The band method and inverse problems for orthogonal matrix functions of Szegő–Kreĭn type

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Dedicated to the memory of Israel Gohberg, a wonderful mathematician and a dear friend. His achievements will be a source of inspiration for many years to come

Abstract

A band method approach for solving inverse problems for certain orthogonal functions is developed. The inverse theorems for Szegő–Kreĭn matrix polynomials and for Kreĭn orthogonal entire matrix functions are obtained as corollaries of the band method results. Other examples, including a non-stationary variant of the Szegő–Kreĭn theorem, are presented to illustrate the scope of the abstract theorems.

1. Introduction

The band method is an abstract scheme that allows one to deal with matrix-valued versions of classical interpolation problems, such as those of Schur, Carathéodory–Toeplitz and Nehari, from one point of view. The method has its origin in papers of Dym and Gohberg from the early eighties \cite{4,5,3}, and has been developed into a more final form in papers by Gohberg and co-authors in \cite{11,12}. A comprehensive introduction to the method and additional references can be found in Chapter XXXIV of the book \cite{7}. For more recent contributions see the article \cite{14} and the references therein.

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In the present paper the inverse theorems for Szegő–Krein matrix polynomials given in [13] are put into the context of the band method using ideas from [1]. We also use our band method results to prove various other inverse theorems, including the one for Krein orthogonal entire matrix functions presented in [10].

To state our main theorem, we first recall some of the basic elements of the band method theory. Let $M$ be a $\ast$-subalgebra of a unital $C^\ast$-algebra $R$ such that the unit $e$ of $R$ belongs to $M$. Assume that $M$ admits a direct sum decomposition

$$M = M_1 + M_2^0 + M_d + M_3^0 + M_4,$$

(1.1)

where the summands $M_1, M_2^0, M_d, M_3^0,$ and $M_4$ are linear submanifolds of $M$. The algebra $M$ is called an algebra with band structure if, in addition, the following three conditions are satisfied

(C1) $e \in M_d$,

(C2) $M_1^* = M_4$, $(M_2^0)^* = M_3^0$, and $M_d = M_d^*$,

(C3) the following multiplication table describes some additional restrictions on the multiplication in $M$:

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_d$</th>
<th>$M_3$</th>
<th>$M_4$</th>
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<tr>
<td>$M_1$</td>
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<td>$M_1^0$</td>
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<td>$M_4$</td>
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</table>

(1.2)

Here

$$M_1^0 = M_1 + M_2^0, \quad M_c = M_2^0 + M_d + M_3^0, \quad M_0 = M_3^0 + M_4.$$

We shall also need the linear submanifolds of $M$ given by

$$M_1^0 = M_1 + M_2^0, \quad M_c = M_2^0 + M_d + M_3^0, \quad M_0 = M_3^0 + M_4.$$

An element $m \in M$ will be called selfadjoint whenever $m = m^\ast$.

The inverse problem we shall be dealing with in this band method setting is the following problem. Given $q \in M_2$ and $a = a^\ast \in M_d$, find $f = f^\ast \in M_c$ such that

$$P_{M_2}(f q) \in M_d, \quad P_{M_d}(q^\ast f q) = a.$$

(1.3)

Here $P_{M_2}$ denotes the projection of $M$ onto $M_2$ along the other spaces in the decomposition (1.1). In a similar way one defines other projections corresponding to subspaces in (1.1). In particular, $P_{M_d}$ is the projection of $M$ onto $M_d$ along $M_1^0 + M_d^0$, and $P_{M_c}$ is the projection of $M$ onto $M_c$ along $M_1$ and $M_4$.

We shall see that under the additional condition that $q$ has an inverse in $M$ the above problem is solvable if and only if the equation

$$uq - q^\ast v = a$$

(1.4)

has a solution $u \in M_2$ and $v \in M_1$. The following theorem is the main result of this paper.
Theorem 1.1. Let \( q \in \mathcal{M}_2 \) and \( a = a^* \in \mathcal{M}_d \). If there exists an element \( f = f^* \in \mathcal{M} \) satisfying conditions (1.3), then there exists a pair \( u \in \mathcal{M}_+ \) and \( v \in \mathcal{M}_1 \) satisfying Eq. (1.4). More precisely, if \( f = f^* \in \mathcal{M} \) satisfies (1.3), then \( u := P_{\mathcal{M}_+}(q^* f) \) and \( v := P_{\mathcal{M}_1}(f q) \) satisfy (1.4) and \( f q = u^* + v \). Moreover, if \( f = f^* \in \mathcal{M}_c \) satisfies (1.3), then \( u = P_{\mathcal{M}_+}(q^* f) \in \mathcal{M}_2 \) and \( v = P_{\mathcal{M}_1}(f q) \in P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2) \). Conversely, if \( q \) is invertible in \( \mathcal{M} \) and (1.4) has a solution pair \( u \in \mathcal{M}_+, v \in \mathcal{M}_1 \), then \( f := (u^* + v)q^{-1} \) is selfadjoint and satisfies conditions (1.3). Furthermore,

(a) if \( f = f^* \in \mathcal{M} \) and \( f_c = P_{\mathcal{M}_c} f \), then \( f_c = f^*_c \), and \( f \) satisfies (1.3) if and only if \( f_c \) satisfies (1.3) in place of \( f \);
(b) if \( q \) is invertible and (1.4) has a solution pair \( u \in \mathcal{M}_+, v \in \mathcal{M}_1 \), then (1.4) has a solution pair \( \tilde{u} \) and \( \tilde{v} \) such that \( \tilde{u} \in \mathcal{M}_2 \) and \( \tilde{v} \in P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2) \);
(c) if \( q \) is invertible and \( f = f^* \in \mathcal{M}_c \) satisfies conditions (1.3), then one can find \( \tilde{u} \) and \( \tilde{v} \) as in (b) such that \( f = (\tilde{u}^* + \tilde{v})q^{-1} \);
(d) if \( q \) is invertible and \( q^{-1} \) belongs to \( \mathcal{M}_+ \), then \( f := q^{-*} a q^{-1} \) satisfies (1.3) and (1.4) holds with \( u = a q^{-1} \) and \( v = 0 \).

The proof of the above theorem appears in the next section. In fact, in the next section we shall derive this theorem as a corollary of its non-symmetric version. In Section 3 we show that the above theorem covers the inverse theorem for Szegő–Krein matrix polynomials given in [13,1], and its non-symmetric version [13, Theorem 7.1]. In Section 4 we use Theorem 1.1 to prove the inverse theorem for Krein orthogonal entire matrix functions given in [10]. We shall also derive its non-symmetric version. One of the special features of the band method is that it not only applies to function theory problems but equally well to problems of so-called non-stationary type, where polynomials and analytic functions are replaced by finite and infinite block matrices. In the final section we illustrate this fact by using Theorem 1.1 in a few examples of non-stationary type. For a general reference on Szegő–Krein polynomials, Krein orthogonal entire functions, and non-stationary variants we refer to the book [6].

2. The non-symmetric version

Throughout this section \( \mathcal{M} \) is an algebra with band structure. In particular, \( \mathcal{M} \) decomposes as in (1.1).

In this section we shall deal with the following problem. Given \( q \in \mathcal{M}_2 \), \( p \in \mathcal{M}_3 \), and \( a \in \mathcal{M}_d \), find \( f \in \mathcal{M}_c \) such that

\[
P_{\mathcal{M}_2}(f q) \in \mathcal{M}_d, \quad P_{\mathcal{M}_3}(p f) \in \mathcal{M}_d, \quad P_{\mathcal{M}_d}(p f q) = a. \tag{2.1}
\]

If \( p = q^* \), then the conditions in (2.1) coincide with those in (1.3). Thus, if \( p = q^* \) and \( a \) is selfadjoint, then the above problem reduces to the problem considered in the previous section provided \( f \) is required to be selfadjoint too.

The next lemma shows that it suffices to look for solutions \( f \) of the equations in (2.1) that just belong to \( \mathcal{M} \).

Lemma 2.1. Let \( q \in \mathcal{M}_2 \), \( p \in \mathcal{M}_3 \), and \( f \in \mathcal{M} \). Put \( f_c = P_{\mathcal{M}_c} f \). Then

\[
P_{\mathcal{M}_2}(f q) = P_{\mathcal{M}_2}(f_c q), \quad P_{\mathcal{M}_3}(p f) = P_{\mathcal{M}_3}(p f_c), \quad P_{\mathcal{M}_d}(p f q) = P_{\mathcal{M}_d}(p f_c q). \tag{2.2}
\]

\[
P_{\mathcal{M}_2}(f q) = P_{\mathcal{M}_2}(f_c q), \quad P_{\mathcal{M}_3}(p f) = P_{\mathcal{M}_3}(p f_c), \quad P_{\mathcal{M}_d}(p f q) = P_{\mathcal{M}_d}(p f_c q). \tag{2.3}
\]
In particular, \( f \in \mathcal{M} \) satisfies the equations in (2.1) if and only if these equations are satisfied with \( f_c \) in place of \( f \).

**Proof.** Since \( f_c = P_{\mathcal{M}_d} f \), we have \( f = f_1 + f_c + f_4 \), where \( f_1 \in \mathcal{M}_1 \) and \( f_4 \in \mathcal{M}_4 \). Using \( q \in \mathcal{M}_2 \), \( p \in \mathcal{M}_3 \), and the multiplication table (1.2) we see that

\[
\begin{align*}
 f_1 q &\in \mathcal{M}_1 \mathcal{M}_2 \subset \mathcal{M}_1, & f_4 q &\in \mathcal{M}_4 \mathcal{M}_2 \subset \mathcal{M}_0, \\
p f_1 &\in \mathcal{M}_3 \mathcal{M}_1 \subset \mathcal{M}_0^0, & p f_4 &\in \mathcal{M}_3 \mathcal{M}_4 \subset \mathcal{M}_4. \\

\end{align*}
\]

(2.4)

From (2.4) it follows that \( P_{\mathcal{M}_d}(f_1 q) \) and \( P_{\mathcal{M}_d}(f_4 q) \) are both zero, and hence, since \( f q = f_1 q + f_c q + f_4 q \), the first identity in (2.2) holds. In a similar way, using (2.5), one proves the second identity in (2.2).

Using (2.4) and the multiplication table (1.2) we see that

\[
\begin{align*}
 p f_1 q &\in \mathcal{M}_3 \mathcal{M}_1 \subset \mathcal{M}_0^0, & p f_4 q &\in \mathcal{M}_4 \mathcal{M}_0^0 \subset \mathcal{M}_0^0. \\

\end{align*}
\]

Hence both \( P_{\mathcal{M}_d}(p f_1 q) \) and \( P_{\mathcal{M}_d}(p f_4 q) \) are zero. Since the element \( p f q \) is equal to \( p f_1 q + p f_c q + p f_4 q \), we conclude that the identity (2.3) is satisfied.

