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Stability for non-hyperbolic fixed points of scalar difference equations

Kouichi Murakami

Department of Mathematical Sciences, Faculty of Integrated Arts and Sciences, The University of Tokushima, Tokushima 770-8502, Japan

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Abstract

We give some new criteria to determine the stability of a non-hyperbolic fixed point of the scalar difference equation

$$x_{n+1} = f(x_n) \quad (n = 0, 1, 2, \ldots),$$

where $x_n \in \mathbb{R}$ and f is a sufficient smooth function. Our results are based on higher order derivative $f^{(k)}(\bar{x})$ at a fixed point of $\bar{x}$.

Keywords: Difference equation; Stability; Non-hyperbolic fixed point

1. Introduction

The purpose of this paper is to give some new criteria to determine the stability of a fixed point of the scalar difference equation

$$x_{n+1} = f(x_n) \quad (n = 0, 1, 2, \ldots), \tag{1.1}$$

where $x_n \in \mathbb{R}$ and f is a sufficient smooth function.

E-mail address: murakami@ias.tokushima-u.ac.jp.

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Consider the higher dimensional difference equation

$$X_{n+1} = F(X_n) \quad (n = 0, 1, 2, \ldots), \tag{1.2}$$

where $X_n \in \mathbb{R}^m$ and F is a sufficient smooth function. Here we let $\bar{X}$ to be a fixed point of (1.2), that is $\bar{X} = f(\bar{X})$. The following is a well-known criterion to see if the fixed point $\bar{X}$ is stable.

Theorem 1.1. *Let $DF(\bar{X})$ denote the Jacobian matrix of F at $\bar{X}$.*

(i) *If all the eigenvalues of $DF(\bar{X})$ have moduli less than one, then $\bar{X}$ is asymptotically stable.*
(ii) *If at least one of the eigenvalues of $DF(\bar{X})$ has modulus greater than one, then $\bar{X}$ is unstable.*

The fixed point $\bar{X}$ is said to be *hyperbolic* if the Jacobian matrix $DF(\bar{X})$ has no eigenvalues with modulus one. Otherwise, it is said to be *non-hyperbolic*. Theorem 1.1 can be applied to the hyperbolic fixed point to determine the stability.

For the stability of the non-hyperbolic fixed point, we need further analysis. In some cases, the center manifold theory is useful to investigate the stability of the non-hyperbolic fixed points [1]. This theory allows us to reduce the dimension of system near the fixed point. Suppose that $DF(\bar{X})$ has c eigenvalues of modulus one, and all other eigenvalues having modulus less than one. Then the dynamics of (1.2) near the fixed point can be described by the c-dimensional difference equation. Particularly, if $c = 1$, the stability of the fixed point of (1.2) can be determined by the stability of the fixed point of the scalar difference equation.

For example, consider the stability of the zero solution of

$$\begin{cases} x_{n+1} = y_n, \\ y_{n+1} = -\frac{1}{2}x_n + \frac{3}{2} - y_n^3. \end{cases} \tag{1.3}$$

The Jacobian matrix of (1.3) at the origin has two eigenvalues $\lambda = 1, \frac{1}{2}$. Thus Theorem 1.1 cannot be applied to (1.3). By the center manifold theory, we can reduce (1.3) to the scalar difference equation

$$u_{n+1} = u_n - 2u_n^3 + O\big(u_n^4\big) \quad (n = 0, 1, 2, \ldots), \tag{1.4}$$

where $u_n \in \mathbb{R}$. For detailed computations, see Section 3.2. Then the stability of the zero solution of (1.3) can be determined by (1.4). Figure 1 shows a solution of (1.3) by numerical computation. In the figure, the dotted line represents one-dimensional center manifold. We see that the dynamics of (1.3) can be approximated by the dynamics on the center manifold.

For another example, we consider the difference equation

$$\begin{cases} x_{n+1} = -x_n + y_n z_n, \\ y_{n+1} = -\frac{1}{2}y_n + x_n^2, \\ z_{n+1} = \frac{1}{2}z_n - x_n y_n. \end{cases} \tag{1.5}$$

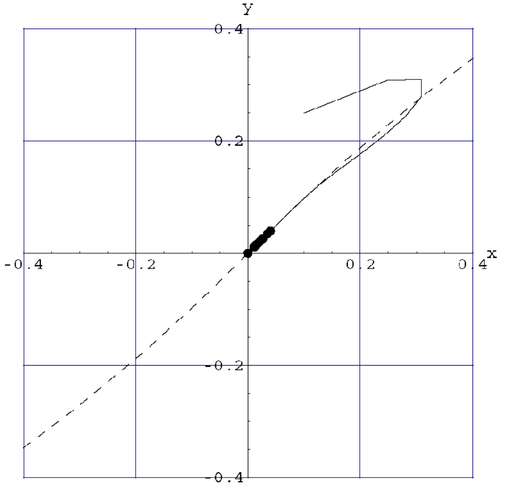

Fig. 1. The solution of (1.3). The dotted line represents the center manifold.

