Resolvable 4-cycle group divisible designs with two associate classes: Part size even

Elizabeth J. Billington\textsuperscript{a}, C.A. Rodger\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, The University of Queensland, Brisbane, Qld. 4072, Australia
\textsuperscript{b}Department of Mathematics and Statistics, 221 Parker Hall, Auburn University, Auburn, AL 36849-5310, USA

Received 15 December 2004; received in revised form 10 October 2005; accepted 27 November 2006

Available online 2 June 2007

Abstract

Let $\lambda_1K_a$ denote the graph on $a$ vertices with $\lambda_1$ edges between every pair of vertices. Take $p$ copies of this graph $\lambda_1K_a$, and join each pair of vertices in different copies with $\lambda_2$ edges. The resulting graph is denoted by $K(a,p;\lambda_1,\lambda_2)$, a graph that was of particular interest to Bose and Shimamoto in their study of group divisible designs with two associate classes. The existence of $z$-cycle decompositions of this graph have been found when $z \in \{3,4\}$. In this paper we consider resolvable decompositions, finding necessary and sufficient conditions for a 4-cycle factorization of $K(a,p;\lambda_1,\lambda_2)$ (when $\lambda_1$ is even) or of $K(a,p;\lambda_1,\lambda_2)$ minus a 1-factor (when $\lambda_1$ is odd) whenever $a$ is even.

© 2007 Elsevier B.V. All rights reserved.

Keywords: Resolvable; Cycle systems; Associate classes; 2-factorizations

1. Introduction

In this paper, graphs usually contain multiple edges. In particular, if $G$ is a simple graph, then let $\lambda G$ denote the multigraph formed by replacing each edge in $G$ with $\lambda$ edges. Let $C_2$ denote a cycle of length 2.

For any two vertex-disjoint graphs $G$ and $H$, define $G \vee H$ to be the graph formed from the union of $G$ and $H$ by joining each vertex in $G$ to each vertex in $H$ with exactly 2 edges; if $\lambda = 1$ then this may be represented by simply $G \vee H$. Let $K(a,p;\lambda_1,\lambda_2)$ denote the graph formed from $p$ vertex-disjoint copies of the multigraph $\lambda_1K_a$ by joining each pair of vertices in different copies of $K_a$ with $\lambda_2$ edges. It will be useful to refer to edges in $K(a,p;\lambda_1,\lambda_2)$ that join vertices in different copies of $\lambda_1K_a$ as being mixed edges, and edges joining vertices in the same copy of $\lambda_1K_a$ as being pure edges. The vertex sets of the $p$ copies of $\lambda_1K_a$ will usually be $V \times \{i\}$ with $0 \leq i < p$ for some set $V$, where $|V| = a$. Then for each pair $\{u,v\} \subseteq V$, with $0 \leq i < j < p$, it will be useful to refer to the edges $\{(u,i),(v,j)\}$ and $\{(u,j),(v,i)\}$ as being the corresponding mixed edges to the edge $\{u,v\}$ in $K_a$. Also, for each $u \in V$ we will refer to the edges $\{(u,i),(u,j)\}$ as being the horizontal mixed edges.

An $H$-decomposition of a graph $G$ is an ordered pair $(V,C)$, where $V$ is the vertex set of $G$ and $C$ is a set of copies of $H$ such that each edge in $G$ occurs in exactly one graph in $C$. When the actual vertex set $V$ is of no interest, it will cause no confusion to refer to the decomposition by simply $C$. There has been considerable interest over the past 20 years in $H$-decompositions of various graphs, such as complete graphs and complete multipartite graphs, especially when $H$ is a...
cycle (see [1,3,8,10,11], for example). More recently, the existence problem for $C_z$-decompositions of $K(a, p; \lambda_1, \lambda_2)$ for $z = 3$ [4,6] and for $z = 4$ [5] has been solved. Such decompositions are known as $C_z$ group divisible designs with two associate classes, following the notation of Bose and Shimamoto who considered the existence problem for $K_z$ group divisible designs [2]. (The reason for this name is that the structure can be thought of as partitioning $ap$ symbols (i.e. vertices) into $p$ sets of size $a$ in such a way that symbols that are in the same set in the partition occur together in $\lambda_1$ blocks, and are known as first associates, whereas symbols that are in different sets in the partition occur together in $\lambda_2$ blocks, and are known as second associates.)