The final statement of the lemma is a direct consequence of the identities in (2.2) and (2.3). \( \square \)

**Proposition 2.2.** Let \( q \in \mathcal{M}_2 \), \( p \in \mathcal{M}_3 \), and \( a \in \mathcal{M}_d \) be given. Assume that there exists \( f \in \mathcal{M} \) such that (2.1) holds. Then there exist \( u_1, u_2 \in \mathcal{M}_+ \) and \( v_1, v_2 \in \mathcal{M}_1 \) such that

\[
\begin{align*}
 u_2 q - p v_1 &= a \quad \text{and} \quad p u_1^* - v_2^* q = a. \\

\end{align*}
\]

(2.6)

If, in addition, \( q \) or \( p \) is invertible in \( \mathcal{M} \), then

\[
\begin{align*}
 f &= (u_1^* + v_1)q^{-1} \quad \text{or} \quad f = p^{-1}(u_2 + v_2^*). \\

\end{align*}
\]

(2.7)

**Proof.** The first two identities in (2.1) show that \( f q \in \mathcal{M}_- + \mathcal{M}_1 \) and \( p f \in \mathcal{M}_+ + \mathcal{M}_4 \). Thus there exist (unique) \( u_1, u_2 \in \mathcal{M}_+ \) and \( v_1, v_2 \in \mathcal{M}_1 \) such that

\[
\begin{align*}
 f q &= u_1^* + v_1 \quad \text{and} \quad p f = u_2 + v_2^*. \\

\end{align*}
\]

(2.8)

If, in addition, \( q \) or \( p \) is invertible in \( \mathcal{M} \), then (2.8) yields (2.7).

It remains to prove (2.6). To do this we use \( p(fq) = (pf)q \) and the two identities in (2.8) to show that \( p u_1^* + v_1 q = u_2 q + v_2^* q \), and thus

\[
\begin{align*}
 u_2 q - p v_1 &= pu_1^* - v_2^* q. \\

\end{align*}
\]

Using the multiplication table (1.2) we see that

\[
\begin{align*}
 u_2 q - p v_1 &\in \mathcal{M}_+ \mathcal{M}_2 + \mathcal{M}_3 \mathcal{M}_1 \subset \mathcal{M}_+ \mathcal{M}_0^0 \subset \mathcal{M}_+. \\
p u_1^* - v_2^* q &\in \mathcal{M}_3 \mathcal{M}_- + \mathcal{M}_4 \mathcal{M}_2 \subset \mathcal{M}_- \mathcal{M}_0^0 \subset \mathcal{M}_-. \\

\end{align*}
\]

Thus

\[
\begin{align*}
 c := u_2 q - p v_1 &= pu_1^* - v_2^* q \in \mathcal{M}_+ \cap \mathcal{M}_- = \mathcal{M}_d. \\

\end{align*}
\]

We have to show that \( c = a \), where \( a \) is determined by the third identity in (2.1). Note that

\[
\begin{align*}
 u_2 q - p v_1 &= (u_2 + v_2^*)q - (p v_1 + v_2^* q) = pf q - (p v_1 + v_2^* q). \\

\end{align*}
\]

(2.9)
Again using the multiplication table (1.2) we see that
\[ pv_1 \in \mathcal{M}_3 \mathcal{M}_1 \subset \mathcal{M}_+^0 \quad \text{and} \quad v_2^* q \in \mathcal{M}_4 \mathcal{M}_2 \subset \mathcal{M}_-^0. \]

This implies that \( P_{\mathcal{M}_d}(pv_1 + v_2^* q) = 0. \) But then, using (2.9), we obtain
\[
c = P_{\mathcal{M}_d} (c) = P_{\mathcal{M}_d} (u_2 q - pv_1)
= P_{\mathcal{M}_d} (pf q) - P_{\mathcal{M}_d} (pv_1 + v_2^* q)
= P_{\mathcal{M}_d} (pf q) = a.
\]

Thus \( c = a \) and the proof is complete. \( \square \)

**Proposition 2.3.** If \( f \in \mathcal{M}_c \) satisfies (2.1), then \( u_1, u_2 \) and \( v_1, v_2 \) in (2.8) can be chosen such that \( u_1, u_2 \in \mathcal{M}_2 \) and \( v_1, v_2 \in P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2) \). Hence, in that case the first equation in (2.6) is solvable with \( u_2 \in \mathcal{M}_2 \) and \( v_1 \in P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2) \), and the second with \( u_1 \in \mathcal{M}_2 \) and \( v_2 \in P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2) \).

**Proof.** We begin with a remark. Recall that \( \mathcal{M}_c^\times = \mathcal{M}_c \) and \( \mathcal{M}_3^\times = \mathcal{M}_2 \). These two identities imply that \((P_{\mathcal{M}_4}(\mathcal{M}_3 \mathcal{M}_c))^\times = P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2)\). Now assume \( f \in \mathcal{M}_c \) satisfies (2.1). Then, using the multiplication table (1.2), we see that
\[
fq \in \mathcal{M}_c \mathcal{M}_2 \subset \mathcal{M}_2 + \mathcal{M}_3^0 + P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2) \subset \mathcal{M}_3 + P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2),
pf \in \mathcal{M}_3 \mathcal{M}_c \subset \mathcal{M}_2 + \mathcal{M}_3^0 + P_{\mathcal{M}_4}(\mathcal{M}_3 \mathcal{M}_c) \subset \mathcal{M}_2^\times + (P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2))^\times.
\]

This implies that in the identity (2.8) we can choose \( u_1, u_2 \in \mathcal{M}_2 \) and \( v_1, v_2 \in P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2) \). Furthermore, the final conclusion (about the two equations in (2.6) being solvable with \( u_2 \in \mathcal{M}_2 \) and \( v_1 \in P_{\mathcal{M}_1}(\mathcal{M}_c \mathcal{M}_2) \), and with \( u_1 \in \mathcal{M}_2 \) and \( v_2 \in P_{\mathcal{M}_4}(\mathcal{M}_c \mathcal{M}_2) \), respectively), now follows by using the same arguments as in the proof of the preceding proposition. \( \square \)

**Proposition 2.4.** Let \( q \in \mathcal{M}_2, p \in \mathcal{M}_3, \) and \( a \in \mathcal{M}_d \), and let \( q \) and \( p \) be invertible in \( \mathcal{M} \). Assume that there exist \( u_1, u_2 \in \mathcal{M}_+ \) and \( v_1, v_2 \in \mathcal{M}_1 \) such that
\[
u_2 q - pv_1 = a \quad \text{and} \quad pu_1^* - v_2^* q = a. \quad (2.10)
\]

Then there exists a \( f \in \mathcal{M} \) satisfying the three equations in (2.1), and one such \( f \) is determined by the data via the formula
\[
f = (u_1^* + v_1)q^{-1} = p^{-1}(u_2 + v_2^*). \quad (2.11)
\]

**Proof.** Put \( f_1 = (u_1^* + v_1)q^{-1} \) and \( f_2 = p^{-1}(u_2 + v_2^*) \). From the two identities in (2.10) it follows that \( (u_2 + v_2^*)q = p(u_1^* + v_1) \). Multiplying the latter identity from the left by \( p^{-1} \) and from the right by \( q^{-1} \) we see that \( f_1 = f_2 \). Hence we can delete the subindices 1, 2, and simply write \( f = f_1 = f_2 \). In particular, (2.11) holds.

It remains to prove that \( f \) satisfies the identities in (2.1). To do this note that
\[
fq = u_1^* + v_1 \in \mathcal{M}_- + \mathcal{M}_1, \quad pf = u_2 + v_2^* \in \mathcal{M}_+ + \mathcal{M}_4.
\]

From these two formulas we see that both \( P_{\mathcal{M}_2}(fq) \) and \( P_{\mathcal{M}_3}(pf) \) belong to \( \mathcal{M}_d \). Next, using the multiplication table (1.2) we see that
\[
pv_1 \in \mathcal{M}_3 \mathcal{M}_1 \subset \mathcal{M}_+^0 \quad \text{and} \quad v_2^* q \in \mathcal{M}_4 \mathcal{M}_2 \subset \mathcal{M}_-^0.
\]
Since \( a = pu_1^* - v_2^*q \), it follows that
\[
pfq = pu_1^* + pv_1 = (pu_1^* - v_2^*q) + v_1^* = a + M_0^0 + M_0^0.
\]

But then \( P_{M_d}(pfq) = a \), as desired. \( \square \)

**Proof of Theorem 1.1.** Throughout this proof we assume that \( q \in \mathcal{M}_2 \) and \( a = a^* \in \mathcal{M}_d \). Furthermore, we set \( p = q^* \).

Assume there exists \( f = f^* \in \mathcal{M} \) satisfying conditions (1.3). Using the fact that both \( f \) and \( a \) are selfadjoint and that \( p = q^* \), we see that \( f \) also satisfies conditions (2.1). But then Proposition 2.2 tells us that there exist \( u_1 \in M_+ \) and \( v_1 \in M_1 \) satisfying the first identity in (2.6). Now put \( u = u_1 \) and \( v = v_1 \). Then, \( u \in \mathcal{M}_+ \), \( v \in \mathcal{M}_1 \), and \( u \) and \( v \) satisfy (1.4) because \( p = q^* \). From (2.8) we see that \( u = u_1 = P_{M_1}(q^*f) \) and \( v = v_1 = P_{M_1}(f) \). Moreover, if \( f = f^* \in \mathcal{M}_c \), then the multiplication table (1.2) shows that \( P_{M_1}(q^*f) \in \mathcal{M}_2 \) and \( P_{M_1}(f) \in \mathcal{M}_1(M_1,M_2) \).

To prove the converse part of the theorem, let \( q \) be invertible, and assume that \( u \in \mathcal{M}_+ \), \( v \in \mathcal{M}_1 \) satisfy Eq. (1.4). Put \( u_1 = u_2 = u \) and \( v_1 = v_2 = v \). Since \( p = q^* \) and \( a = a^* \), it follows that the equations in (2.10) are satisfied. Furthermore, using \( q \) is invertible, we can apply Proposition 2.4 to show that there exists \( f \in \mathcal{M} \) satisfying the three equations in (2.1), and this \( f \) is uniquely determined by the data via formula (2.11). Hence,
\[
fq = u^* + v \quad \text{and} \quad q^* f = u + v^*.
\]
This implies that \( f_1 = (q^* f)_1 = f^* \). Thus \( f = f^* \), because \( q \) is invertible. But then (2.1) implies (1.3).

It remains to prove items (a)–(d). To prove (a) note that the symmetry properties of an algebra with band structure (see (C2) in Section 1) yield \( P_{M_c} f = P_{M_c} f^* \). Thus the fact that \( f \) is selfadjoint implies that the same is true for \( f_c = P_{M_c} f \). With this remark it is straightforward to see that (a) follows from Lemma 2.1.

To prove (b), let \( q \) be invertible, and assume that (1.4) has a solution pair \( u \in \mathcal{M}_+ \), \( v \in \mathcal{M}_1 \). Then, as we have seen, there exists \( f = f^* \in \mathcal{M} \) satisfying (1.3). Using (a) this implies that the element \( f_c = P_{M_c} f \) is selfadjoint and that (1.3) holds with \( f_c \) in place of \( f \). Thus by the first part of the theorem \( \tilde{u} := P_{M_1}(q^*f_c) \) and \( \tilde{v} := P_{M_1}(f_cq) \) satisfy (1.4). Moreover, as we have seen earlier, because of the multiplication table (1.2), we have \( \tilde{u} \in \mathcal{M}_2 \) and \( \tilde{v} \in P_{M_1}(M_c,M_2) \).