The Jacobian matrix of (1.5) at the origin has three eigenvalues $\lambda = -1, -\frac{1}{2}, \frac{1}{2}$. In this case, the stability of the zero solution of (1.5) can be determined by the scalar difference equation

$$u_{n+1} = -u_n - \frac{8}{27}u_n^5 + O\left(u_n^6\right) \quad (n = 0, 1, 2, \ldots), \tag{1.6}$$

where $u_n \in \mathbb{R}$. For computations, see also Section 3.2.

Now our problem is to find some criteria to determine the stability of the fixed point of the scalar difference equation such as (1.4) and (1.6). The following results partially answer to this problem [2].

Theorem 1.2. *Let $\bar{x}$ to be a fixed point of* (1.1). *Suppose that $f \in C^3(\mathbb{R})$ and $f'(\bar{x}) = 1$.*

(i) *If $f''(\bar{x}) \neq 0$, then $\bar{x}$ is unstable.*
(ii) *If $f''(\bar{x}) = 0$ and $f'''(\bar{x}) > 0$, then $\bar{x}$ is unstable.*
(iii) *If $f''(\bar{x}) = 0$ and $f'''(\bar{x}) < 0$, then $\bar{x}$ is asymptotically stable.*

Theorem 1.3. *Let $\bar{x}$ to be a fixed point of* (1.1). *Suppose that $f \in C^3(\mathbb{R})$ and $f'(\bar{x}) = -1$.*

(i) *If $-3\{f''(\bar{x})\}^2 - 2f'''(\bar{x}) < 0$, then $\bar{x}$ is asymptotically stable.*
(ii) *If $-3\{f''(\bar{x})\}^2 - 2f'''(\bar{x}) > 0$, then $\bar{x}$ is unstable.*

Theorem 1.2 implies that the zero solution of (1.4) is asymptotically stable. So the zero solution of (1.3) is asymptotically stable, too. On the other hand, the stability of the solution of (1.6) cannot be determined by the above theorems. These theorems use the values of $f''(\bar{x})$ and $f'''(\bar{x})$ to determine the stability of $\bar{x}$, whereas such values vanish for (1.6).

The goal of this paper is to extend the above theorems. In the next section, we give some new criteria based on more higher order derivative $f^{(k)}(\bar{x})$. Using our results, we can determine the stability of the fixed points for more general cases. To illustrate our results, we give some applications in the last section.

2. Main result

2.1. The case $f'(\bar{x}) = 1$

Theorem 2.1. *Let $\bar{x}$ to be a fixed point of* (1.1). *Suppose that $f \in C^k(\mathbb{R})$ $(k \geqslant 2)$ and*

$$f'(\bar{x}) = 1, \qquad f^{(j)}(\bar{x}) = 0 \quad (j = 2, 3, \ldots, k-1), \qquad f^{(k)}(\bar{x}) \neq 0.$$

(i) *If k is even, then $\bar{x}$ is unstable.*
(ii) *If k is odd and $f^{(k)}(\bar{x}) > 0$, then $\bar{x}$ is unstable.*
(iii) *If k is odd and $f^{(k)}(\bar{x}) < 0$, then $\bar{x}$ is asymptotically stable.*

Proof. From Taylor's theorem, we have

$$\begin{aligned} f(\bar{x}+h) &= f(\bar{x}) + f'(\bar{x})h + \frac{f''(\bar{x})}{2!}h^2 + \cdots + \frac{f^{(k-1)}(\bar{x})}{(k-1)!}h^{k-1} + \frac{f^{(k)}(\bar{x}+\theta h)}{k!}h^k \\ &= \bar{x} + h + \frac{f^{(k)}(\bar{x}+\theta h)}{k!}h^k \quad (0 < \theta < 1) \end{aligned}$$

for sufficient small h. Since $f \in C^k$ and $f^{(k)}(\bar{x}) \neq 0$, there exists $\varepsilon_0 > 0$ such that

$$0 < m \leqslant \left| \frac{f^{(k)}(\bar{x}+\theta h)}{k!} \right| \leqslant M$$

for all $|h| < \varepsilon_0$. Here m and M are positive constants.

Define $h_n = x_n - \bar{x}$ $(n = 0, 1, 2, \ldots)$. Then we have

$$x_{n+1} = \bar{x} + h_{n+1}$$

and

$$f(x_n) = f(\bar{x} + h_n) = \bar{x} + h_n + \frac{f^{(k)}(\bar{x}+\theta h_n)}{k!}h_n^k.$$

Since x_n satisfies (1.1), we have

$$h_{n+1} = h_n + \frac{f^{(k)}(\bar{x}+\theta h_n)}{k!}h_n^k.$$

(i) Assume that k is even. We only consider the case that $f^{(k)}(\bar{x}) > 0$. Then we have

$$\frac{f^{(k)}(\bar{x}+\theta h)}{k!} \geqslant m > 0$$

for $|h| < \varepsilon_0$. Also we have $h_n^k > 0$ for $h_n \neq 0$. If we choose h_0 to be positive, then we have

$$h_{n+1} \geqslant h_n + m h_n^k > h_n$$

whenever $|h_n| < \varepsilon_0$. Hence there exists $n_0 > 0$ such that $|h_{n_0}| \geqslant \varepsilon_0$. We can choose h_0 to be arbitrary small. Thus we conclude that $\bar{x}$ is unstable. In case of $f^{(k)}(\bar{x}) < 0$, we may choose h_0 to be negative.