In an $H$-decomposition $(V, C)$ of a graph $G$, a parallel class is a subset $S$ of $C$ such that each vertex in $V$ occurs in exactly one copy of $H$ in $S$. The decomposition $(V, C)$ is said to be resolvable if $C$ can be partitioned into parallel classes. In this paper we completely settle the existence problem for resolvable $C_4$-decompositions of $K(a, p; \lambda_1, \lambda_2)$, or of $K(a, p; \lambda_1, \lambda_2)$ minus a 1-factor, when $a$ is even. A resolvable $C_4$-decomposition is also known as a $C_4$-factorization, and a parallel class in a resolvable $C_4$-decomposition is also known as a $C_4$-factor.

Let $G[V]$ denote the subgraph of $G$ induced by the vertex set $V$.

### 2. Some preliminary results

A near $C_4$-factor of $G$ is a spanning subgraph of $G$ in which one component is $K_2$ and all others are $C_4$. A partition of $E(G)$ in which each element induces a near $C_4$-factor is called a near $C_4$-factorization of $G$. We can easily obtain the following result, which is of some interest in its own right.

**Theorem 1.** For all $n \geq 1$ there exists a near $C_4$-factorization of $K_{4n+2}$.

**Proof.** Let $V(K_{4n+2}) = \mathbb{Z}_{2n+1} \times \mathbb{Z}_2$. Then

$$B = \bigcup_{i=0}^{2n} \{(i, 0), (i, 1), (j + i, 0), (j + i + 1, 1), (j + 1, 0), (-j + i, 0)|1 \leq j \leq n\}$$

provides the required resolvable decomposition. □

In any near $C_4$-factorization $S$ of $K_{4n+2}$, it is clear that the set $F(S)$ of copies of $K_2$ forms a 1-factor of $K_{4n+2}$. Let $\{F_0, F_1, \ldots, F_{4n}\}$ be a 1-factorization of $K_{4n+2}$. By simply renaming vertices in $S$, it is clear that for $0 \leq i \leq 4n$ we can form a near $C_4$-factorization $S_i$ of $K_{4n+2}$ in which $F(S_i) = F_i$.

This observation is especially useful here for the following reason. Let $a = 4n + 2$. For $0 \leq i \leq 4n$ and for $j = 0, 1$, let $\mathcal{F}_j(S_i)$ be the set of 4-cycles formed by deleting $F_i$ from $S_i$ and renaming each vertex $u$ in each 4-cycle with the vertex $(u, j)$. Then

$$\mathcal{F}(S_i) = \mathcal{F}_0(S_i) \cup \mathcal{F}_1(S_i) \cup \{(x, 0), (y, 1), (x, 1), (y, 0)|x, y \in E(F_i)\}$$

is a $C_4$-factorization of the graph formed by joining two copies of $K_a$ with the 1-factor formed by the mixed edges corresponding to the edges in $F_i$. Also note then that $\bigcup_{i=0}^{4n} \mathcal{F}(S_i)$ is a $C_4$-factorization of $(4n + 1)K_a \cup (4n + 1)K_a - \{(u, 0), (u, 1)|u \in V\}$, where $V = V(K_a)$. So we can easily get the following result.

**Lemma 1.** Let $a = 4n + 2$. There exists a $C_4$-factorization of $aK_a \cup aK_a$.

**Proof.** Extending the notation developed in the previous paragraph, define

$$\mathcal{F}'(S_i) = \mathcal{F}_0(S_i) \cup \mathcal{F}_1(S_i) \cup \{(x, 0), (x, 1), (y, 1), (y, 0)|x, y \in E(F_i)\}$$

Then $\bigcup_{i=0}^{4n} \mathcal{F}'(S_i) \cup \mathcal{F}(S_0)$ is a $C_4$-factorization of $aK_a \cup aK_a$. □

We can also obtain Lemma 3 below with a much more sophisticated use of this observation together with the following extremely useful result, essentially proved by Stern and Lenz (see [12] and [9] p.158)).
Lemma 2. Let \( G \) be a regular graph. Suppose there exists a partition of the vertex set of \( G \) into two sets of equal size, \( V_1 \) and \( V_2 \) such that

- \( G_1 = G[V_1] \) and \( G_2 = G[V_2] \) are isomorphic simple regular graphs, and
- there exists an isomorphism \( f \) from \( G_1 \) to \( G_2 \) such that \( \{v, f(v)\} \) is an edge in \( G \) for each vertex \( v \) in \( G_1 \).