Next, note that the first part of the theorem covers item (c). Finally, let \( q \) be invertible and assume that \( q^{-1} \in \mathcal{M}_+ \). Put \( f = q^{-1}a \). Then \( f_q = q^{-1}a \in \mathcal{M}_- \mathcal{M}_d \subset \mathcal{M}_- \), and hence \( P_{M_2} f_q \in \mathcal{M}_d \). Furthermore, \( q^* f_q = a \in \mathcal{M}_d \). Thus (d) holds. \( \square \)

3. Szegő–Krein orthogonal polynomials

We begin with some notation. Let \( f \) be an \( r \times r \) matrix function from the Wiener algebra \( \mathcal{W}^{r \times r}(\mathbb{T}) \). Thus \( f(\xi) = \sum_{\nu=-\infty}^{\infty} \xi^\nu f_\nu \), where the coefficients \( f_{-1}, f_0, f_1, \ldots \) are \( r \times r \) matrices summable in norm, that is, \( \sum_{\nu=-\infty}^{\infty} \|f_\nu\| < \infty \). The adjoint of \( f \) is the function \( f^* \) defined by \( f^*(\xi) = f(\xi)^* \) for \( \xi \) on the unit circle \( \mathbb{T} \). If \( f \) is a trigonometric polynomial, then \( f^*(\xi) = f(\xi)^* \) for each \( 0 \neq \xi \in \mathbb{C} \). The function \( f \) is said to be hermitian whenever \( f^* = f \). The latter is equivalent to \( f_{-\nu} = f_{\nu}^* \) for \( \nu = 0, 1, 2, \ldots \).
In what follows we fix a positive integer \( n \). Given a hermitian \( f \in W^{r \times r} \), an \( r \times r \) matrix polynomial \( \varphi(z) = \varphi_0 + z\varphi_1 + \cdots + z^n\varphi_n \) is called an \( n \)-th Szegő–Kreĭn orthogonal polynomial generated by \( f \) whenever

\[
\begin{bmatrix}
  f_0 & f_{-1} & \cdots & f_{-n} \\
  f_1 & f_0 & \cdots & f_{-n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_n & f_{n-1} & \cdots & f_0
\end{bmatrix}
\begin{bmatrix}
  \varphi_0 \\
  \varphi_1 \\
  \vdots \\
  x_n
\end{bmatrix}
= \begin{bmatrix}
  I_r \\
  0 \\
  \vdots \\
  0
\end{bmatrix}.
\] (3.1)

Here \( I_r \) is the \( r \times r \) identity matrix. Note that (3.1) only requires the knowledge of the Fourier coefficients \( f_{-n}, \ldots, f_n \).

For the scalar case with a positive definite weight \( f \) the Szegő–Kreĭn orthogonal polynomials have been introduced by Szegő (see [19]), for indefinite scalar weights they go back to Kreĭn [16] and for the matrix-valued case they are studied in [13]. We shall prove the following result.

**Theorem 3.1.** Let \( \varphi(z) = \varphi_0 + z\varphi_1 + \cdots + z^n\varphi_n \) be an \( r \times r \) matrix polynomial. If \( \varphi \) is an \( n \)-th Szegő–Kreĭn orthogonal polynomial, then \( \varphi_0 \) is hermitian and there exist \( r \times r \) matrix polynomials \( u \) and \( w \), with degree \( u \) at most \( n \) and degree \( w \) at most \( n - 1 \), such that

\[
u(z)\varphi(z) - z^{n+1}\varphi^*(z)w(z) = \varphi_0, \quad \text{where} \quad \varphi^*(z) = \varphi(\bar{z}^{-1})^*.
\] (3.2)

Conversely, if \( \varphi_0 \) is positive definite and there exist \( r \times r \) matrix polynomials \( u \) and \( w \), with degree \( u \) at most \( n \) and degree \( w \) at most \( n - 1 \), such that (3.2) holds, then \( \varphi \) is an \( n \)-th Szegő–Kreĭn orthogonal polynomial. Furthermore, \( \varphi \) is invertible in the Wiener algebra \( W^{r \times r}(\mathbb{T}) \) and a hermitian \( f \in W^{r \times r}(\mathbb{T}) \) generating \( \varphi \) is given by

\[
f(\xi) = (u^*(\xi) + \xi^{n+1}w(\xi))\varphi(\xi)^{-1}, \quad \xi \in \mathbb{T}.
\] (3.3)

We shall derive the above theorem as a corollary of a non-symmetric version of Theorem 3.1, where \( f \in W^{r \times r} \) is not required to be hermitian. To state this non-symmetric version we need some additional notation. As before, \( \varphi \) is an \( r \times r \) matrix polynomial of degree at most \( n \). Apart from \( \varphi \) we have an additional \( r \times r \) matrix polynomial in \( z^{-1} \) of degree at most \( n \), denoted by \( \psi \). Thus we shall be working with

\[
\varphi(z) = \varphi_0 + z\varphi_1 + \cdots + z^n\varphi_n, \quad \psi(z) = \psi_0 + z^{-1}\psi_1 + \cdots + z^{-n}\psi_n.
\] (3.4)

In addition to (3.1) we shall consider the equation

\[
\begin{bmatrix}
  \psi_0 & \psi_1 & \cdots & \psi_n
\end{bmatrix}
\begin{bmatrix}
  f_0 & f_{-1} & \cdots & f_{-n} \\
  f_1 & f_0 & \cdots & f_{-n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_n & f_{n-1} & \cdots & f_0
\end{bmatrix}
= \begin{bmatrix}
  I_r \\
  0 \\
  \vdots \\
  0
\end{bmatrix}.
\] (3.5)

We shall deal with the following question. Under what conditions on \( \varphi \) and \( \psi \) does there exists a \( f \in W^{r \times r} \) such that (3.1) and (3.5) hold. The following theorem is our non-symmetric version of Theorem 3.1.

**Theorem 3.2.** Let \( \varphi \) and \( \psi \) be the \( r \times r \) matrix functions in (3.4). If there exists a \( f \in W^{r \times r} \) such that (3.1) and (3.5) hold, then \( \varphi_0 = \psi_0 \) and there exist \( r \times r \) matrix polynomials \( u_1, u_2 \) of
degree at most $n$ and $r \times r$ matrix polynomials $w_1$, $w_2$ of degree at most $n - 1$ such that

\[ u_2(z)\varphi(z) - z^{n+1}\psi(z)w_1(z) = \varphi_0, \quad (3.6) \]
\[ \psi(z)u_1^*(z) - z^{-(n+1)}w_2^*(z)\varphi(z) = \varphi_0. \quad (3.7) \]

Conversely, if $\varphi_0 = \psi_0$ is non-singular, $\det \varphi(z)$ and $\det \psi(z)$ have no zero on the unit circle, and there exist $r \times r$ matrix polynomials $u_1, u_2$ of degree at most $n$ and $r \times r$ matrix polynomials $w_1, w_2$ of degree at most $n - 1$ such that (3.6) and (3.7) are satisfied, then there exists $f \in \mathcal{W}^{r \times r}$ such that (3.1) and (3.5) hold. Moreover, one such $f$ is given by

\[ f(\zeta) = (u_1^*(\zeta) + \zeta^{n+1}w_1(\zeta))\varphi(\zeta)^{-1} \]
\[ = \psi(\zeta)^{-1}(u_2(\zeta) + \zeta^{n+1}w_2^*(\zeta)), \quad \zeta \in \mathbb{T}. \quad (3.8) \]

To prove the above theorem we shall apply results of Section 2 with $\mathcal{R} = C^{r \times r}(\mathbb{T})$, the algebra of all continuous $r \times r$ matrix functions on $\mathbb{T}$, and for $\mathcal{M}$ we take the Wiener algebra $\mathcal{W}^{r \times r}(\mathbb{T})$. The involution $*$ on $C^{r \times r}(\mathbb{T})$ is given by $f^*(\zeta) = f(\zeta)^*$ for each $\zeta \in \mathbb{T}$, and the unit $e$ of $C^{r \times r}(\mathbb{T})$ is the $r \times r$ matrix function which is identically equal to the $r \times r$ identity matrix $I_r$. Clearly, $\mathcal{W}^{r \times r}(\mathbb{T})$ is a $*$-subalgebra of $C^{r \times r}(\mathbb{T})$, and $e \in \mathcal{W}^{r \times r}(\mathbb{T})$.

In what follows we simply write $\mathcal{V}$ in place of $\mathcal{W}^{r \times r}(\mathbb{T})$. As before, the $v$-th Fourier coefficient of $f \in \mathcal{V}$ is denoted by $f_v$. Now define

\[ \mathcal{V}_1 = \{ f \in \mathcal{V} \mid f_v = 0 \text{ for } v \leq n \}, \]
\[ \mathcal{V}_0^2 = \{ f \in \mathcal{V} \mid f_v = 0 \text{ for } v \leq 0 \text{ or } v \geq n + 1 \}, \]
\[ \mathcal{V}_d = \{ f \in \mathcal{V} \mid f_v = 0 \text{ for } v \neq 0 \}, \]
\[ \mathcal{V}_3^0 = \{ f \in \mathcal{V} \mid f_v = 0 \text{ for } v \geq 0 \text{ or } v \leq -n - 1 \}, \]
\[ \mathcal{V}_4 = \{ f \in \mathcal{V} \mid f_v = 0 \text{ for } v \geq -n \}. \]

Then $\mathcal{V}$ decomposes as

\[ \mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2^0 + \mathcal{V}_d + \mathcal{V}_3^0 + \mathcal{V}_4. \quad (3.9) \]

In this case the space $\mathcal{V}_2 = \mathcal{V}_2^0 + \mathcal{V}_d$ and $\mathcal{V}_3 = \mathcal{V}_3^0 + \mathcal{V}_d$. Thus $\mathcal{V}_2$ is the space consisting all $r \times r$ matrix polynomials of degree at most $n$. In particular, $\varphi \in \mathcal{V}_2$. Similarly, $\psi \in \mathcal{V}_3$. A straightforward calculation shows that

\[ (3.1) \iff P_{\mathcal{V}_2}(f\varphi) = e \quad \text{and} \quad (3.5) \iff P_{\mathcal{V}_3}(\psi f) = e. \quad (3.10) \]

**Proof of Theorem 3.2.** Throughout $\varphi$ and $\psi$ are the $r \times r$ matrix functions in (3.4). We shall prove the theorem by applying the results of Section 2 with $q = \varphi$ and $p = \psi$. For $a$ we shall take the function identically equal to $\varphi_0$ on $\mathbb{T}$. Clearly, $q \in \mathcal{V}_2$, $p \in \mathcal{V}_3$, and $a \in \mathcal{V}_d$. We split the proof into two parts.