(ii) Assume that k is odd and $f^{(k)}(\bar{x}) > 0$. Then we have

$$\frac{f^{(k)}(\bar{x} + \theta h)}{k!} \geqslant m > 0$$

for $|h| < \varepsilon_0$. Also we have $h_n^k > 0$ for $h_n > 0$, and $h_n^k < 0$ for $h_n < 0$. If we choose h_0 to be positive, then we have

$$h_{n+1} \geqslant h_n + m h_n^k > h_n$$

whenever $|h_n| < \varepsilon_0$. Hence there exists n_0 such that $|h_{n_0}| \geqslant \varepsilon_0$. We can choose h_0 to be arbitrary small. Consequently we have that $\bar{x}$ is unstable.

(iii) Assume that k is odd and $f^{(k)}(\bar{x}) < 0$. Then we have

$$\frac{f^{(k)}(\bar{x} + \theta h)}{k!} \leqslant -m < 0$$

for $|h| < \varepsilon_0$. Also we have $h_n^k > 0$ for $h_n > 0$, and $h_n^k < 0$ for $h_n < 0$. Moreover, we can find $\delta > 0$ such that

$$\left| \frac{f^{(k)}(\bar{x} + \theta h)}{k!} h^k \right| \leqslant M|h|^k < |h|$$

for $|h| < \delta$. Thus we have that h_n has definite sign and

$$|h_{n+1}| = \left| h_n + \frac{f^{(k)}(\bar{x} + \theta h_n)}{k!} h_n^k \right| < |h_n|$$

whenever $|h_n| < \delta$. So h_n has a limit. Let $\lim_{n\to\infty} h_n = \bar{h}$, then we have

$$\bar{h} = \bar{h} + \frac{f^{(k)}(\bar{x} + \theta \bar{h})}{k!} \bar{h}^k.$$

Thus we have that $\bar{h} = 0$. Hence we conclude that $|h_0| < \delta$ implies that $|h_n| < |h_0|$ and $\lim_{n\to\infty} h_n = 0$. This shows that $\bar{x}$ is asymptotically stable. □

Remark 2.1. In case of $k = 2$ or $k = 3$, Theorem 2.1 coincides with Theorem 1.1. Moreover, Theorem 2.1 guarantees that the zero solutions of

$$x_{n+1} = x_n - x_n^5 + O\left(x_n^6\right),$$
$$x_{n+1} = x_n - x_n^7 + O\left(x_n^8\right)$$

are asymptotically stable, and that the zero solutions of

$$x_{n+1} = x_n + x_n^4 + O\big(x_n^5\big),$$
$$x_{n+1} = x_n - x_n^4 + O\big(x_n^5\big),$$
$$x_{n+1} = x_n + x_n^5 + O\big(x_n^6\big)$$

are unstable.

2.2. *The case* $f'(\bar{x}) = -1$

Consider the difference equation

$$y_{n+1} = g(y_n), \quad \text{where } g(y) = f\big(f(y)\big). \tag{2.1}$$

Lemma 2.2. *Suppose that* $f(x) \in C(\mathbb{R})$.

(i) *If* $\bar{x}$ *is a fixed point of* (1.1), *then it is the fixed point of* (2.1).
(ii) *If the fixed point* $\bar{x}$ *of* (1.1) *is asymptotically stable with respect to* (2.1), *then it is asymptotically stable with respect to* (1.1).
(iii) *If the fixed point* $\bar{x}$ *of* (1.1) *is unstable with respect to* (2.1), *then it is unstable with respect to* (1.1).

Proof. Let x_n be a solution of (1.1), and y_n be a solution of (2.1). In the following, we assume that $x_0 = y_0$. Then we have $y_n = x_{2n}$.

