knot* is a link with one connected component. Two knots K_0 and K_1 are *smoothly* (respectively, *topologically*) *cobordant* if there is an oriented annulus C *smoothly* (respectively, *topologically locally-flatly*) embedded in $S^3 \times [0,1]$ such that $C \cap S^3 \times \{0,1\} = K_0 \cup -K_1^*$, where $-K_1^*$ is the mirror image of K_1 with reversed string orientation. We denote by $\mathcal{K}_{\mathbf{DIFF}}$ (respectively, $\mathcal{K}_{\mathbf{TOP}}$) the set of *smooth* (respectively, *topological*) *cobordism classes* of knots which is endowed with an abelian group structure under the operation of *connected sum*, denoted $\sharp$. Every knot in the identity class of $\mathcal{K}_{\mathbf{DIFF}}$ (respectively, $\mathcal{K}_{\mathbf{TOP}}$) is called *smoothly* (respectively, *topologically*) *slice*. Knots $\{K_i\}$ are *linearly independent* if the smooth cobordism classes of $\{K_i\}$ are linearly independent in $\mathcal{K}_{\mathbf{DIFF}}$. If K_0 and K_1 are smoothly cobordant, so are they topologically. Thus, there is a natural epimorphism $i: \mathcal{K}_{\mathbf{DIFF}} \to \mathcal{K}_{\mathbf{TOP}}$. By making use of a result of Donaldson [5], Gompf [6] showed that the kernel of i contains a free abelian group of infinite rank. More recently, by making

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use of a result of Furuta concerning the homology cobordism groups of homology 3-spheres, Endo [10] gave an infinite family of knots of infinite order which are linearly independent in the kernel of i.

In this note, we define the *four-genus* (respectively, *topological four-genus*) for a knot K as the *minimum genus* for an oriented 2-manifold, without closed component, which is smoothly (respectively, topologically locally-flatly) embedded in 4-disk D^4 with boundary K, and denote it by $G_{\mathbf{S}}(K)$ (respectively, $G_{\mathbf{T}}(K)$). It is obvious that $G_{\mathbf{S}}(K) \geqslant G_{\mathbf{T}}(K)$ for any K.

We consider the following question:

Question. For each pair (l, m) of non-negative integers such that $m > l$, is there a knot K which satisfies that $G_{\mathbf{T}}(K) = l$ and $G_{\mathbf{S}}(K) = m$?

In the recent works of Rudolph [8,9], it was shown that there are infinitely many knots $\{K_i\}$ satisfying $G_{\mathbf{T}}(K_i) = 0$ and $G_{\mathbf{S}}(K_i) = 1$. On the other hand, Yasuhara showed that there are infinitely many knots $\{K_j\}$ satisfying $G_{\mathbf{T}}(K_j) = 0$ and $G_{\mathbf{S}}(K_j) \geqslant 3$ [11].

Our main result is:

Theorem 1.1. *For each pair (l, m) of non-negative integers such that $m > l$, there exists an infinite family of prime knots $\{K_i\}$ which are linearly independent, and satisfy that $G_{\mathbf{T}}(K_i) = l$ and $G_{\mathbf{S}}(K_i) = m$.*

Remark 1.2. In particular, the above theorem says that there is an infinite family of linearly independent knots in the kernel of i which have the four-genera as large as desired and also says that there is an properly embedding in D^4 of oriented connected 2-manifold, with arbitrary genus, which is topologically locally-flatly, but not smoothly.

In Section 2, we review Rudolph's works [9] on the *quasipositive link* and Endo's result [11] which are used, in Section 3, to prove the main theorem.

2. Quasipositivity and four-genera of links

In this section we survey a work of Rudolph [9] on the link which is called *quasipositive link* defined as following:

Let $\mathbb{R}^4$ be identified with

$$\mathbb{C}^2 \supset S^3 := \{(z, w)\colon\ |z| + |w| = 1\}.$$

Definition 2.1. In the *n-string braid group*

$$B_n := gp\left(\sigma_i, 1 \leqslant i \leqslant n-1 \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = \sigma_j^{-1}\sigma_i, & |i-j| = 1 \\ [\sigma_i, \sigma_j] = 1, & |i-j| \neq 1 \end{array}\right),$$

a *positive band* is any conjugate $\omega\sigma_i\omega^{-1}$ ($\omega \in B_n$); a *quasipositive braid* is any product of positive bands. A *quasipositive* oriented link is one which can be realized as the closure of a quasipositive braid.

Four-genus. Let K be a knot in S^3. Let $\chi_{\mathbf{S}}(K)$ (respectively, $\chi_{\mathbf{T}}(K)$) be the greatest *Euler characteristic* $\chi(F)$ of an oriented 2-manifold F, without closed components, smoothly (respectively, topologically locally-flatly) embedded in D^4 with boundary K.