**Part 1.** In this part we assume that there exists a function $f \in \mathcal{V} = \mathcal{W}^{r \times r}$ such that (3.1) and (3.5) hold. Without loss of generality we may assume that $f$ is a trigonometric polynomial of degree at most $n$. In other words, $f \in \mathcal{V}_c$, where $\mathcal{V}_c = \mathcal{V}_2^0 + \mathcal{V}_d + \mathcal{V}_3^0$. From (3.10) we know that
Propositions 2.2 and 2.3, we see that

\[ \psi_0 = \begin{bmatrix} \psi_0 & \psi_1 & \cdots & \psi_n \end{bmatrix} \begin{bmatrix} f_0 & f_{-1} & \cdots & f_{-n} \\ f_1 & f_0 & \cdots & f_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n-1} & \cdots & f_0 \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_n \end{bmatrix} = \psi_0. \] (3.11)

Thus \( \psi_0 = \psi_0 \). The identity (3.11) also shows that \( P_{V_2}(q f q) = a \). Summarizing we have

\[ P_{V_2}(f q) \in V_d, \quad P_{V_1}(p f) \in V_d \quad P_{V_3}(q^* f q) = a. \] (3.12)

But then we can apply Propositions 2.2 and 2.3 to show that there exist functions \( u_1, u_2 \in V_2 \) and \( v_1, v_2 \in P_{V_1}(V_c V_2) \) such that (2.6) holds in the present setting. Note that in the present setting

\[ u \in V_2 \iff u \text{ is an } r \times r \text{ matrix polynomial of degree at most } n \] (3.13)

\[ v \in P_{V_1}(V_c V_2) \iff v(\zeta) = \zeta^{n+1} w(\zeta), \text{ where } w \text{ is an } r \times r \text{ matrix polynomial of degree at most } n - 1. \] (3.14)

Thus (3.6) and (3.7) hold, and the direct part of the theorem is proven.

**Part 2.** In this part we assume that \( \det \varphi(z) \) and \( \det \psi(z) \) have no zero on the unit circle. Furthermore, we assume that \( \varphi_0 = \psi_0 \), and there exist \( r \times r \) matrix polynomials \( u_1, u_2 \) of degree at most \( n \) and \( r \times r \) matrix polynomials \( w_1, w_2 \) of degree at most \( n - 1 \) such that (3.6) and (3.7) are satisfied. By Wiener’s theorem, the first assumption implies that \( q = \varphi \) and \( p = \psi \) are invertible in \( V \).

For \( j = 1, 2 \) let \( v_j \) be the \( r \times r \) matrix polynomial defined by the formula \( v_j(\zeta) = \zeta^{n+1} w_j(\zeta) \). From (3.14) we see that \( v_j \in P_{V_1}(V_c V_2) \subset V_1 \). According to (3.13) we have \( u_j \in V_2 \) for \( j = 1, 2 \). Using \( q = \varphi, p = \psi \), and \( a \equiv \varphi_0 \), the identities (3.6) and (3.7) can be rewritten in the form (2.6). But then Proposition 2.4 tells us that there exists \( f \in V \) such that (2.1) hold. Moreover, one such \( f \) is given by (2.11). In the present setting, with \( M = V \), this means that \( f \in V \) satisfies (3.12) and \( f \) is given by (3.8). Finally, note that

\[ P_{V_2}(f q) \in V_d \iff \begin{bmatrix} f_0 & f_{-1} & \cdots & f_{-n} \\ f_1 & f_0 & \cdots & f_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n-1} & \cdots & f_0 \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} = \begin{bmatrix} \star \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \] (3.15)

Here \( \star \) denotes an unspecified \( r \times r \) matrix. Using \( q = \varphi, p = \psi \), and \( a \equiv \varphi_0 \), the formulas in (3.12) then show that \( \psi_0 \star = \varphi_0 = \psi_0 \). Since \( \psi_0 \) is invertible, this yields \( \star = I_m \), and hence \( f \) satisfies (3.1) and (3.5). \( \square \)

Next we derive Theorem 3.1 as a corollary of Theorem 3.2. For this purpose we need the following lemma, which follows from [13, Theorem 2.1].

**Lemma 3.3.** Let \( \varphi(z) = \varphi_0 + z \varphi_1 + \cdots + z^n \varphi_n \) be an \( r \times r \) matrix polynomial, and assume there exist \( r \times r \) matrix polynomials \( u \) and \( w \) such that (3.2) holds. If, in addition, \( \varphi_0 \) is positive definite, then \( \det \varphi(z) \) has no zero on the unit circle.

**Proof of Theorem 3.1.** We apply Theorem 3.2 with \( \psi = \varphi^* \). First assume that \( \varphi \) is an \( n \)-th Szegő–Kreǐn orthogonal polynomial generated by some \( f = f^* \in W_{r \times r}^f \). The fact that \( f \) is hermitian and \( \psi = \varphi^* \) implies that both (3.1) and (3.5) are satisfied. It follows that \( \varphi_0 = \psi_0 \) and
there exist $r \times r$ matrix polynomials $u_1, u_2$ of degree at most $n$ and $r \times r$ matrix polynomials $w_1, w_2$ of degree at most $n - 1$ such that (3.6) and (3.7) hold. Since $\psi = \phi^*$, the equality $\phi_0 = \psi_0$ yields $\phi_0$ is hermitian. Furthermore, with $\psi = \phi^*, u = u_1$ and $w = w_1$ the identity (3.6) is just (3.2). This proves the direct part of Theorem 3.1.

Next assume that $\phi_0$ is positive definite and there exist $r \times r$ matrix polynomials $u$ and $w$, with degree $u$ at most $n$ and degree $w$ at most $n - 1$, such that (3.2) holds. From Lemma 3.3 we know $\det \phi(z)$ has no zero on the unit circle. Then the same holds true for $\psi = \phi^*$. Moreover, both $\phi_0$ and $\psi_0$ are invertible. Now put $u_1 = u_2 = u$ and $w_1 = w_2 = w$. Then (3.6) is just (3.2) and (3.7) is the adjoint of (3.2). The converse part of Theorem 3.2 then shows that there exists $f \in W^{r \times r}$ such that (3.1) holds. Moreover, by (3.8), using $u_1 = u_2 = u$, $w_1 = w_2 = w$, $\psi = \phi^*$, one such $f$ is given by

$$f(\zeta) = (u^*(\zeta) + \zeta^{n+1} w(\zeta))\phi(\zeta)^{-1} = \phi^*(\zeta)^{-1}(u(\zeta) + \zeta^{n+1} w^*(\zeta)), \quad \zeta \in \mathbb{T}.$$ 

The latter implies that $f$ is hermitian, and thus $\phi$ is an $n$-th Szegő–Krein orthogonal polynomial generated by $f$. □

Let us consider Eq. (3.2) in more detail. Recall that $\phi(z)$ is an $r \times r$ matrix polynomial, $\phi(z) = \phi_0 + z\phi_1 + \cdots + z^n\phi_n$, and let us assume that $\phi_0$ is positive definite. Rewrite Eq. (3.2) as

$$u(z)\phi(z) + \eta(z)w(z) = \phi_0^*, \quad \text{where } \eta(z) = -z^{n+1}\phi^*(z).$$

Note that $\eta(z)$ is an $r \times r$ matrix polynomial of degree $n + 1$ and with leading coefficient $\phi_0^*$. In particular, the leading coefficient of $\eta(z)$ is non-singular. Next observe that $\phi(0) = \phi_0$ is also non-singular. Hence both $\phi$ and $\eta$ are regular matrix polynomials and the common zeros of $\det \phi(z)$ and $\det \eta(z)$ are non-zero.

Now let $z_0 \neq 0$, and assume $z_0$ is a common zero of $\det \phi(z)$ and $\det \eta(z)$. Then both $z_0$ and $z_0^{-1}$ are zeros of $\det \phi(z)$. Furthermore, if $y_0, \ldots, y_{\beta - 1}$ is a right Jordan chain of $\phi$ at $z_0^{-1}$, then $y_0^*, \ldots, y_{\beta - 1}^*$ is a left Jordan chain of $\eta(z)$ at $z_0$, and any left Jordan chain of $\eta(z)$ at $z_0 \neq 0$ is obtained in this way. Using this connection the following result can be obtained as a corollary of [13, Theorem 4.1].

**Proposition 3.4.** Let $\phi(z) = \phi_0 + z\phi_1 + \cdots + z^n\phi_n$ be an $r \times r$ matrix polynomial, and assume that $\phi_0$ is positive definite. Then there exist $r \times r$ matrix polynomials $u$ and $w$, with degree $u$ at most $n$ and degree $w$ at most $n - 1$, satisfying (3.2) if and only if for every symmetric pair of zeros $z_0, z_0^{-1}$ of $\det \phi(z)$, we have

$$\sum_{j=0}^k y_{k-j}^* \phi_0 x_j = 0, \quad k = 0, 1, \ldots, \min(\alpha, \beta) - 1,$$

where $x_0, \ldots, x_{\alpha-1}$ is any right Jordan chain of $\phi$ at $z_0$ and $y_0, \ldots, y_{\beta-1}$ is any right Jordan chain of $\phi$ at $z_0^{-1}$.

By combining the above result with Theorem 3.1 we arrive at the following corollary, which is covered by [13, Theorem 6.2].

**Corollary 3.5.** Let $\phi(z) = \phi_0 + z\phi_1 + \cdots + z^n\phi_n$ be an $r \times r$ matrix polynomial, and assume that $\phi_0$ is positive definite. Then $\phi$ is an $n$-th Szegő–Krein orthogonal polynomial if and only if
for every symmetric pair of zeros $z_0, \bar{z}_0^{-1}$ of $\det \varphi(z)$, we have

$$\sum_{j=0}^{k} y_{k-j}^* \varphi_0 x_j = 0, \quad k = 0, 1, \ldots, \min\{\alpha, \beta\},$$

where $x_0, \ldots, x_\alpha$ is any right Jordan chain of $\varphi$ at $z_0$ and $y_0, \ldots, y_\beta$ is any right Jordan chain of $\varphi$ at $\bar{z}_0^{-1}$.

We conclude this section with a few comments. The first is an addition to Theorem 3.1, the second concerns Eqs. (3.6) and (3.7), and the third is a remark about Theorem 3.2 and its relation to Theorem 7.1 in [13].

**Comment 3.6.** Theorem 1.1 provides the following addition to Theorem 3.1. Let $\varphi(z) = \varphi_0 + z \varphi_1 + \cdots + z^n \varphi_n$ be an $r \times r$ matrix polynomial, and assume that $\varphi_0$ is positive definite. Then $\varphi$ is an $n$-th Szegő–Krein orthogonal polynomial if and only if there exists $u, w \in \mathcal{W}_+^{r \times r}(\mathbb{T})$ satisfying the equation

$$u(z) \varphi(z) - z^{n+1} \varphi^*(z) w(z) = \varphi_0. \tag{3.17}$$

Moreover in that case the set of all hermitian $f \in \mathcal{W}_+^{r \times r}(\mathbb{T})$ generating $\varphi$ is parameterized by the set

$$\mathcal{E} = \{(u, v) \mid u, w \in \mathcal{W}_+^{r \times r}(\mathbb{T}), \text{ and } u, v \text{ is a solution pair of (3.17)}\},$$

using the map $(u, v) \mapsto f$, with $f$ given by $f(\zeta) = (u^*(\zeta) + \varepsilon^{n+1} w(\zeta)) \varphi(\zeta)^{-1}$, where $\zeta \in \mathbb{T}$.