Then there exists a 1-factorization of \( G \).

Lemma 3. Let \( a = 4n + 2 \). Let \( 1 \leq r < a \). There exists a \( C_4 \)-factorization of \( G = rK_a \cup \cdots \cup rK_a - E(F) \), where \( F \) is a 1-factor of \( G \) when \( r \) is odd and \( F \) has no edges when \( r \) is even.

Proof. Let the vertex set of \( rK_a \cup \cdots \cup rK_a \) be \( \mathbb{Z}_{2n+1} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \). Let the edges joining vertices with different third coordinates be known as mixed edges, and all other edges be known as pure edges. Let \( r = 4s + \varepsilon \) where \( \varepsilon \in \{0, 1, 2, 3\} \).

Let \( \{H_1, \ldots, H_\beta\} \) be a Hamilton decomposition of \( K_{2n+1} \) on the vertex set \( \mathbb{Z}_{2n+1} \). For \( 1 \leq j \leq n-s-[(\varepsilon-2)/4]=\nu \), if \( H_j = (v_0, v_1, \ldots, v_{2n}) \) then define

\[
C_{2j-1} = \{((v_{\ell}, 0, 0), (v_{\ell+1}, 0, 1), (v_\ell, 1, 0), (v_{\ell+1}, 1, 1))|0 \leq \ell \leq 2n\}
\]

and

\[
C_{2j} = \{((v_{\ell+1}, 0, 0), (v_\ell, 0, 1), (v_{\ell+1}, 1, 0), (v_\ell, 1, 1))|0 \leq \ell \leq 2n\}.
\]

Then each of \( C_1, C_2, \ldots, C_{2\nu} \) is a \( C_4 \)-factor of \( K_{2n+1} \) which contains only mixed edges; in fact, the mixed edges in \( C_{2j-1} \cup C_{2j} \) are precisely the mixed edges corresponding to the edges in \( \{((u, 0), (v, 0)), ((u, 0), (v, 1)), ((u, 1), (v, 0)), ((u, 1), (v, 1))\} \times \{v, u\} \in E(H_j) \).

Now let \( G \) be the graph with vertex set \( \mathbb{Z}_{2n+1} \times \mathbb{Z}_2 \) and edge set

\[
E(G) = \{((u, 0), (v, 0)), ((u, 0), (v, 1)), ((u, 1), (v, 0)), ((u, 1), (v, 1))\} \cup \{((u, 0), (u, 1))|u \in \mathbb{Z}_{2n+1}\}
\]

Then, by the definition of \( \nu \), \( G \) is a \( \beta \)-regular graph, where

\[
\beta = \begin{cases} 
4s + 1 & \text{if } \varepsilon \leq 2 \\
4s + 5 & \text{if } \varepsilon = 3
\end{cases}
\]

Furthermore, the function \( f((u, 0)) = (u, 1) \) is clearly an isomorphism from \( G[\mathbb{Z}_{2n+1} \times \{0\}] \) to \( G[\mathbb{Z}_{2n+1} \times \{1\}] \). Therefore, by Lemma 2, there exists a 1-factorization \( \{F_1, \ldots, F_{\beta}\} \) of \( G \). For \( 1 \leq i \leq \min[\beta, r] \) let \( S_i \) be a new \( C_4 \)-factorization of \( K_{4n+2} \) on the vertex set \( \mathbb{Z}_{2n+1} \times \mathbb{Z}_2 \) in which \( F(S_i) = F_i \). Now use this to form \( \mathcal{F} (S_i) \), a \( C_4 \)-factorization of the graph formed by joining two copies of \( K_{4n+2} \) (on the vertex set \( \mathbb{Z}_{2n+1} \times \mathbb{Z}_2 \)) by the 1-factor consisting of the mixed edges corresponding to \( F_i \). Let \( B = (\bigcup_{1 \leq j \leq 2n} C_j) \cup (\bigcup_{1 \leq i \leq \min[\beta, r]} \mathcal{F} (S_i)) \).