(i) We can easily see that

$$g(\bar{x}) = f\big(f(\bar{x})\big) = f(\bar{x}) = \bar{x}.$$

(ii) Suppose that $\bar{x}$ is stable with respect to (2.1). Then for any $\varepsilon_1 > 0$ there exists $\delta_1(\varepsilon_1) > 0$ such that $|y_0 - \bar{x}| = |x_0 - \bar{x}| < \delta_1$ implies that $|y_n - \bar{x}| = |x_{2n} - \bar{x}| < \varepsilon_1$ for all $n \geqslant 0$. From the continuity of $f(x)$ at $\bar{x}$, we have that for any $\varepsilon_2 > 0$ there exists $\delta_2(\varepsilon_2) > 0$ such that $|x - \bar{x}| < \delta_2$ implies that $|f(x) - f(\bar{x})| < \varepsilon_2$. Now we choose ε_1 as $\varepsilon_1 = \delta_2(\varepsilon_2) > 0$, then there exists $\delta_1(\delta_2(\varepsilon_2)) > 0$ such that $|x_0 - \bar{x}| < \delta_1$ implies that $|f(x_{2n}) - f(\bar{x})| = |x_{2n+1} - \bar{x}| < \varepsilon_2$ for all $n \geqslant 0$. Hence, for any $\varepsilon > 0$, if we let $\delta = \min(\delta_1(\varepsilon), \delta_1(\delta_2(\varepsilon)))$, then $|x_0 - \bar{x}| < \delta$ implies that $|x_n - \bar{x}| < \varepsilon$ for all $n \geqslant 0$. This shows that $\bar{x}$ is stable with respect to (1.1).

Suppose that there exists $\eta > 0$ such that $|y_0 - \bar{x}| < \eta$ implies that $\lim_{n\to\infty} y_n = \bar{x}$. Then for any $\varepsilon_1 > 0$ there exists $N_1(\varepsilon_1)$ such that $|y_n - \bar{x}| = |x_{2n} - \bar{x}| < \varepsilon_1$ for any $n \geqslant N_1$. From the continuity of $f(x)$ at $\bar{x}$, we have that for any $\varepsilon_2 > 0$ there exists $\delta(\varepsilon_2) > 0$ such that $|x - \bar{x}| < \delta$ implies that $|f(x) - f(\bar{x})| < \varepsilon_2$. Now we choose ε_1 as $\varepsilon_1 = \delta(\varepsilon_2) > 0$, then there exists $N_1(\delta(\varepsilon_2))$ such that $|f(x_{2n}) - f(\bar{x})| = |x_{2n+1} - \bar{x}| < \varepsilon_2$ for any $n \geqslant N_1$. Hence, for any $\varepsilon > 0$, if we let $N = \max(N_1(\varepsilon), N_1(\delta(\varepsilon)))$, then $|x_n - \bar{x}| < \varepsilon$ for any $n \geqslant N$. Consequently $\bar{x}$ is asymptotically stable with respect to (1.1).

(iii) Assume that $\bar{x}$ is unstable with respect to (2.1). Then there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists x_0 ($|x_0 - \bar{x}| < \delta$) and $n \geqslant 0$ such that $|y_n - x_0| \geqslant \varepsilon$. Since $y_n = x_{2n}$, we can conclude that $\bar{x}$ is unstable with respect to (1.1). □

Theorem 2.3. *Let $\bar{x}$ to be a fixed point of* (1.1). *Suppose that $f \in C^{2k-1}(\mathbb{R})$ and*

$$f'(\bar{x}) = -1, \qquad f^{(j)}(\bar{x}) = 0 \quad (j = 2, 3, \dots, k-1), \qquad f^{(k)}(\bar{x}) \neq 0.$$

(i) *If k is odd and $f^{(k)}(\bar{x}) > 0$, then $\bar{x}$ is asymptotically stable.*
(ii) *If k is odd and $f^{(k)}(\bar{x}) < 0$, then $\bar{x}$ is unstable.*
(iii) *Assume that k is even, and there exists an integer $l < k$ such that*

$$f^{j}(\bar{x}) = 0 \quad (j = k+1,\ k+3, \dots,\ 2l-3), \qquad f^{(2l-1)}(\bar{x}) \neq 0.$$

  (a) *If $f^{(2l-1)}(\bar{x}) > 0$, then $\bar{x}$ is asymptotically stable.*
  (b) *If $f^{(2l-1)}(\bar{x}) < 0$, then $\bar{x}$ is unstable.*

(iv) *Assume that k is even, and*

$$f^{j}(\bar{x}) = 0 \quad (j = k+1, k+3, \dots, 2k-3).$$

  (a) *If $\frac{k}{2}\left(\frac{f^{(k)}(\bar{x})}{k!}\right)^2 + \frac{f^{(2k-1)}(\bar{x})}{(2k-1)!} > 0$, then $\bar{x}$ is asymptotically stable.*
  (b) *If $\frac{k}{2}\left(\frac{f^{(k)}(\bar{x})}{k!}\right)^2 + \frac{f^{(2k-1)}(\bar{x})}{(2k-1)!} < 0$, then $\bar{x}$ is unstable.*