Now we define the following:

$$G_{\mathbf{S}}(K) \overset{\text{def}}{\equiv} (1 - \chi_{\mathbf{S}}(K))/2, \qquad G_{\mathbf{T}}(K) \overset{\text{def}}{\equiv} (1 - \chi_{\mathbf{T}}(K))/2.$$

It is obvious that K is smoothly (respectively, topologically) slice if and only if $G_{\mathbf{S}}(K) = 0$ (respectively, $G_{\mathbf{T}}(K) = 0$). We call $G_{\mathbf{S}}(K)$ (respectively, $G_{\mathbf{T}}(K)$) the *four-genus* (respectively, *topological four-genus*) of K.

Rudolph showed the following:

Theorem 2.2 (Rudolph [9]). *If* $\beta = \omega_1 \sigma_{i_1} \omega_1^{-1} \ldots \omega_k \sigma_{i_k} \omega_k^{-1} \in B_n$ *is quasipositive, then* $\chi_{\mathbf{S}}(\hat{\beta}) = n - k$.

Quasipositive pretzel. Let $p, q, r \in \mathbb{Z}$. A diagram for the *pretzel* link $\varphi(p, q, r)$ is obtained from a braid diagram for $\beta_{p,q,r} := \sigma_1^{-p} \sigma_3^{-q} \sigma_5^{-r} \in B^6$ by forming the *plat* of $\beta_{p,q,r}$ as shown in Fig. 1.

If p, q, r are all odd, then $\varphi(p, q, r)$ is a knot. Rudolph showed that *for* p, q, r *all odd,* $\varphi(p, q, r)$ *is quasipositive iff* $\min\{p + q, p + r, q + r\} > 0$, and he also showed that $\varphi(p, q, r)$ satisfies $G_{\mathbf{T}}(\varphi(p, q, r)) = 0$, $G_{\mathbf{S}}(\varphi(p, q, r)) = 1$ for a triple (p, q, r) of odd integers satisfying

$$pq + pr + qr = -1, \quad |p|, |q|, |r| \neq 1.$$

All the pretzel knots of the following Endo's theorem are of this type.

By using a result of Furuta concerning the homology cobordism group of homology 3-spheres, Endo showed the following theorem:

Theorem 2.3 (Endo [10]). *Each family of infinitely many pretzel knots exhibited below are linearly independent in the kernel of* i.

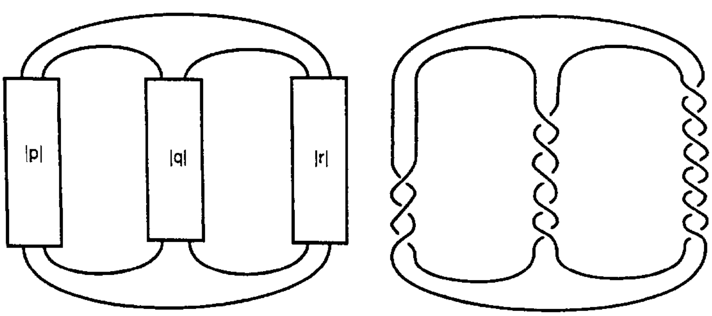

Fig. 1. Pretzel link $\varphi(p, q, r)$.

$$\varphi(-2k-1, 4k+1, 4k+3) \quad (k = 1, 2, \ldots), \tag{1}$$

$$\varphi(-2k-1, 2k+3, 2k^2+4k+1) \quad (k = 1, 2, \ldots), \tag{2}$$

$$\varphi(-2k-1, 2k+5, k^2+3k+1) \quad (k = 1, 2, \ldots), \tag{3}$$

$$\varphi(-4k-1, 6k+1, 12k+5) \quad (k = 1, 2, \ldots), \tag{4}$$

$$\varphi(-4k-3, 6k+5, 12k+7) \quad (k = 1, 2, \ldots). \tag{5}$$

3. Proof of the main theorem

In this section, we prove the Theorem 1.1 by making use of the following three lemmas.

Lemma 3.1. *Let K_0 and K_1 be two quasipositive knots. Put $K = K_0 \sharp K_1$. Then*

$$G_{\mathrm{S}}(K) = G_{\mathrm{S}}(K_0) + G_{\mathrm{S}}(K_1).$$

Proof. Let β_0 and β_1 be the quasipositive braids corresponding to K_0 and K_1, respectively (i.e., $\beta_0 \in B_m$, $\beta_1 \in B_n$). Let the lengths of β_0 and β_1 be to k and l, respectively. Put $\beta = \beta_0 \sigma_m \beta_1'$ (i.e., β_1' is the braidsword obtained from β_1 adding m to all its indices). Then β is quasipositive and $\hat{\beta} = K$. By Theorem 2.2, we note that

$$\chi_{\mathrm{S}}(\hat{\beta}) = m + n - (k + l + 1) = (m - k) + (n - l) - 1 = \chi_{\mathrm{S}}(\hat{\beta}_0) + \chi_{\mathrm{S}}(\hat{\beta}_1) - 1.$$

So

$$\begin{aligned} G_{\mathrm{S}}(K) &= (1 - \chi_{\mathrm{S}}(\hat{\beta}))/2 = (1 - \chi_{\mathrm{S}}(\hat{\beta}_0))/2 + (1 - \chi_{\mathrm{S}}(\hat{\beta}_1))/2 \\ &= G_{\mathrm{S}}(K_0) + G_{\mathrm{S}}(K_1). \qquad \square \end{aligned}$$

Corollary 3.2. *There is no quasipositive knot which has order two in $\mathcal{K}_{\mathbf{DIFF}}$, so there are many knots which are not quasipositive.*

Proof. For a quasipositive knot K, suppose that $K \sharp K$ is slice. By the Lemma 3.1,

$$G_{\mathrm{S}}(K) + G_{\mathrm{S}}(K) = G_{\mathrm{S}}(K \sharp K) = 0.$$

So $G_{\mathrm{S}}(K) = 0$. □

A knot K in S^3 is *prime* if any 2-sphere in S^3, which meets K transversely in two points, bounds in S^3 a ball meeting K in an unknotted spanning arc.