The only case where \( \beta < r \) is when \( r = 4s+2 \), so \( \varepsilon = 2 \) and so \( \beta = 4s+1 \). So cycles in \( B \) cover each pure edge exactly \( r \) times except when \( \varepsilon = 2 \); in this exceptional case each pure edge is covered exactly \( r - 1 \) times. All mixed edges are covered by cycles in \( B \) exactly once except for the mixed edges corresponding to edges in \( \{F_{\min[\beta, r]+1}, \ldots, F_{\beta}\} \) and except for the horizontal mixed edges.

If \( \varepsilon \neq 2 \), now consider the use of the mixed edges corresponding to the edges in \( F_{r+1}, \ldots, F_{\beta} \). If \( \varepsilon = 0 \) then let \( D \) be the \( C_4 \)-factor formed by the corresponding mixed edges to those in \( F_{\beta} = F_{r+1} \) together with the horizontal mixed edges. Then \( B \cup D \) is a \( C_4 \)-factorization of \( rK_a \cup \cdots \cup rK_a \). If \( \varepsilon = 1 \) then \( B \) is a \( C_4 \)-factorization of the graph formed from \( rK_a \cup \cdots \cup rK_a \) by removing the edges in the 1-factor consisting of the horizontal edges. If \( \varepsilon = 3 \) then let \( D \) be the \( C_4 \)-factor formed by the corresponding mixed edges to those in \( F_{\beta} \) together with the horizontal mixed edges. Then \( B \cup D \) is a \( C_4 \)-factorization of the graph formed from \( rK_a \cup \cdots \cup rK_a \) by removing the edges in the 1-factor consisting of the corresponding mixed edges to those in \( F_{\beta-1} \).

Finally, consider the case where \( \varepsilon = 2 \). As stated above, all pure edges still need to be used once, as do the horizontal mixed edges. But, as in the proof of Lemma 1, \( \mathcal{F} (S_{\beta}) \) covers precisely these edges. So \( B \cup \mathcal{F} (S_{\beta}) \) is a \( C_4 \)-factorization of \( rK_a \cup \cdots \cup rK_a \).
There is one glaring omission in what we have proved so far, namely the case in Lemma 3 where \( r = 0 \), so there are no pure edges; but that is easily handled by the following lemma.

**Lemma 4.** Let \( a \) be even. There exists a \( C_4 \)-factorization of the complete bipartite graph \( K_{a,a} \).

**Proof.** Let \( V \) be partitioned into parts in \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). Define the parallel class \( C_i = \{(2j,0),(2j+2i,1),(2j+1,0),(2j+2i+1,1)\} \) \( j \in \mathbb{Z}_{a/2} \) for each \( i \in \mathbb{Z}_{a/2} \). Then \( \{C_1, \ldots, C_{a/2}\} \) is the required decomposition. \( \square \)

Finally, we easily handle the case where there are no mixed edges.

**Lemma 5.** Let \( a = 4n \). There exists a \( C_4 \)-factorization of \( \lambda_1 K_a - E(F) \), where \( F \) is a 1-factor when \( \lambda_1 \) is odd if and only if \( F \) has no edges when \( \lambda_1 \) is even.

**Proof.** Let \( K_a \) have vertex set \( \mathbb{Z}_{2n} \times \mathbb{Z}_2 \). Let \( \{F_0, \ldots, F_{2n-2}\} \) be a 1-factorization of \( K_{2n} \) on the vertex set \( \mathbb{Z}_{2n} \). Define \( C(i) = \{(u,0),(v,1),(u,1),(v,0)\} \) \( i \in \mathbb{Z}_{2n-1} \). Then \( C(i) \) is a parallel class in \( K_a \), and \( C = \bigcup_{i \in \mathbb{Z}_{2n-1}} C(i) \) is a \( C_4 \)-factorization of \( K_a - F \).