**Comment 3.7.** Using Theorem 4.1 in [13] and replacing $z$ by $z^{-1}$ in (3.7) one can prove that the two Eqs. (3.6) and (3.7) are equivalent in the following sense. There exist an $r \times r$ matrix polynomial $u_2$ of degree at most $n$ and an $r \times r$ matrix polynomial $w_1$ of degree at most $n - 1$ such that the pair $(u_2, w_1)$ is a solution pair Eq. (3.6) if and only if there exist an $r \times r$ matrix polynomial $u_1$ of degree at most $n$ and an $r \times r$ matrix polynomial $w_2$ of degree at most $n - 1$ such that the pair $(u_1, w_2)$ is a solution pair Eq. (3.7). See also the next comment.

**Comment 3.8.** We comment on Theorem 3.2 and its relation to Theorem 7.1 in [13]. The latter theorem tells us that the existence of a function $f$ such that (3.1) and (3.5) hold is equivalent with $\varphi_0 = \psi_0$ is non-singular and (3.6) has polynomial solutions, and in that case there is a formula for $f$ different from the one given in (3.8). Thus given Theorem 7.1 in [13], the addition to the existing literature by Theorem 3.2 is rather modest. On the other hand, Theorem 3.2 contains Theorem 3.1 as a special case.

### 4. Krein orthogonal entire matrix functions

Let $\varphi \in L_1^{-\times r}[0, \omega]$, and let $\Phi$ be the entire matrix function given by

$$\Phi(\lambda) = I_r + \int_{0}^{\omega} e^{i\lambda t} \varphi(t) \, dt, \quad \lambda \in \mathbb{C}. \tag{4.1}$$

The function $\Phi$ is called a Krein orthogonal function whenever there exists a hermitian $n \times n$ matrix function $k$ in the space $L_1^{-\times r}[-\omega, \omega]$ such that

$$\varphi(t) - \int_{0}^{\omega} k(t-s)\varphi(s) \, ds = k(t), \quad 0 \leq t \leq \omega. \tag{4.2}$$
In that case we say that $\Phi$ is generated by $k$. Recall that $k \in L^1_{r \times r}[-\omega, \omega]$ is called hermitian if $k(t) = k(-t)^*$ for each $t \in [-\omega, \omega]$. See [15,17,18] for the first papers on this type of orthogonal functions. The following result is covered by [10, Theorem 1.2].

**Theorem 4.1.** Let $\varphi \in L^1_{1 \times r}[0, \omega]$. If $\Phi$ defined by (4.1) is a Kreĭn orthogonal function, then there exist $u \in L^1_{1 \times r}[0, \omega]$ and $v \in L^1_{1 \times r}[\omega, 2\omega]$ such that the entire matrix functions

$$U(\lambda) = \mathbb{I}_r + \int_0^\omega e^{i\lambda t} u(t) \, dt, \quad V(\lambda) = \int_0^{2\omega} e^{i\lambda t} v(t) \, dt$$

(4.3)

satisfy the equation

$$U(\lambda) \Phi(\lambda) - \Phi(\lambda)^* V(\lambda) = \mathbb{I}_r, \quad \lambda \in \mathbb{C}. \tag{4.4}$$

Conversely, if $U$ and $V$ in (4.3) satisfy (4.4), then $\det(\Phi(\lambda)) \neq 0$ for each real $\lambda$ and the function $k$ defined by

$$k = \lambda |_{-\omega, \omega}, \quad \text{where} \int_{-\infty}^\infty e^{i\lambda s} \ell(s) \, ds = (U(\lambda)^* + V(\lambda)) \Phi(\lambda)^{-1} (\lambda \in \mathbb{R}),$$

(4.5)

is a hermitian matrix function in $L^1_{1 \times r}[-\omega, \omega]$ and $\Phi$ is a Kreĭn orthogonal function generated by $k$.

We shall derive the above theorem as a corollary of a non-symmetric version of Theorem 4.1, where $k \in L^1_{r \times r}[-\omega, \omega]$ is not required to be hermitian. For the non-symmetric version we start with two $r \times r$ matrix functions, $\varphi_1 \in L^1_{1 \times r}[0, \omega]$ and $\varphi_2 \in L^1_{1 \times r}[-\omega, 0]$, and we look for a $k \in L^1_{r \times r}[-\omega, \omega]$ such that $\varphi_1$ and $\varphi_2$ satisfy the following integral equations

$$\varphi_1(t) - \int_0^\omega k(t-s) \varphi_1(s) \, ds = k(t), \quad 0 \leq t \leq \omega, \tag{4.6}$$

$$\varphi_2(t) - \int_{-\omega}^0 k(t-s) \varphi_2(s) \, ds = k(t), \quad -\omega \leq t \leq 0. \tag{4.7}$$

To find such a $k$ we associate with $\varphi_1$ and $\varphi_2$ the following entire matrix functions:

$$\Phi_1(\lambda) = \mathbb{I}_r + \int_0^\omega e^{i\lambda s} \varphi_1(s) \, ds, \quad \Phi_2(\lambda) = \mathbb{I}_r + \int_{-\omega}^0 e^{i\lambda s} \varphi_2(s) \, ds. \tag{4.8}$$

The following result is our non-symmetric version of Theorem 4.1.

**Theorem 4.2.** Let $\Phi_1$ and $\Phi_2$ be defined by (4.8), where $\varphi_1 \in L^1_{1 \times r}[0, \omega]$ and $\varphi_2 \in L^1_{1 \times r}[-\omega, 0]$. If there exists a matrix function $k \in L^1_{r \times r}[-\omega, \omega]$ such that (4.6) and (4.7) hold, then there exist $u_1, u_2 \in L^1_{1 \times r}[0, \omega]$ and $v_1, v_2 \in L^1_{1 \times r}[\omega, 2\omega]$ such that the entire matrix functions

$$U_j(\lambda) = \mathbb{I}_r + \int_0^\omega e^{i\lambda t} u_j(t) \, dt, \quad V_j(\lambda) = \int_0^{2\omega} e^{i\lambda t} v_j(t) \, dt \quad (j = 1, 2) \tag{4.9}$$

satisfy the equations

$$U_2(\lambda) \Phi_1(\lambda) - \Phi_2(\lambda) V_1(\lambda) = \mathbb{I}_r, \quad \lambda \in \mathbb{C}, \tag{4.10}$$

$$\Phi_2(\lambda) U_1(\lambda)^* - V_2(\lambda)^* \Phi_1(\lambda) = \mathbb{I}_r, \quad \lambda \in \mathbb{C}. \tag{4.11}$$
Conversely, if \( \det \Phi_1(\lambda) \) and \( \det \Phi_2(\lambda) \) have no zero on the real line, and there exist matrix functions \( u_1, u_2 \in L^r_{1} \times r[0, \omega] \) and \( v_1, v_2 \in L^r_{1} \times r[\omega, 2\omega] \) such that the entire matrix functions in (4.9) satisfy (4.10) and (4.11), then there exists a matrix function \( k \in L^r_{1} \times r[-\omega, \omega] \) such that (4.6) and (4.7) hold. Furthermore, one such \( k \in L^r_{1} \times r[-\omega, \omega] \) is obtained by \( k = \ell|_{-\omega, \omega} \), where the function \( \ell \) is given by

\[
\int_{-\infty}^{\infty} e^{i\lambda s} \ell(s) \, ds = (U_1(\tilde{\lambda})^* + V_1(\lambda)) \Phi_1(\lambda)^{-1} \quad (\lambda \in \mathbb{R})
\]

\[
= \Phi_2(\lambda)^{-1}(U_2(\lambda) + V_2(\tilde{\lambda})^*) \quad (\lambda \in \mathbb{R}).
\]

(4.12)

We shall derive the above theorem by using the results of Section 2. For this purpose we need the Wiener algebra \( \mathcal{W} = \mathcal{W}^r_{\times r}(\mathbb{R}) \) of \( r \times r \) matrix functions on the real line. Recall that

\[
\mathcal{W} = \left\{ H \mid H(\lambda) = M + \int_{-\infty}^{\infty} e^{i\lambda t} h(t) \, dt, \quad M \in \mathbb{C}^r_{\times r}, \quad h \in L^r_{1}(\mathbb{R}) \right\}.
\]

Note that \( \mathcal{W} \) is a subalgebra of the unital \( C^* \)-algebra \( C(\mathbb{R})^r_{\times r} \) which consists of all \( r \times r \) matrix functions \( H \) that are continuous on the extended real line \( \mathbb{R} = \mathbb{R} \cup \{\infty\} \). The unit of \( C(\mathbb{R})^r_{\times r} \) is the function \( E \) which is identically equal to the \( r \times r \) identity matrix and the involution * in \( C(\mathbb{R})^r_{\times r} \) is given by

\[
H^*(\lambda) = H(\lambda)^*, \quad \lambda \in \mathbb{R}.
\]

Observe that the unit \( E \) belongs to \( \mathcal{W} \), and \( \mathcal{W} \) is *-closed.

The Wiener algebra \( \mathcal{W} \) has a natural band structure. Indeed, put

\[
\mathcal{W}_1 = \left\{ H \mid H(\lambda) = \int_{\omega}^{\infty} e^{i\lambda t} h(t) \, dt, \quad h \in L^r_{1}[\omega, \infty) \right\},
\]

\[
\mathcal{W}_2 = \left\{ H \mid H(\lambda) = \int_{0}^{\infty} e^{i\lambda t} h(t) \, dt, \quad h \in L^r_{1}[0, \omega) \right\},
\]

\[
\mathcal{W}_d = \left\{ H \mid H(\lambda) = M, \quad M \in \mathbb{C}^r_{\times r} \right\},
\]

\[
\mathcal{W}_3 = \left\{ H \mid H(\lambda) = \int_{0}^{\omega} e^{i\lambda t} h(t) \, dt, \quad h \in L^r_{1}[-\omega, 0] \right\},
\]

\[
\mathcal{W}_4 = \left\{ H \mid H(\lambda) = \int_{-\omega}^{0} e^{i\lambda t} h(t) \, dt, \quad h \in L^r_{1}(-\infty, -\omega) \right\}.
\]

Then \( \mathcal{W} \) decomposes as

\[
\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2^0 + \mathcal{W}_d + \mathcal{W}_3^0 + \mathcal{W}_4,
\]

(4.13)

and with respect to this decomposition \( \mathcal{W} \) is an algebra with band structure. As usual we set

\[
\mathcal{W}_2 = \mathcal{W}_2^0 + \mathcal{W}_d, \quad \mathcal{W}_3 = \mathcal{W}_3^0 + \mathcal{W}_d, \quad \mathcal{W}_c = \mathcal{W}_2^0 + \mathcal{W}_d + \mathcal{W}_3^0.
\]

By the Riemann–Lebesgue lemma, for \( H \in \mathcal{W} \), we have

\[
P_{\mathcal{W}_d} H = \lim_{\lambda \to \infty} H(\lambda) = H(\infty).
\]

(4.14)
Furthermore,

\[ U \in \mathcal{W}_2 \iff U(\lambda) = U(\infty) + \int_{0}^{\infty} e^{i\lambda t} h(t) dt, \quad h \in L_1^{r \times r}[0, \omega], \quad (4.15) \]

\[ V \in P_{\mathcal{W}_1}(\mathcal{W}_2, \mathcal{W}_2) \iff V(\lambda) = \int_{-\infty}^{\omega} e^{i\lambda t} h(t) dt, \quad h \in L_1^{r \times r}[\omega, 2\omega]. \quad (4.16) \]

We are now ready to prove Theorem 4.2.