Proof. From Taylor's theorem, we have

$$\begin{aligned} f(x) &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{f''(\bar{x})}{2!}(x-\bar{x})^2 \\ &\quad + \cdots + \frac{f^{(2k-1)}(\bar{x})}{(2k-1)!}(x-\bar{x})^{2k-1} + o(x-\bar{x})^{2k-1} \\ &= \bar{x} - (x-\bar{x}) + \sum_{i=k}^{2k-1} a_i (x-\bar{x})^i + o(x-\bar{x})^{2k-1}, \end{aligned}$$

where $a_k = \dfrac{f^{(k)}(\bar{x})}{k!}$. Then we have

$$\begin{aligned} g(x) &= f\big(f(x)\big) \\ &= \bar{x} - \big(f(x)-\bar{x}\big) + \sum_{i=k}^{2k-1} a_i \big(f(x)-\bar{x}\big)^i + o\big(f(x)-\bar{x}\big)^{2k-1} \\ &= \bar{x} - \left\{-(x-\bar{x}) + \sum_{i=k}^{2k-1} a_i (x-\bar{x})^i\right\} \\ &\quad + \sum_{i=k}^{2k-1} \left(a_i \left\{-(x-\bar{x}) + \sum_{j=k}^{2k-1} a_j (x-\bar{x})^j\right\}^i\right) + o(x-\bar{x})^{2k-1} \\ &= x + \sum_{i=k}^{2k-1} \big(a_i \{-1 + (-1)^i\}(x-\bar{x})^i\big) \\ &\quad + k a_k^2 (-1)^{k-1} (x-\bar{x})^{2k-1} + o(x-\bar{x})^{2k-1}. \end{aligned}$$

If k is odd, then we have

$$g(x) = x - 2a_k(x - \bar{x})^k + o(x - \bar{x})^k.$$

Thus we have

$$g'(\bar{x}) = 1, \qquad g^{(i)}(\bar{x}) = 0 \quad (i = 2, 3, \ldots, k-1), \qquad g^{(k)}(\bar{x}) = -2f^{(k)}(\bar{x}) \neq 0.$$

Therefore Theorem 2.1 and Lemma 2.2 imply that (i) and (ii) are valid.

Assume that k is even. Then we have

$$g(x) = x - 2\sum_{i=1}^{k/2} a_{k+2i-1}(x - \bar{x})^{k+2i-1} - ka_k^2(x - \bar{x})^{2k-1} + o(x - \bar{x})^{2k-1}.$$

In case (iii), we have

$$g(x) = x - 2a_{2l-1}(x - \bar{x})^{2l-1} + o(x - \bar{x})^{2l-1}.$$

Thus we have

$$g'(\bar{x}) = 1, \qquad g^{(i)}(\bar{x}) = 0 \quad (i = 2, 3, \ldots, 2l-2), \qquad g^{(2l-1)}(\bar{x}) = -2f^{(2l-1)}(\bar{x}).$$

From Theorem 2.1 and Lemma 2.2, we can conclude that (iii) are valid. In case (iv), we have

$$g(x) = x - \left(2a_{2k-1} + ka_k^2\right)(x - \bar{x})^{2k-1} + o(x - \bar{x})^{2k-1}.$$

Thus we have

$$g'(\bar{x}) = 1, \qquad g^{(i)}(\bar{x}) = 0 \quad (i = 2, 3, \ldots, 2k-2),$$
$$g^{(2k-1)}(\bar{x}) = -2(2k-1)!\left(\frac{k}{2}a_k^2 + a_{2k-1}\right).$$

From Theorem 2.1 and Lemma 2.2, we can conclude that (iv) are valid. □

Remark 2.2. In case of $k = 2$, Theorem 2.3(iv) coincides with Theorem 1.2. Moreover, Theorem 2.3 guarantees that the zero solutions of

$$x_{n+1} = -x_n + x_n^5 + O\left(x_n^6\right),$$
$$x_{n+1} = -x_n + x_n^4 + x_n^5 + O\left(x_n^6\right),$$
$$x_{n+1} = -x_n + x_n^4 + 3x_n^7 + O\left(x_n^8\right)$$

are asymptotically stable, and that the zero solutions of

$$x_{n+1} = -x_n - x_n^5 + O\left(x_n^6\right),$$
$$x_{n+1} = -x_n + x_n^4 - x_n^5 + O\left(x_n^6\right),$$
$$x_{n+1} = -x_n + x_n^4 - 3x_n^7 + O\left(x_n^8\right)$$

are unstable.

3. Application

3.1. Scalar case

We consider the stability of the zero solution of

$$x_{n+1} = x_n e^{-x_n^k} \quad (n = 1, 2, 3, \ldots), \tag{3.1}$$

where $x_n \in \mathbb{R}$ and k is a positive integer.

Theorem 3.1. *If k is even, then the zero solution of* (3.1) *is asymptotically stable. If k is odd, then the zero solution of* (3.1) *is unstable.*

Proof. Let $f(x) = xe^{-x^k}$. Then we have

$$\begin{aligned} f(x) &= x\left(1 + (-x^k) + \frac{1}{2!}(-x^k)^2 + \frac{1}{3!}(-x^k)^3 + \cdots\right) \\ &= x - x^{k+1} + \frac{1}{2!}x^{2k+1} - \frac{1}{3!}x^{3k+1} + \cdots. \end{aligned}$$

Thus we have

$$f'(0) = 1, \qquad f^{(j)}(0) = 0 \quad (j = 2, 3, \ldots, k), \qquad f^{(k+1)}(0) = -(k+1)! < 0.$$

Using Theorem 2.1, we complete the proof. □

3.2. Higher dimensional case

We consider the stability of the zero solution of the higher dimensional systems

$$X_{n+1} = AX_n + F(X_n) \quad (n = 0, 1, 2, \ldots), \tag{3.2}$$

where $X_n \in \mathbb{R}^m$ and $F(x) = O(\|X\|^2)$.