Define \( C'(0) = \{(u,0),(u,1),(v,1),(v,0)\} \) \( (u,0),(v,1),(v,0),(u,1)\} \) \( i \in \mathbb{Z}_{2n} \). Then

\[
\lambda_1 \bigcup_{i=1}^{2n-2} C(i) \cup \lfloor \lambda_1/2 \rfloor C(0) \cup \lfloor \lambda_1/2 \rfloor C'(0)
\]

is a \( C_4 \)-factorization of \( \lambda_1 K_a - F \), where \( F \) is the 1-factor \( \{(u,0),(u,1)\} u \in \mathbb{Z}_{2n} \) if \( \lambda_1 \) is odd, and \( F \) is empty if \( \lambda_1 \) is even. \( \square \)

3. The main result

**Theorem 2.** Let \( \lambda_1, \lambda_2 \geq 1, a \) be even and \( p \geq 2 \). Let \( G = K(a,p; \lambda_1,\lambda_2) \).

There exists a \( C_4 \)-factorization of \( G \) (of \( G - F \), where \( F \) is a 1-factor of \( G \)) if and only if

1. \( 4 \) divides \( ap \);
2. \( \lambda_1 \) is even (for \( G \)); \( \lambda_1 \) is odd (for \( G - F \));
3. if \( a \equiv 2 \pmod{4} \) then \( \lambda_2 a(p-1) \geq \lambda_1 \), unless \( a = 2 \) and \( \lambda_1 \) is odd, in which case \( \lambda_2 a(p-1) \geq \lambda_1 - 1 \).

**Proof.** Let the vertex set \( V \) of \( G \) be partitioned into parts \( V_1, \ldots, V_p \), each of size \( a \). We begin by proving the necessity of conditions (1)–(3).

Condition (1) follows because each parallel class naturally induces a partition of \( V \) into sets of size 4. Condition (2) holds because each vertex has degree \( \lambda_1(a-1) + \lambda_2 a(p-1) \) in \( G \), which must clearly be even for a \( C_4 \)-factorization of \( G \), and odd for a \( C_4 \)-factorization of \( G - F \). Notice that conditions (1)–(2) imply that \( \lambda_1 a(p-1)/2 + \lambda_2 a^2 p(p-1)/2 \), the number of edges in \( G \), is divisible by 4. Similarly, conditions (1)–(2) imply that the number of edges in \( G - F \), namely, \( \lambda_1 (a-1)/2 + \lambda_2 a^2 p(p-1)/2 \), is also divisible by 4.

To see that condition (3) is necessary, note that when \( a \equiv 2 \pmod{4} \), \( p \) must be even so that the number of vertices in each parallel class is divisible by 4. Also, each parallel class \( P \) must contain at least two vertices in each part that are incident with a mixed edge in \( P \), so \( P \) must contain at least \( p \) mixed edges. Since each parallel class contains \( ap \) edges altogether, the number of parallel classes in \( G \) is \( \lambda_1 (a-1)/2 + \lambda_2 a(p-1)/2 \), the total number of edges divided by \( ap \). Similarly, the number of parallel classes in \( G - F \) is \( \lambda_1 (a-1)/2 + \lambda_2 a(p-1)/2 \). So the number of mixed edges in \( G \) satisfies

\[
\lambda_2 a^2 p(p-1)/2 \geq \lambda_1 (a-1)/2 + \lambda_2 a(p-1)/2 \quad \text{so}
\]

\[
\lambda_2 a(a-1)(p-1) \geq \lambda_1 (a-1) \quad \text{and so}
\]

\[
\lambda_2 a(p-1) \geq \lambda_1.
\]
Similarly, the number of mixed edges in \( G - F \) satisfies \( \lambda_2 a (p - 1) \geq \lambda_1 - 1 / (a - 1) \). Since the left-hand side of this inequality is an integer, this inequality is the same as the one for \( G \) unless \( a = 2 \), so condition (3) is necessary.