**Proof of Theorem 4.2.** We apply the results proved in Section 2. For \( \mathcal{M} \) we take the Wiener algebra \( \mathcal{W} = \mathcal{W}_{r \times r}(\mathbb{R}) \), and we use the band structure defined by (4.13). Furthermore, we take \( q = \phi_1, p = \phi_2, \) and \( a = E. \) Note that \( \phi_1 \in \mathcal{W}_2 \) and \( \phi_2 \in \mathcal{W}_3. \)

**Part 1.** Assume that there exists a matrix function \( k \in L_1^{r \times r}[-\infty, \omega] \) such that (4.6) and (4.7) hold. Put

\[ F(\lambda) = I_r - \int_{-\omega}^{\omega} e^{i\lambda t} k(t) dt, \quad \lambda \in \mathbb{C}. \quad (4.17) \]

Since \( k \) has its support in \([−\omega, \omega]\), we have \( F \in \mathcal{W}_c. \) We claim that

\[ P_{\mathcal{W}_2}(F \phi_1) \in \mathcal{W}_d, \quad P_{\mathcal{W}_3}(\phi_2 F) \in \mathcal{W}_d, \quad P_{\mathcal{W}_d}(\phi_2 F \phi_1) = E. \quad (4.18) \]

In what follows we use the convention that a function on a finite interval is considered as a function on the full line by defining it to be zero outside the given interval. This allows us to rewrite the identities (4.6) and (4.7) in the following form

\[ \varphi_1(t) - \int_{-\infty}^{\infty} k(t-s) \varphi_1(s) ds = k(t) + w_1(t), \quad t \in \mathbb{R}, \quad (4.19) \]

\[ \varphi_2(t) - \int_{-\infty}^{\infty} \varphi_2(t-s) k(s) ds = k(t) + w_2(t), \quad t \in \mathbb{R}, \quad (4.20) \]

where \( w_1(t) = 0 \) for \( 0 \leq t \leq \omega \) and \( w_2(t) = 0 \) for \( -\omega \leq t \leq 0. \) Taking Fourier transforms in (4.19) and (4.20) we obtain

\[ \hat{\varphi}_1(\lambda) - \hat{k}(\lambda) \hat{\varphi}_1(\lambda) = \hat{k}(\lambda) + \hat{w}_1(\lambda), \quad \lambda \in \mathbb{R}, \quad (4.21) \]

\[ \hat{\varphi}_2(\lambda) - \hat{\varphi}_2(\lambda) \hat{k}(\lambda) = \hat{k}(\lambda) + \hat{w}_2(\lambda), \quad \lambda \in \mathbb{R}. \quad (4.22) \]

According to (4.6), (4.7) and (4.17) we have

\[ \Phi_1 = E + \hat{\varphi}_1, \quad \Phi_2 = E + \hat{\varphi}_2, \quad F = E - \hat{k}. \]

Thus (4.21) and (4.22) yield

\[ F \Phi_1 = E + \hat{w}_1 \quad \text{and} \quad \Phi_2 F = E + \hat{w}_2. \quad (4.23) \]

Since the support of \( w_1 \) is contained in the set \((−\infty, 0] \cup [0, \infty)\) and the support of \( w_2 \) in \((−\omega, −\omega] \cup [0, \infty), \) we see that

\[ \hat{w}_1 \in \mathcal{W}^0_\omega + \mathcal{W}_1 \quad \text{and} \quad \hat{w}_2 \in \mathcal{W}^0_\omega + \mathcal{W}_4. \]

Thus \( P_{\mathcal{W}_3} \hat{w}_1 \) and \( P_{\mathcal{W}_3} \hat{w}_2 \) are both equal to zero, and the identities in (4.23) show that

\[ P_{\mathcal{W}_2}(F \Phi_1) = E \in \mathcal{W}_d \quad \text{and} \quad P_{\mathcal{W}_3}(\Phi_2 F) = E \in \mathcal{W}_d. \]

The identities in (4.23) also imply that

\[ \Phi_2 F \Phi_1 = \Phi_2 + \Phi_2 \hat{w}_1 = E + \hat{\varphi}_2 + \hat{w}_1 + \hat{\varphi}_2 \hat{w}_1. \]
\[ E + \mathcal{W}_+^0 + \mathcal{W}_+^0 + \mathcal{W}_1 + \mathcal{W}_1^0 (\mathcal{W}_1^0 + \mathcal{W}_1) \]

Hence \( P_{\mathcal{W}_2}(\Phi^* F \Phi) = E \), and (4.18) is proved.

Since \( F \in \mathcal{W}_+ \) and (4.18) holds, Propositions 2.2 and 2.3 tell us that there exist \( u_1, u_2 \in L^r_1[0, \omega] \) and \( v_1, v_2 \in L^r_1[\omega, 2\omega] \) such that the entire matrix functions \( U_1, U_2 \) and \( V_1, V_2 \) defined by (4.9) satisfy (4.10) and (4.11). This completes the proof of the direct part.

**Part 2.** In this part we assume that \( \det \Phi_1(\lambda) \) and \( \det \Phi_2(\lambda) \) have no zero on the real line. Furthermore, \( U_1, U_2 \) and \( V_1, V_2 \) are entire matrix functions as in (4.9), and these functions satisfy the identities in (4.10) and (4.11).

Since \( \det \Phi_1(\lambda) \) and \( \det \Phi_2(\lambda) \) have no zero on the real line the functions \( \Phi_1 \) and \( \Phi_2 \) are invertible in the Wiener algebra. Put

\[
F_1(\lambda) = (U_1(\tilde{\lambda})^* + V_1(\lambda)) \Phi_1(\lambda)^{-1}, \quad \lambda \in \mathbb{R},
\]

\[
F_2(\lambda) = \Phi_2(\lambda)^{-1}(U_2(\lambda) + V_2(\tilde{\lambda})^*), \quad \lambda \in \mathbb{R}.
\]

Then the argument used in the first paragraph of the proof of Proposition 2.4 shows that \( F_1 = F_2 \). Furthermore, Proposition 2.4 implies that for \( F = F_1 = F_2 \) the identities in (4.18) hold.

Since \( \Phi_1(\infty) = I_r \) and \( \Phi_2(\infty) = I_r \), the formulas for \( U_j \) and \( V_j \), \( j = 1, 2 \), in (4.9) imply that \( F(\infty) = F_1(\infty) = F_2(\infty) = I_r \). It follows that \( F(\lambda) = I_r - \ell(\lambda) \) for some \( \ell \in L^r_1(-\infty, \infty) \). Put \( k = \ell_{[-\omega, \omega]} \). Then \( k \in L^r_1[\omega, \omega] \). We claim that with this \( k \) the identities (4.6) and (4.7) hold.

Put \( F_c(\lambda) = I_r - \mathcal{K}(\lambda) \). Then \( F_c = P_{\mathcal{W}_2} F \), and we know (see Lemma 2.1) that (4.18) remains true with \( F_c \) in place of \( F \). But then, using Proposition 2.3, we see that there exist entire matrix functions \( \bar{U}_1, \bar{U}_2 \) and \( \bar{V}_1, \bar{V}_2 \) as in (4.9) satisfying the identities (4.10) and (4.11) such that

\[
F_c(\lambda) = (\bar{U}_1(\tilde{\lambda})^* + \bar{V}_1(\lambda)) \Phi_1(\lambda)^{-1}
\]

\[
= \Phi_2(\lambda)^{-1}(\bar{U}_2(\lambda) + \bar{V}_2(\tilde{\lambda})^*), \quad \lambda \in \mathbb{R}.
\]

It follows that

\[
\hat{\varphi}_1 - \hat{\kappa}\hat{\varphi}_1 - \hat{k} = (E - \mathcal{K})(E + \hat{\varphi}_1) - E
\]

\[
= F_c \Phi_1 - E = (\bar{U}_1^* - E) + \bar{V}_1 \in \mathcal{W}_+^0 + \mathcal{W}_1,
\]

\[
\hat{\varphi}_2 - \hat{\kappa}\hat{\varphi}_2 - \hat{k} = (E + \hat{\varphi}_2)(E - \mathcal{K}) - E
\]

\[
= \Phi_2 F_c - E = (\bar{U}_2 - E) + \bar{V}_2^* \in \mathcal{W}_+^0 + \mathcal{W}_4.
\]

Taking inverse Fourier transforms we see that

\[
\varphi_1(t) - \int_{-\infty}^{\infty} k(t-s)\varphi_1(s) \, ds = k(t) + w_1(t), \quad t \in \mathbb{R},
\]

\[
\varphi_2(t) - \int_{-\infty}^{\infty} \varphi_2(s)k(t-s) \, ds = k(t) + w_2(t), \quad t \in \mathbb{R},
\]

where \( w_1(t) = 0 \) on \( 0 \leq t \leq \omega \) and \( w_2(t) = 0 \) on \( -\omega \leq t \leq 0 \). Restricting (4.27) to the interval \([0, \omega]\) yields (4.6), and restricting (4.28) to the interval \([-\omega, 0]\) yields (4.7). Hence \( \Phi_1 \) and \( \Phi_2 \) are generated by \( k \). \( \square \)
Next we derive Theorem 4.1 as corollary of Theorem 4.2. For this purpose we need the following continuous time analogue of Lemma 3.3. For the proof see Part 2 of the proof of Theorem 1.2 in [10].

**Lemma 4.3.** Let $\Phi$ be defined by (4.1), where $\varphi \in L_1^{rxr}[0, \omega]$. Assume there exist $u \in L_1^{rxr}[0, \omega]$ and $v \in L_1^{rxr}[\omega, 2\omega]$ such that the entire matrix functions $U$ and $V$ in (4.3) satisfy (4.4), then $\det \Phi(\lambda) \neq 0$ for each real $\lambda$.

**Proof of Theorem 4.1.** We apply Theorem 4.2 with $\Phi = \Phi(\lambda)$ and $\varphi = \varphi(\lambda)$.

First assume that $\Phi$ is a Krešn orthogonal function generated by $k = k^* \in L_1^{rxr}[-\omega, \omega]$. Since $\varphi_1 = \varphi$, $\varphi_2 = \varphi^*$, and $k$ is hermitian, (4.2) implies that both (4.6) and (4.7) are satisfied. But then Theorem 4.2 tells us that there exist $u_1, u_2 \in L_1^{rxr}[0, \omega]$ and $v_1, v_2 \in L_1^{rxr}[\omega, 2\omega]$ such that the entire matrix functions given by (4.9) satisfy the Eqs. (4.10) and (4.11). Now put $u = u_2$, $v = v_1$, and define $U$ and $V$ by (4.3). Then (4.4) is just (4.10), and hence $U$ and $V$ satisfy (4.4).