First of all we prepare the center manifold theory. We let λ be a simple eigenvalue of A, and the corresponding eigenvectors be $p \in \mathbb{R}^{1\times m}$ and $q \in \mathbb{R}^m$ which satisfy

$$Aq = \lambda q, \qquad pA = \lambda p, \qquad pq = 1.$$

Then $X \in \mathbb{R}^m$ can be decomposed by

$$X = qu + v,$$

where $u = pX \in \mathbb{R}$ and $v = X - qu$. Also, (3.2) can be decomposed as

$$\begin{cases} u_{n+1} = \lambda u_n + pF(qu_n + v_n), \\ v_{n+1} = Av_n + (I - qp)F(qu_n + v_n). \end{cases} \tag{3.3}$$

Assume that $|\lambda| = 1$, and A has no other eigenvalues with modulus one. Then there exists a function $v = h(u)$ where $h(0) = 0$, $Dh(0) = 0$ such that the dynamics of (3.2) can be described by

$$u_{n+1} = \lambda u_n + pF\big(qu_n + h(u_n)\big). \tag{3.4}$$

We let the center manifold as

$$v = h(u) = C_2u^2 + C_3u^3 + \cdots$$

and assume that

$$F\big(qu + h(u)\big) = F_2u^2 + F_3u^3 + \cdots,$$

where $C_i, F_i \in \mathbb{R}^m$. We note that F_i depends on C_j $(j = 2, 3, \dots, i-1)$. Using the first equation of (3.3), we have

$$\begin{aligned} v_{n+1} &= h(u_{n+1}) \\ &= h\big(\lambda u_n + pF\big(qu_n + h(u_n)\big)\big) \\ &= C_2\big(\lambda u_n + p\big(F_2u_n^2 + F_3u_n^3 + \cdots\big)\big)^2 \\ &\quad + C_3\big(\lambda u_n + p\big(F_2u_n^2 + F_3u_n^3 + \cdots\big)\big)^3 + \cdots \\ &= \lambda^2 C_2 u_n^2 + \big(2\lambda pF_2C_2 + \lambda^3 C_3\big)u_n^3 + \cdots. \end{aligned}$$

On the other side, the second equation of (3.3) implies that

$$\begin{aligned} v_{n+1} &= Ah(u_n) + (I - qp)F\big(qu_n + h(u_n)\big) \\ &= A\big(C_2u_n^2 + C_3u_n^3 + \cdots\big) + (I - qp)\big(F_2u_n^2 + F_3u_n^3 + F_4u_n^4 + \cdots\big) \\ &= \big(AC_2 + (I - qp)F_2\big)u_n^2 + \big(AC_3 + (I - qp)F_3\big)u_n^3 + \cdots. \end{aligned}$$

Comparing the coefficients of u_n^i $(i = 2, 3, \dots)$ of the above equations, we can compute C_i such as

$$\begin{cases} C_2 \;=\; (\lambda^2 I - A)^{-1}(I - qp)F_2, \\ C_3 \;=\; (\lambda^3 I - A)^{-1}((I - qp)F_3 - 2\lambda pF_2C_2), \\ \qquad \vdots \end{cases} \tag{3.5}$$

Now we give some concrete examples. First, we consider the equation

$$\begin{cases} x_{n+1} = -x_n + 3y_nz_n, \\ y_{n+1} = -\frac{1}{2}y_n + 3x_n^2, \\ z_{n+1} = \frac{1}{2}z_n - x_ny_n. \end{cases} \tag{3.6}$$

Let

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad F(X) = \begin{pmatrix} 3yz \\ 3x^2 \\ -xy \end{pmatrix}.$$

Then we can write (3.6) as

$$X_{n+1} = AX_n + F(X_n).$$

The matrix A has eigenvalues $\lambda = -1,\ -\frac{1}{2},\ \frac{1}{2}$, and eigenvectors associated with the eigenvalue $\lambda = -1$ are given by

$$q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad p = (1\ 0\ 0).$$

We let the expression of the center manifold $h(x)$ as

$$v = h(x) = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} x^2 + \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} x^3 + \cdots.$$

In this case, we have that $I - qp = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$. Then (3.5) implies that $a_i = 0$. Thus, on the center manifold, the solution can be expressed by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = qx + h(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ b_2 \\ c_2 \end{pmatrix} x^2 + \begin{pmatrix} 0 \\ b_3 \\ c_3 \end{pmatrix} x^3 + \cdots.$$

Therefore we can reduce (3.6) to

$$x_{n+1} = -x_n + 3y_n z_n = -x_n + 3\bigl(b_2 x_n^2 + b_3 x_n^3 + \cdots\bigr)\bigl(c_2 x_n^2 + c_3 x_n^3 + \cdots\bigr)$$

on the center manifold.

