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Reprint of: Extension of the near acoustic field of a jet to the far field☆

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Abstract

The problem of extending the near acoustic field of a high-speed jet to the far field is considered. This is akin to the problem of analytic continuation in complex variable. Analytic continuation is the extension of an analytic function in a limited domain to a larger domain. The general continuation problem involving acoustic sources enclosed by a surface is first analyzed. It is shown that a unique continuation to the far field is possible provided either the pressure or the pressure gradient normal to the surface is given on the surface. In this paper, the continuation is carried out by means of an adjoint Green's function. One significant advantage of using adjoint Green's function is that for a given direction of radiation, one is required to solve only a single acoustic scattering problem. Because the present method requires measuring only the fluctuation pressure on the bounding surface whereas other popular methods such as the Kirchhoff integral method or the Ffowcs-Williams and Hawkings equation requires measuring three or more variables, it appears that the present method has definite advantage when it is used to extend an experimentally measured near acoustic field to the far field.

Keywords: Computational aeroacoustics; jet noise; surface Green's function; adjoint Green's function.

1. Introduction

In Computational Aeroacoustics (CAA), extension of a computed near field solution to the far field constitutes an important problem. Numerical simulations of high-speed jets are, invariably, performed in a finite computational domain. This is because of finite computer resources and computing time. The need to have a reliable, accurate and robust method to extend a solution in a limited space to a much larger space was well recognized since the

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beginning of research effort in jet noise simulation. Primarily, three methods have been used. Some investigators e. g. Colonius and Freund [1], Wang, Lele and Moin [2], recast the problem in the tradition of Lighthill's Acoustic Analogy [3,4]. This led to integrals of the sources over the volume of the jet flow. Owing to the fact that volume integrals are very costly to carry out, this approach is very seldom used nowadays. The other two approaches are the Ffowcs-Williams and Hawkings integral [5] and the Kirchhoff integral (see Lyrintzis [6]) methods. Both methods result in integrals over a closed surface enclosing the noise sources. If the integral surface is taken in the near acoustic field outside a turbulent jet where the fluctuations are essentially linear, then the two methods are effectively very similar.

The Kirchhoff integral method and the Ffowcs-Williams and Hawkings method are derived by means of the free-space Green's function of the wave equation. In order to compute the sound pressure at a far field point using the Kirchhoff integral method the pressure, the normal and time derivatives of the pressure (or the velocity component normal to the surface) on the integral surface must be prescribed. To use the Ffowcs-Williams and Hawkings method more than three variables have to be specified at the integral surface.

Maestrello [7] appears to be the first to consider using a surface Green's function on a plane surface placed near a jet to calculate the far field sound pressure. The input required by such a calculation is the measured pressure gradient fluctuations normal to the plane. Unfortunately, measuring the fluctuating pressure gradient is not an easy task. Because the data quality is somewhat compromised the predicted result can only match qualitatively the measured far field data. Recently, in a series of papers, Reba, Simonich and Schlinker [8] , Reba, Narayanan, Colonius and Suzuki [9] proposed to surround a jet by a conical surface placed just outside the turbulent jet flow. They developed an analytical surface Green's function to extend the pressure fluctuations on the conical surface to the far field. Detailed construction of the analytical surface Green's function is provided in a more recent paper [10]. On using limited data measured on the conical surface at the United Technology Research Center, they were able to show good comparisons with measured far field directivity at a few selected Strouhal numbers.

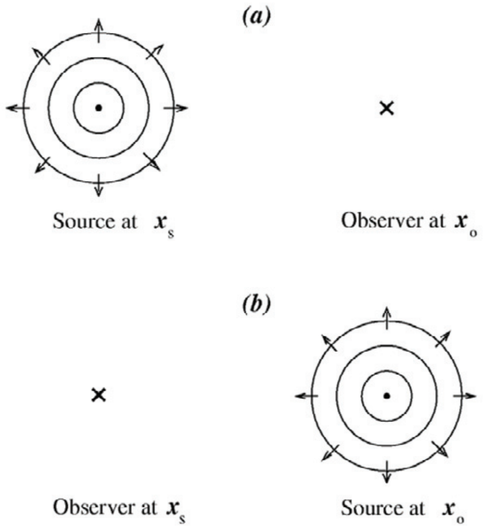

Figure 1. A monopole sound source and an observer forming a self-adjoint system.