Conversely, assume that there exist $u \in L_1^{rxr}[0, \omega]$ and $v \in L_1^{rxr}[\omega, 2\omega]$ such that the entire matrix functions defined by (4.3) satisfy (4.4). Then Lemma 4.3 tells us that $\det \Phi(\lambda) \neq 0$ for each real $\lambda$. Furthermore, with $u_1 = u_2 = u$ and $v_1 = v_2 = v$, we see that (4.10) and (4.11) are satisfied. In fact, (4.10) is just the same as (4.4), and (4.11) is the adjoint of (4.4). But then we can apply the converse part of Theorem 4.2 to show that there exists a $k \in L_1^{rxr}[-\omega, \omega]$ such that (4.6) and (4.7) hold. Furthermore, such a $k$ is obtained by $k = \ell|_{[-\omega, \omega]}$, where the function $\ell$ is given by (4.12).

In the present case (4.12) means that

$$\int_{-\infty}^{\infty} e^{i\lambda s} \ell(s) \, ds = (U(\lambda)^* + V(\lambda)) \Phi(\lambda)^{-1}$$

$$= \Phi(\lambda)^{-*}(U(\lambda) + V(\lambda)^*) \quad (\lambda \in \mathbb{R}).$$

Thus (4.5) holds and is hermitian. Since $k = \ell|_{[-\omega, \omega]}$, it follows that $k$ is also hermitian. Thus $\Phi$ is a Krešn orthogonal matrix function generated by $k$. \qed

From Section 3 in [10] we know the following result which is based on the main theorem in [8].

**Proposition 4.4.** In order that there exist $U$ and $V$ as in (4.3) satisfying (4.4) it is necessary and sufficient that for any symmetric pair of zeros $\lambda_0$, $\bar{\lambda}_0$ of $\det \Phi(\lambda)$ the following condition is satisfied:

(C) If $\varphi_0, \varphi_1, \ldots, \varphi_{p-1}$ and $\psi_0, \psi_1, \ldots, \psi_{q-1}$ are arbitrary right Jordan chains for $\Phi$ at $\lambda_0$ and $\bar{\lambda}_0$, respectively, then

$$\sum_{j=0}^{k} \psi_j^* \varphi_{k-j} = 0, \quad k = 0, 1, \ldots, \min\{p, q\} - 1. \quad (4.29)$$

Note that the condition on the symmetric pairs of zeros $\lambda_0$, $\bar{\lambda}_0$ of $\det \Phi(\lambda)$ in the above proposition implies that $\det \Phi(\lambda)$ has no real zero.

By combining the above proposition with Theorem 4.2 we obtain the following corollary, which is [10, Theorem 1.1].

**Corollary 4.5.** Let $\varphi \in L_1^{rxr}[0, \omega]$. Then the function $\Phi$ defined by (4.1) is a Krešn orthogonal function if and only if for any symmetric pair of zeros $\lambda_0$, $\bar{\lambda}_0$ of $\det \Phi(\lambda)$ condition (C) above is satisfied.
For the proof of the sufficiency part in the above proposition we refer to [10]. The scalar version (i.e., when \( r = 1 \)) of Proposition 4.4 can be derived as a corollary of Theorem 2 in [9].

Finally, we note that with appropriate modifications comments analogous to Comments 3.6 and 3.7 are valid in the context of the present section. For the analog of Comment 3.7 in the present setting the role of Theorem 4.1 in [13] has to be taken over by Theorem 1.1 in [8]. We omit further details.

5. A non-stationary example

In this section we present a non-stationary variant of Theorem 3.1, inspired by [2]. Throughout this section \( r \) is a positive integer. By \( \mathcal{N} \) we denote the set of all doubly infinite block matrices \( T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \) with matrix entries \( \tau_{j,k} \) of size \( r \times r \) and such that

\[
\sum_{j=-\infty}^{\infty} \sup_{v \in \mathbb{Z}} \| \tau_{j,j-v} \| < \infty.
\]

(5.1)

With the usual addition and multiplication of block matrices the set \( \mathcal{N} \) is an algebra. Furthermore, the map

\[
T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \mapsto T^* = [\tau_{k,j}]_{j,k=-\infty}^{\infty}
\]

defines an involution on \( \mathcal{N} \). If \( T = T^* \), then \( T \) is called selfadjoint.

Each \( T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N} \) induces a bounded linear operator on \( \ell^2(\mathbb{C}^r) \), also denoted by \( T \), via de formula

\[
T \begin{bmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix}, \quad y_j = \sum_{k=-\infty}^{\infty} \tau_{j,k}x_k.
\]

This allows us to view \( \mathcal{N} \) as a \( * \)-subalgebra of the \( C^* \)-algebra of all bounded linear operators on \( \ell^2(\mathbb{C}^r) \). Note that the identity operator \( E \) on \( \ell^2(\mathbb{C}^r) \) and the forward shift \( S \) on \( \ell^2(\mathbb{C}^r) \) are determined by block matrices, \( E = [\delta_{j,k} I_r]_{j,k=-\infty}^{\infty} \) and \( S = [\delta_{j-1,k} I_r]_{j,k=-\infty}^{\infty} \), where \( \delta \) is the Kronecker delta. Hence both \( E \) and \( S \) belong to \( \mathcal{N} \), and \( E \) is the unit of the algebra \( \mathcal{N} \).

We say that \( T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \) is lower triangular (upper triangular) if \( \tau_{j,k} = 0 \) for \( j - k < 0 \) (\( j - k > 0 \)). If \( \tau_{j,k} = 0 \) for \( |j - k| > n \), where \( n \) is a non-negative integer, then \( T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \) is called \( n \)-banded.

Now let \( T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N} \) be selfadjoint, and let \( \Phi = [\varphi_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N} \) be an \( n \)-banded lower triangular matrix. We say that \( \Phi \) is an \( n \)-th non-stationary orthogonal matrix polynomial generated by \( T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \) if

\[
\begin{bmatrix}
\tau_{k,k} & \tau_{k,k+1} & \cdots & \tau_{k,k+n} \\
\tau_{k+1,k} & \tau_{k+1,k+1} & \cdots & \tau_{k+1,k+n} \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{k+n,k} & \tau_{k+n,k+1} & \cdots & \tau_{k+n,k+n}
\end{bmatrix}
\begin{bmatrix}
\varphi_{k,k} \\
\varphi_{k+1,k} \\
\vdots \\
\varphi_{k+n,k}
\end{bmatrix}
= \begin{bmatrix}
I_r \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad k \in \mathbb{Z}.
\]

(5.2)

If the entries of \( T \) and \( \Phi \) depend only on the difference \( j - k \) of the indices \( j, k \), then the Eqs. (5.2) reduce to one equation, namely (3.1).
Theorem 5.1. Let $\Phi = [\varphi_{j,k}]_{j,k=-\infty}^{\infty}$ be $n$-banded and lower triangular. If $\Phi$ is an $n$-th non-stationary orthogonal matrix, then $\Phi_{k,k}$ is hermitian for each $k \in \mathbb{Z}$, and there exist lower triangular matrices $U$ and $W$ in $\mathcal{N}$ satisfying the equation

$$U^* \Phi - \Phi^* S^{n+1} W = [\delta_{j,k} \varphi_{k,k}]_{j,k=-\infty}^{\infty}. \quad (5.3)$$

Conversely, if $\Phi$ is invertible in $\mathcal{N}$ and there exist lower triangular matrices $U$ and $W$ in $\mathcal{N}$ satisfying the Eq. (5.3), then $\Phi$ is an $n$-th non-stationary orthogonal matrix polynomial and a selfadjoint $T = [\tau_{j,k}]_{j,k=-\infty}^{\infty}$ in $\mathcal{N}$ generating $\Phi$ is given by

$$T = (U^* + S^{n+1} W) \Phi^{-1}. \quad (5.4)$$

Proof. We shall sketch, without going into details, how to obtain the above result as a corollary of Theorem 1.1. The first step is to introduce a band structure on $\mathcal{N}$. Fix a positive integer $n$. Put

$$\mathcal{N}_1 = \{ T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N} \mid \tau_{j,k} = 0 \text{ for } j - k \leq n \},$$

$$\mathcal{N}_2^0 = \{ T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N} \mid \tau_{j,k} = 0 \text{ for } j - k \leq 0 \text{ or } j - k \geq n + 1 \},$$

$$\mathcal{N}_d = \{ T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N} \mid \tau_{j,k} = 0 \text{ for } j \neq k \},$$

$$\mathcal{N}_3 = \{ T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N} \mid \tau_{j,k} = 0 \text{ for } j - k \geq 0 \text{ or } j - k \leq -n - 1 \},$$

$$\mathcal{N}_4 = \{ T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N} \mid \tau_{j,k} = 0 \text{ for } j - k \geq -n \}.$$ 

Then

$$\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2^0 + \mathcal{N}_d + \mathcal{N}_3 + \mathcal{N}_4,$$

and this direct sum decomposition is a band structure on $\mathcal{N}$. Note that $\mathcal{N}_c = \mathcal{N}_2^0 + \mathcal{N}_d + \mathcal{N}_3$ is precisely the space of $n$-banded matrices in $\mathcal{N}$. Furthermore, $\mathcal{N}_+ := \mathcal{N}_1 + \mathcal{N}_2^0 + \mathcal{N}_d$ and $\mathcal{N}_- := \mathcal{N}_d + \mathcal{N}_3 + \mathcal{N}_4$ are respectively equal to the sets of all lower triangular and upper triangular matrices in $\mathcal{N}$. Next, note that the set $\mathcal{N}_2 := \mathcal{N}_2^0 + \mathcal{N}_d$ is equal to the space of all $n$-banded lower triangular matrices in $\mathcal{N}$, and $\mathcal{N}_1 = S^{n+1} \mathcal{N}_+$. The following lemma puts (5.2) into the band context; the proof is straightforward.

Lemma 5.2. Let $T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N}$, and let $\Phi = [\varphi_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N}$ be an $n$-banded lower triangular matrix. Then the equations in (5.2) are satisfied if and only if $P_{\mathcal{N}_2}(T \Phi) = E$. If, in addition, $T$ is selfadjoint, then (5.2) implies that

$$[\delta_{j,k} \varphi_{k,k}^*]_{j,k=-\infty}^{\infty} = P_{\mathcal{N}_d} \left( \Phi^* (T \Phi) \right) = P_{\mathcal{N}_d} \left( (\Phi^* T) \Phi \right) = [\delta_{j,k} \varphi_{k,k}]_{j,k=-\infty}^{\infty}.$$

Hence the matrix $\varphi_{k,k}$ is hermitian for each $k$.

Given the above lemma, Theorem 5.1 follows by applying the first part of Theorem 1.1. We omit further details. □

Next we further specify Theorem 5.1 for the case when $T = [\tau_{j,k}]_{j,k=-\infty}^{\infty}$ has the property that the number of pairs $\{ j, k \}$ such that $\tau_{j,k} \neq 0$ and $j \neq k$ is finite. The set of all such $T$ will denoted by $\mathcal{F}$. Clearly, $\mathcal{F} \subset \mathcal{N}$. Furthermore, $T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{N}$ belongs to $\mathcal{F}$ if and only if $T$ is banded and each diagonal of $T$, except for the main diagonal, has only a finite number of non-zero entries. The set $\mathcal{F}$ is a subalgebra of $\mathcal{N}$, $\mathcal{F}^* = \mathcal{F}$, the identity $E$ belongs to $\mathcal{F}$, and $\mathcal{F}$ is inverse closed, that is, if $T \in \mathcal{F}$ is invertible as an operator on $\ell^2(\mathbb{C}^n)$, then $T^{-1}$ also belongs...
to $\mathcal{F}$. The following proposition describes non-stationary orthogonal matrices generated by an operator $T \in \mathcal{F}$.