To prove the sufficiency of conditions (1)–(3), we begin by assuming that \( a \equiv 0 \pmod{4} \). Let \( a = 4n \). Then by Lemma 5 there exists a \( C_4 \)-factorization \( C(i) \) of \( \lambda_1 K_a \) (if \( \lambda_1 \) is even) or of \( \lambda_1 K_a - E(F) \) where \( F \) is a 1-factor of \( K_a \) (if \( \lambda_1 \) is odd) on the vertex set \( V_i \) for \( a \leq i \leq p \). So clearly \( \bigcup_{1 \leq i \leq a} C_i \) can be used to form a set \( C \) of parallel classes of \( G \) (or \( G - F \)) which uses each pure edge \( \lambda_1 \) times.

By [7] there exists a 1-factorization \( \{ F_1, \ldots, F_p \} \) of the complete multipartite graph with \( p \) parts, each of size \( 2n \) (so \( z = 2n(p - 1) \)). Then \( C(F_k) = \{(u, v, u + 2n, v + 2n)|\{u, v\} \in F_k\} \) is a \( C_4 \)-factor of \( G \) that contains only mixed edges. Furthermore, the multiset \( C' \) consisting of \( \lambda_2 \) copies of \( \bigcup_{1 \leq k \leq p} C(F_k) \) is a \( C_4 \)-factorization of the complete multipartite graph with \( p \) parts and \( 4n \) vertices in each part. Then \( C \cup C' \) is a \( C_4 \)-factorization of \( G \) (or of \( G - F \)).

Next suppose that \( a \equiv 2 \pmod{4} \). Let \( a = 4n + 2 \). Then by (1), \( p \) is even, so let \( \{ F_1, \ldots, F_{p-1} \} \) be a 1-factorization of \( K_p \) on the vertex set \( \{1, \ldots, p\} \). If \( \lambda_1 \) is even, by (3), there exist even integers \( l_{i,j} \) for \( 1 \leq i \leq \frac{p}{2} \) and \( 1 \leq j \leq p - 1 \) such that:

(a) \( 0 \leq l_{i,j} < a \) for all \( i, j \), and
(b) \( \sum_{i=1}^{l_{i,j}} \sum_{j=1}^{p-1} l_{i,j} = \lambda_1 \).

And if \( \lambda_1 \) is odd, other than in the special case where \( a = 2 \) and \( \lambda_2 a (p - 1) = \lambda_1 - 1 \), then by (3) there exist integers \( l_{i,j} \) for \( 1 \leq i \leq \frac{p}{2} \) and \( 1 \leq j \leq p - 1 \) with all but one of the integers \( l_{i,j} \) even, and one odd, so that (a) and (b) above also hold. (Of course in this case, at least one integer \( l_{i,j} \) is strictly less than \( a \), since \( \lambda_1 \) is odd and (3) holds.) If we are in the special case where \( a = 2 \) and \( \lambda_2 a (p - 1) = \lambda_1 - 1 \), then we can instead find a \( C_4 \)-factorization of \( G' = K(a, p; \lambda_1 - 1, \lambda_2) \) (for which (3) is clearly satisfied) as described next, then set \( F \) to be the pure edges in \( G \) that are not in \( G' \).

For \( 1 \leq i \leq \frac{p}{2} \) and \( 1 \leq j \leq p - 1 \), and for each \( e = \{u, v\} \in F_j \), define a set \( C_{i,j}(e) \) of partial parallel classes of \( G \) formed by applying Lemma 1 if \( l_{i,j} = a \), Lemma 3 if \( 1 \leq l_{i,j} < a \), and Lemma 4 if \( l_{i,j} = 0 \), where the vertex sets in the two copies of \( K_a \) are \( V_a \) and \( V_v \). Then \( \bigcup_{e \in F_j} C_{i,j}(e) \) is a \( C_4 \)-factorization of the subgraph of \( G \) formed by joining each pair of vertices in the same part of \( G \) with \( l_{i,j} \) edges, and joining each pair of vertices in parts \( V_a \) and \( V_v \) with one edge, for each \( e = \{u, v\} \in F_j \). Therefore

\[
\bigcup_{1 \leq j \leq p-1} \left( \bigcup_{1 \leq i \leq \frac{p}{2}} \bigcup_{e \in F_j} C_{i,j}(e) \right)
\]

is the required \( C_4 \)-factorization of \( G \), or of \( G - F \) when \( \lambda_1 \) is odd. \( \square \)

References