Oftentimes, the surface geometry of the enclosing surface is not simple. In such cases, the surface Green's function cannot be easily found. For such surface geometries, one may use an adjoint Green's function and the reciprocity relation. In many branches of mechanics, the reciprocity principle applies. However, the existence of a reciprocity principle has not been fully exploited in the field of acoustics and fluid dynamics. Some earlier works that utilized reciprocity are due to Cho [11], Howe [12,13], Dowling [14], and Tam and Auriault [15] in acoustics; Roberts [16], Eckhaus [17], Chandrasekhar [18] in hydrodynamic stability; and Hill [19] in receptivity problems. To fix ideas on reciprocity, consider a time-periodic point source of sound located at $\boldsymbol{x}_s$ as shown in Fig. 1 (a). Let $G(\boldsymbol{x}_o, \boldsymbol{x}_s, \omega)$ be the pressure associated with the sound field measured by an observer at $\boldsymbol{x}_0$. The time factor $e^{-i\omega t}$ is omitted. Mathematically, $G(\boldsymbol{x}_o, \boldsymbol{x}_s, \omega)$ is the Green's function of the Helmholtz equation and ω is the angular frequency of oscillations (Note: the notation that the first argument of the Green's function is the location of the observer and the second argument is the location of the source is adopted here). Now suppose that the location of the sound source and that of the observer is interchanged as shown in Fig. 1 (b). Clearly, by symmetry, the pressure measured by the observer, now at $\boldsymbol{x}_s$ while the source is at $\boldsymbol{x}_0$, is the same as before. That is,

$$G(\boldsymbol{x}_0, \boldsymbol{x}_s, \omega) = G(\boldsymbol{x}_s, \boldsymbol{x}_0, \omega) \tag{1}$$

Eq. 1 is simply a statement that the Green's function $G(\boldsymbol{x}_0, \boldsymbol{x}_s, \omega)$ is self-adjoint. It is the reciprocity relation.

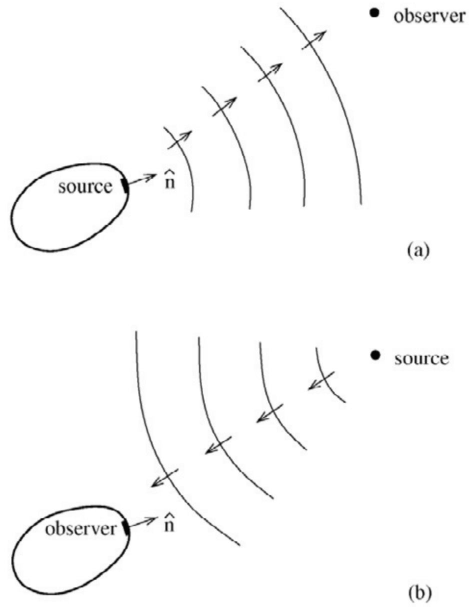

Figure 2. A surface source and a distant observer forming a non-self adjoint system.

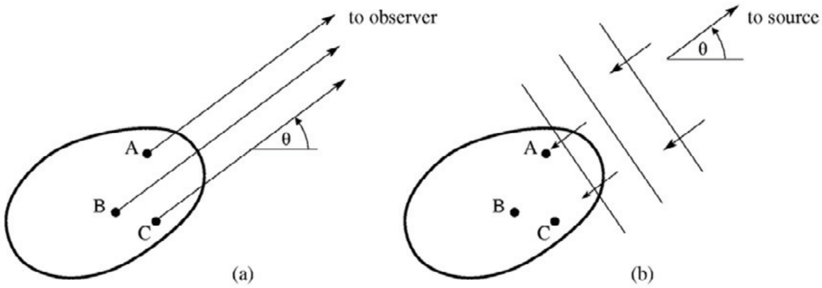

Figure 3. (a) The direct surface Green's function, (b) the adjoint Green's function.

For a surface source and a far field observer as shown in Fig. 2, a reciprocity relation does exist. The problem is, however, non-self adjoint. The adjoint Green's function is not governed by the same equations governing the direct surface Green's function. The boundary condition is also different. But the adjoint equations and boundary condition may easily be derived.

There is a significant advantage in using the adjoint Green's function instead of the direct Green's function when a simple analytical formula for the direct Green's function cannot be found. Suppose the far field sound in the direction θ produced by all surface sources, as shown in Fig. 3(a), is to be found. To determine the total acoustic radiation to the far field direction θ, the radiation from each surface source elements such as A, B, C in Fig. 3(a) have to be calculated and then summed. That is, the surface Green's functions at A, B, C and the other surface elements have to be separately computed. In the absence of a simple analytical formula, this would be a tedious and laborious effort. On the other hand, if the adjoint formulation is used, the situation is different. Since the source point is now in the far field, the sound waves from the far field source near the closed surface are plane waves. The adjoint Green's function is, therefore, the solution of a simple plane wave scattering problem by the closed surface. The scattering problem needs to be computed only once. By the reciprocity relation, the values of the direct Green's function for radiation from every point on the closed surface to the far field direction is found simultaneously in one single calculation of the scattered wave solution.