We compute the coefficients of the center manifold. From

$$F\bigl(qx + h(x)\bigr) = \begin{pmatrix} 3yz \\ 3x^2 \\ -xy \end{pmatrix} = \begin{pmatrix} 3(b_2x^2 + b_3x^3 + \cdots)(c_2x^2 + c_3x^3 + \cdots) \\ 3x^2 \\ -x(b_2x^2 + b_3x^3 + \cdots) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} x^2 + \begin{pmatrix} 0 \\ 0 \\ -b_2 \end{pmatrix} x^3 + \cdots,$$

we have

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = (I - A)^{-1}(I - qp)\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = (-I - A)^{-1}(I - qp)\begin{pmatrix} 0 \\ 0 \\ -b_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{4}{3} \end{pmatrix}.$$

Consequently we have the equation on the center manifold as

$$\begin{aligned} x_{n+1} &= -x_n + 3\bigl(b_2 x_n^2 + b_3 x_n^3 + \cdots\bigr)\bigl(c_2 x_n^2 + c_3 x_n^3 + \cdots\bigr) \\ &= -x_n + 3\bigl(2x_n^2 + \cdots\bigr)\left(\frac{4}{3}x_n^3 + \cdots\right) \\ &= -x_n + 8x_n^5 + O\bigl(x_n^6\bigr). \end{aligned} \tag{3.7}$$

Let $f(x) = -x + 8x^5 + O(x^6)$, then we have

$$f'(0) = -1, \qquad f^{(j)}(0) = 0 \quad (j = 2, 3, 4), \qquad f^{(5)}(0) = 8 \cdot 5! > 0.$$

Thus Theorem 2.3(i) implies that $x_n = 0$ of (3.7) is asymptotically stable. Hence the zero solution of (3.6) is asymptotically stable. Figure 2 shows a solution of (3.6) by numerical computation.

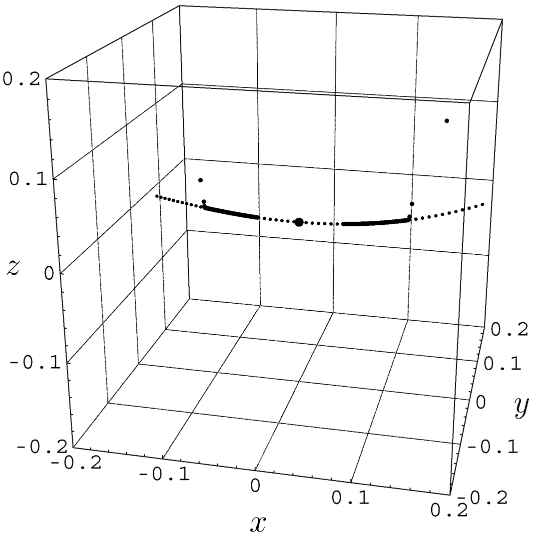

Fig. 2. The solution of (3.6). The dotted line represents the center manifold.

Next we consider the equation

$$\begin{cases} x_{n+1} = -x_n - 2x_n^2 y_n, \\ y_{n+1} = -\frac{1}{2} y_n + 3x_n^2, \\ z_{n+1} = \frac{1}{2} z_n - x_n y_n. \end{cases} \tag{3.8}$$

Except for the non-linear term $-2x_n^2 y_n$ of the first equation, (3.8) is much the same as (3.6). In the same way, we let

$$F(X) = \begin{pmatrix} -2x^2 y \\ 3x^2 \\ -xy \end{pmatrix}$$

and the expression of the center manifold $h(x)$ as

$$v = h(x) = \begin{pmatrix} 0 \\ b_2 \\ c_2 \end{pmatrix} x^2 + \begin{pmatrix} 0 \\ b_3 \\ c_3 \end{pmatrix} x^3 + \begin{pmatrix} 0 \\ b_4 \\ c_4 \end{pmatrix} x^4 + \begin{pmatrix} 0 \\ b_5 \\ c_5 \end{pmatrix} x^5 + \cdots.$$

Then we can reduce (3.8) to

$$x_{n+1} = -x_n - 2x_n^2 \left(b_2 x_n^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + \cdots \right).$$