**Proposition 5.3.** Assume that $\Phi = [\varphi_{j,k}]_{j,k=-\infty}^{\infty}$ is $n$-banded and lower triangular. If $\Phi$ is an $n$-th non-stationary orthogonal matrix polynomial generated by an operator $T \in \mathcal{F}$, then $\Phi \in \mathcal{F}$ and for each $k \in \mathbb{Z}$ the diagonal entry $\varphi_{k,k}$ is hermitian. Conversely, if $\Phi \in \mathcal{F}$ and for each $k \in \mathbb{Z}$ the diagonal entry $\varphi_{k,k}$ is hermitian and non-singular, then $\Phi$ is invertible in $\mathcal{F}$, the inverse $\Phi^{-1}$ is block lower triangular, and $\Phi$ is a non-stationary orthogonal matrix polynomial generated by $T = \Phi^{-*} \Delta \Phi^{-1}$. Here $\Delta$ is the block diagonal matrix given by

$$
\Delta = [\delta_{j,k}\varphi_{k,k}]_{j,k=-\infty}^{\infty}, \quad \text{where } \delta_{j,k} \text{ is the Kronecker delta}.
$$

**Proof.** We split the proof into two parts. In the first part we show that an $n$-th non-stationary orthogonal matrix polynomial generated by an operator $T \in \mathcal{F}$ belongs to $\mathcal{F}$. The second part deals with the converse statement.

**Part 1.** Let $T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{F}$, and assume that $\Phi = [\varphi_{j,k}]_{j,k=-\infty}^{\infty}$ is an $n$-th non-stationary orthogonal matrix polynomial generated by $T$. Thus the identities in (5.2) are satisfied. Since $T \in \mathcal{F}$, there exists a positive integer $N$ such that $\tau_{j,k} = 0$ whenever $j \neq k$ and $|k| > N$.

Now fix $k > N$. Then $k+i > N$ for $i = 0, 1, \ldots, n$. It follows that for all $j \in \mathbb{Z}$ the matrices $\tau_{j,k}, \tau_{j,k+1}, \ldots, \tau_{j,k+n}$ are zero with the matrices $\tau_{k,k}, \tau_{k+1,k+1}, \ldots, \tau_{k+n,k+n}$ excluded. This implies that the square matrix in the left hand side of (5.2) is a diagonal matrix and the identities in (5.2) reduce to

$$
\tau_{k,k} \Phi_{k,k} = I_r, \quad \tau_{k+i,k+i} \Phi_{k+i,k} = 0 \quad (i = 1, \ldots, n).
$$

The first identity in (5.6) shows that $\tau_{k,k}$ is non-singular. Recall that $k$ is any integer strictly larger than $N$. Thus the preceding result also applies to the matrices $\tau_{k+i,k+i}, i = 1, \ldots, n$, that is, all these matrices are non-singular too. But then the right hand side of (5.6) implies that $\Phi_{k+i,k} = 0$ for $i = 1, \ldots, n$. Since $\Phi$ is $n$-banded and lower triangular, we conclude that

$$
\Phi_{j,k} = 0 \quad \text{for } k > N \text{ and all } j \in \mathbb{Z}, j \neq k.
$$

Next, take $k < -N - n$. Then $k+i < -N$ for $i = 0, 1, \ldots, n$. This implies that for all $j \in \mathbb{Z}$ the matrices $\tau_{j,k}, \tau_{j,k+1}, \ldots, \tau_{j,k+n}$ are zero with the matrices $\tau_{k,k}, \tau_{k+1,k+1}, \ldots, \tau_{k+n,k+n}$ excluded. As in the previous paragraph the identities in (5.2) reduce to those in (5.6), and we conclude that $\tau_{k,k}$ is non-singular for each $k < -N - n$.

We proceed with $k < -N - 2n$, then $k, k+1, \ldots, k+n$ are all strictly less than $-N - n$, and the result of previous paragraph implies that the matrices $\tau_{k+i,k+i}, i = 1, \ldots, n$, are again all non-singular. On the other hand, since $k < N - n$, the previous paragraph also tells us that the identities in the right hand side of (5.6) are satisfied. It follows that $\Phi_{k+i,k} = 0$ for $i = 1, \ldots, n$. Since $\Phi$ is $n$-banded and lower triangular, we conclude that

$$
\Phi_{j,k} = 0 \quad \text{for } k < -N - 2n \text{ and for all } j \in \mathbb{Z}, j \neq k.
$$

Using that $\Phi$ is $n$-banded, and statements (5.7) and (5.8), we see that $\Phi \in \mathcal{F}$. From Theorem 5.1 we know that the diagonal entries $\varphi_{k,k}, k \in \mathbb{Z}$, are hermitian.

**Part 2.** Let $\Phi \in \mathcal{F}$, and assume that for each $k \in \mathbb{Z}$ the diagonal entry $\varphi_{k,k}$ is hermitian and non-singular. Since $\Phi \in \mathcal{F}$, there exists a positive integer $N$ such that $\varphi_{j,k} = 0$ for $|j| > N$ and/or $|k| > N$ with $j = k$ excluded. Using this positive integer $N$ we write $\ell^2(\mathbb{C}^r)$ as a Hilbert
spaces direct sum, $\ell^2(\mathbb{C}^r) = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+$, where

\[
\mathcal{H}_- = \{ x = \text{col} [x_j]_{j \in \mathbb{Z}} \in \ell^2(\mathbb{C}^r) \mid x_j = 0 \text{ for } j \geq -N \}, \\
\mathcal{H}_0 = \{ x = \text{col} [x_j]_{j \in \mathbb{Z}} \in \ell^2(\mathbb{C}^r) \mid x_j = 0 \text{ for } j < -N \text{ or } j > N \}, \\
\mathcal{H}_+ = \{ x = \text{col} [x_j]_{j \in \mathbb{Z}} \in \ell^2(\mathbb{C}^r) \mid x_j = 0 \text{ for } j \leq N \}.
\]

Note that $\mathcal{H}_0$ is finite dimensional. Since $\varphi_{j,k} = 0$ for $|j| > N$ and/or $|k| > N$ with $j = k$ excluded, we can write $\Phi$ as a $3 \times 3$ diagonal operator matrix $\Phi = \text{diag} \{ \Phi_-, \Phi_0, \Phi_+ \}$ relative to the above Hilbert space direct sum. Moreover, using the standard basis of $\ell^2(\mathbb{C}^r)$ one sees that $\Phi_-$ and $\Phi_+$ are represented by block diagonal matrices with matrices $\varphi_{k,k}$ as diagonal entries. Thus our hypotheses imply that both $\Phi_-$ and $\Phi_+$ are invertible and their inverses are again block diagonal. Recall that $\Phi$ is block lower triangular. It follows that with respect to the standard basis of $\ell^2(\mathbb{C}^r)$ the operator $\Phi_0$ is block lower triangular with invertible matrices $\varphi_{k,k}$ as diagonal entries, where $-N \leq k \leq N$. Using the finite dimensionality of $\mathcal{H}_0$ and the invertibility of the diagonal entries $\varphi_{k,k}$, we conclude that $\Phi_0$ is invertible and that $\Phi_0^{-1}$ is again block lower triangular. Since $\Phi = \text{diag} \{ \Phi_-, \Phi_0, \Phi_+ \}$, we see that $\Phi$ is invertible, and $\Phi^{-1} = \text{diag} \{ \Phi_-^{-1}, \Phi_0^{-1}, \Phi_+^{-1} \}$. This implies that $\Phi^{-1}$ is in $\mathcal{F}$ and that $\Phi^{-1}$ is block lower triangular. In particular, $\Phi$ is invertible in $\mathcal{F}$.

Next, let $T = \Phi^{-\ast} \Delta \Phi^{-1}$, where $\Delta$ is the block diagonal matrix defined by (5.5). Since the entries $\varphi_{k,k}$ are hermitian, $\Delta$ is selfadjoint, and hence the same holds true for $T$. Note that $T \Phi = \Phi^{-\ast} \Delta$. Since $\Phi^{-\ast}$ is block upper triangular and the diagonal entries of $\Phi^{-\ast}$ are equal to $\varphi_{k,k}^{-\ast} = \varphi_{k,k}^{-1}$ for $k \in \mathbb{Z}$, we see that $\Phi^{-\ast} \Delta$ is block upper triangular and its diagonal entries are all equal to $I_r$. Thus $P_{N_k}(T \Phi) = P_{N_k}(\Phi^{-\ast} \Delta) = E$. This shows that $\Phi$ is a non-stationary orthogonal matrix polynomial generated by $T$. \hfill \Box

We conclude this section with the non-symmetric version of Proposition 5.3. For this purpose we need the following equations:

\[
\begin{bmatrix}
\psi_{k,k} & \psi_{k,k+1} & \cdots & \psi_{k,k+n}
\end{bmatrix}
\begin{bmatrix}
\tau_{k,k} & \tau_{k,k+1} & \cdots & \tau_{k,k+n} \\
\tau_{k+1,k} & \tau_{k+1,k+1} & \cdots & \tau_{k+1,k+n} \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{k+n,k} & \tau_{k+n,k+1} & \cdots & \tau_{k+n,k+n}
\end{bmatrix}
= \begin{bmatrix} I_r & 0 & \cdots & 0 \end{bmatrix}, \quad k \in \mathbb{Z}.
\]

The following result is the desired non-symmetric version of Proposition 5.3.

**Proposition 5.4.** Let $\Phi = [\varphi_{j,k}]_{j,k=-\infty}^{\infty}$ and $\Psi = [\psi_{j,k}]_{j,k=-\infty}^{\infty}$ be $n$-banded $r \times r$ block matrices, with $\Phi$ lower triangular and $\Psi$ upper triangular. Assume that there exists $T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{F}$ such that the identities in (5.2) and (5.9) are satisfied. Then $\Phi$ and $\Psi$ belong to $\mathcal{F}$ and $\varphi_{k,k} = \psi_{k,k}$. Conversely, if $\Phi$ and $\Psi$ belong to $\mathcal{F}$ and for each $k \in \mathbb{Z}$ we have $\varphi_{k,k} = \psi_{k,k}$ is non-singular, then $\Phi$ and $\Psi$ are invertible in $\mathcal{F}$, the inverses $\Phi^{-1}$ and $\Psi^{-1}$ are (block) lower triangular and (block) upper triangular, respectively, and there exists $T = [\tau_{j,k}]_{j,k=-\infty}^{\infty} \in \mathcal{F}$ such that the identities in (5.2) and (5.9) are satisfied. Furthermore, one such $T$ is given by $T = \Psi^{-1} \Delta \Phi^{-1}$. Here $\Delta$ is the block diagonal matrix given by

\[
\Delta = [\delta_{j,k} \varphi_{j,k}]_{j,k=-\infty}^{\infty}, \quad \text{where } \delta_{j,k} \text{ is the Kronecker delta.}
\]
In the proof of Proposition 5.3 the fact that $T$ is self-adjoint plays a minor role, and with some appropriate modifications the proof of Proposition 5.3 also yields the above result. We omit the details.

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References