Lemma 3.3. *Let K be a knot, then there is a prime knot K' which satisfies $G_{\mathrm{S}}(K) = G_{\mathrm{S}}(K')$.*

Proof. By the Kirby and Lickorish's theorem [4] we have that *any knot is cobordant to a prime knot*. The lemma follows from the cobordism invariance of four-genera of knots. □

Let a knot K have a Seifert surface F, and let the Seifert form $H_1(F) \times H_1(F) \to Z$ be represented, with respect to some basis by the Seifert matrix V. Then, the signature of the symmetric matrix $V + V^T$ is the *signature of* K, denoted $\sigma(K)$. Murasugi [1] proved of the smooth cobordism invariance of the signature. It is well-known that the following lemma holds.

Lemma 3.4 (for example, see [3]). *Let $\sigma(K)$ be the signature of K. Then the following holds.*

$$|\sigma(K)|/2 \leqslant G_{\mathbf{T}}(K).$$

Remark 3.5. We can prove the Lemma 3.4 by making use of the Wall's topological version [2] of the G-signature theorem.

Proof of Theorem 1.1. Let $\{K_i\}$ be the family of an infinitely many linearly independent quasipositive knots with $G_{\mathbf{T}}(K_i) = 0$ and $G_{\mathbf{S}}(K_i) = 1$ (for example, let $\{K_i\}$ be a family of the pretzel knots of the Theorem 2.5) and K_T be a quasipositive knot with $G_{\mathbf{T}}(K_T) = G_{\mathbf{S}}(K_T) = |\sigma(K_T)|/2 = 1$ (for example, let K_T be a trefoil knot). Put

$$\sharp^{m-l} K_i = \underbrace{K_i \sharp \cdots \sharp K_i}_{m-l}, \qquad \sharp^l K_T = \underbrace{K_T \sharp \cdots \sharp K_T}_{l}.$$

By Lemma 3.1, we can show that

$$G_{\mathbf{S}}(\sharp^{m-l} K_i) = m - l, \qquad G_{\mathbf{T}}(\sharp^{m-l} K_i) = 0.$$

Thus, by Lemma 3.4, we obtain

$$G_{\mathbf{T}}((\sharp^{m-l} K_i)\sharp(\sharp^l K_T)) = l, \qquad G_{\mathbf{S}}((\sharp^{m-l} K_i)\sharp(\sharp^l K_T)) = m.$$

We show that the knots $\{(\sharp^{m-l} K_i)\sharp(\sharp^l K_T)\}$ are linearly independent in $\mathcal{K}_{\mathbf{DIFF}}$. For any integers p, $\{\alpha_i\}_{i=1}^p$ with $p \geqslant 2$ and $\alpha_1^2 + \cdots + \alpha_p^2 \neq 0$, assume that

$$\begin{aligned}&\alpha_1 \left[(\sharp^{m-l} K_1)\sharp(\sharp^l K_T)\right] + \cdots + \alpha_p \left[(\sharp^{m-l} K_p)\sharp(\sharp^l K_T)\right] \\ &\quad \big(= (m-l)\alpha_1[K_1] + \cdots + (m-l)\alpha_p[K_p] + l(\alpha_1 + \cdots + \alpha_p)[K_T]\big) = 0.\end{aligned}$$

Put $\omega = l(\alpha_1 + \cdots + \alpha_p)$. If $\omega = 0$, then $\alpha_1 = \cdots = \alpha_p = 0$. It is contrary to assumption. Suppose that $\omega \neq 0$; then

$$\begin{aligned}&\sigma\big((\sharp^{(m-l)\alpha_1} K_1)\sharp \cdots \sharp(\sharp^{(m-l)\alpha_p} K_p)\sharp(\sharp^{\omega} K_T)\big) \\ &\quad = (m-l)(\alpha_1\sigma(K_1) + \cdots + \alpha_p\sigma(K_p)) + \omega\sigma(K_T) \\ &\quad = \omega\sigma(K_T) \neq 0.\end{aligned}$$

It is also contrary to assumption.

By the proof of Lemma 3.3, the cobordism classes of these knots can be represented by the prime knots, denoted by $\{K_{P_i}^{m,l}\}$. From the cobordism invariance of the four-genus and the signature of knots, the four-genera and the topological four-genera of $\{K_{P_i}{}^{m,l}\}$ equal to m and l, respectively. Thus we have completed the proof. □

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