Using $h(\lambda x + pF(qx + h(x))) = Ah(x) + (I - qp)F(qx + h(x))$, we compute the coefficients of the center manifold. Form

$$F\big(qx+h(x)\big)=\begin{pmatrix}-2x^2(b_2x^2+b_3x^3+\cdots)\\3x^2\\-x(b_2x^2+b_3x^3+b_4x^4+\cdots)\end{pmatrix}$$
$$=\begin{pmatrix}0\\3\\0\end{pmatrix}x^2+\begin{pmatrix}0\\0\\-b_2\end{pmatrix}x^3+\begin{pmatrix}-2b_2\\0\\-b_3\end{pmatrix}x^4+\begin{pmatrix}-2b_3\\0\\-b_4\end{pmatrix}x^5+\cdots,$$

we have

$$\lambda x+pF\big(qx+h(x)\big)=-x-2b_2x^4-2b_3x^5+\cdots.$$

Thus we have

$$\begin{aligned}&h\big(\lambda x+pF\big(qx+h(x)\big)\big)\\&\quad=\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}\big(-x-2b_2x^4+\cdots\big)^2+\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix}\big(-x-2b_2x^4+\cdots\big)^3+\cdots\\&\quad=\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}x^2-\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix}x^3+\begin{pmatrix}a_4\\b_4\\c_4\end{pmatrix}x^4+\left\{-\begin{pmatrix}a_5\\b_5\\c_5\end{pmatrix}+4b_2\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}\right\}x^5+\cdots.\end{aligned}$$

Therefore we have

$$\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}=(I-A)^{-1}(I-qp)\begin{pmatrix}0\\3\\0\end{pmatrix}=\begin{pmatrix}\frac12&0&0\\0&\frac23&0\\0&0&2\end{pmatrix}\begin{pmatrix}0\\3\\0\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix},$$
$$\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix}=(-I-A)^{-1}(I-qp)\begin{pmatrix}0\\0\\-b_2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&-2&0\\0&0&-\frac23\end{pmatrix}\begin{pmatrix}0\\0\\-2\end{pmatrix}=\begin{pmatrix}0\\0\\\frac43\end{pmatrix},$$
$$\begin{pmatrix}a_4\\b_4\\c_4\end{pmatrix}=(I-A)^{-1}(I-qp)\begin{pmatrix}-2b_2\\0\\-b_3\end{pmatrix}=\begin{pmatrix}\frac12&0&0\\0&\frac23&0\\0&0&2\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$
$$\begin{aligned}\begin{pmatrix}a_5\\b_5\\c_5\end{pmatrix}&=(-I-A)^{-1}\left((I-qp)\begin{pmatrix}-2b_3\\0\\-b_4\end{pmatrix}-4b_2\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}\right)\\&=\begin{pmatrix}0&0&0\\0&-2&0\\0&0&-\frac23\end{pmatrix}\begin{pmatrix}0\\-16\\0\end{pmatrix}=\begin{pmatrix}0\\32\\0\end{pmatrix}.\end{aligned}$$

Consequently the equation on the center manifold is given by

$$\begin{aligned}x_{n+1}&=-x_n-2x_n^2\big(b_2x_n^2+b_3x^3+b_4x^4+b_5x^5+\cdots\big)\\&=-x_n-2x_n^2\big(2x_n^2+32x_n^5+\cdots\big)\\&=-x_n-4x_n^4-64x_n^7+O\big(x_n^8\big).\end{aligned}\tag{3.9}$$

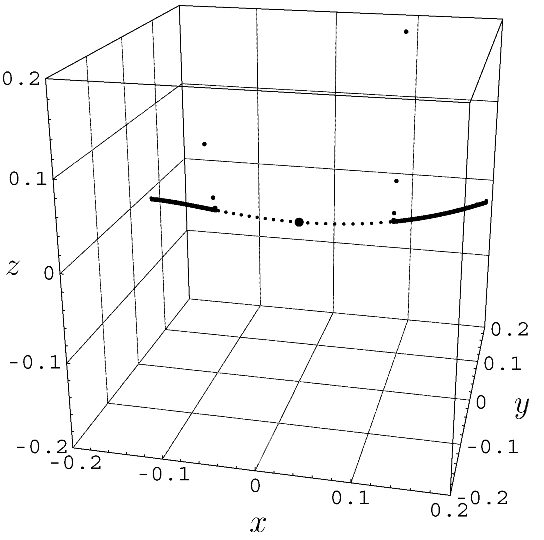

Fig. 3. The solution of (3.7). The dotted line represents the center manifold.

Let $f(x) = -x - 4x^4 - 64x^7 + O(x^8)$, then we have

$$f'(0) = -1, \qquad f^{(j)}(0) = 0 \quad (j = 2, 3), \qquad f^{(4)}(0) = -4 \cdot 4! \neq 0.$$

Also we have

$$f^{(5)}(0) = 0, \qquad \frac{4}{2}\left(\frac{f^{(4)}(0)}{4!}\right)^2 + \frac{f^{(7)}(0)}{7!} = 2(-4)^2 - 64 = -32 < 0.$$

Thus Theorem 2.3(iv) implies that $x_n = 0$ of (3.9) is unstable. Hence the zero solution of (3.8) is unstable. Figure 3 shows a solution of (3.9) by numerical computation.

References

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