The objective of this investigation is to develop an adjoint Green's function method to perform the continuation of the near acoustic field of a jet to the far field. The method has been extended to apply to broadband noise sources such as that of high-speed jets. The method has also been validated experimentally by using measured jet near and far acoustic field data. Space limitation does not allow the inclusion of these materials in this paper. Interested readers may consult the recent work of Tam *et al* [20], [21].

2. The Continuation Problem

Consider the acoustic field generated by some localized noise sources as shown in Fig. 4. The sound field is governed by the compressible Navier-Stokes equation. Suppose the solution is known. To identify that it is the solution of the Navier-Stokes equations, a subscript NS will be added to the pressure and velocity field, i.e. p_{NS} and $\mathbf{v}_{NS}$.

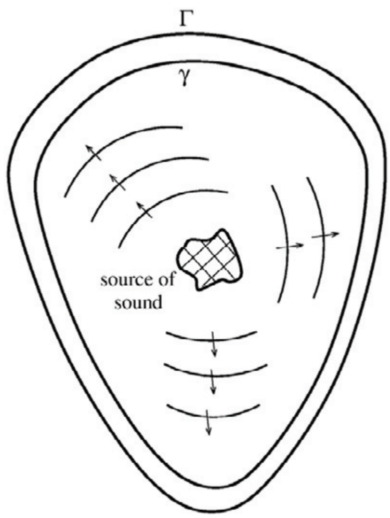

Figure 4. Sound generated by a localized source.

Now let γ be a convex surface enclosing the sources and the near sound field. It will be assumed that γ is far enough away from the sources that the disturbances at γ are, for all intents and purpose, linear and inviscid. Under these conditions, the equations of motion outside γ are the linearized Euler Equations,

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} + \nabla p = 0 \tag{2}$$

$$\frac{\partial p}{\partial t} + \gamma p_0 \nabla \bullet \boldsymbol{v} = 0 \tag{3}$$

Thus, in the region outside γ , $p_{NS}, \boldsymbol{v}_{NS}$ and the solution of the linearized Euler equations, denoted by $p_{Euler}, \boldsymbol{v}_{Euler}$ are practically the same, namely $p_{NS} \longrightarrow p_{Euler}, \boldsymbol{v}_{NS} \longrightarrow \boldsymbol{v}_{Euler}$.

Suppose Γ is a closed convex surface enclosing γ as shown in Fig. 4. On Γ, $p = p_{Euler}$ is known. Now as a first step toward establishing a way to continue the solution from surface Γ to the far field, consider the following initial boundary value problem in the space outside Γ. The governing equations are the linearized Euler equations (2) and (3). The boundary and initial conditions are,

$$\text{on } \Gamma, \qquad p = p_{Euler} \tag{4}$$

$$\text{As } |\boldsymbol{x}| \longrightarrow \infty, \qquad p, \boldsymbol{v} \text{ behave like outgoing waves} \tag{5}$$

$$\text{At } t = 0, \qquad p = p_{Euler}(\boldsymbol{x},0), \quad \boldsymbol{v} = \boldsymbol{v}_{Euler}(\boldsymbol{x},0) \tag{6}$$

By eliminating $\boldsymbol{v}$ from Eqs. (2) and (3), the equation for p is the simple wave equation,

$$\frac{\partial^2 p}{\partial t^2} - a_0^2 \nabla^2 p = 0 \tag{7}$$

Now, it is known that the solution of the simple wave equation (7) satisfying boundary conditions (4) and (5) and initial conditions (6) is unique. But the original solution p_{NS}, $\boldsymbol{v}_{NS}$ which become p_{Euler}, $\boldsymbol{v}_{Euler}$ outside Γ is also a solution of Eq. (7) and boundary and initial conditions (4) to (6). Therefore, the two solutions must be equal. In other words, the continuation solution from surface Γ to the far field is given by the solution of Eqs. (2) to (6). Hence a way to continue a near field solution to the far field is to solve the initial, boundary value problem stated above. How to construct such a solution is the subject of this paper

Instead of specifying $p = p_{Euler}$ on Γ as boundary condition, the solution of the simple wave equation with $\frac{\partial p}{\partial n} = \frac{\partial p_{Euler}}{\partial n}$ (the normal derivative) specified as boundary condition on Γ is also unique. By Eq. (2), the normal derivative of p is related to the velocity component in the normal direction as follows,

$$\nabla p \bullet \hat{n} = -\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} \bullet \hat{n} = -\rho_0 \frac{\partial v_n}{\partial t} \tag{8}$$

However, if $v_n(\boldsymbol{x},t)$ is known on Γ, then $\frac{\partial v_n}{\partial t}$ is known. Therefore, it is possible to extend or continue a solution beyond surface Γ by solving Eqs. (2) and (3) with appropriate initial conditions and matching v_n to $v_{n,Euler}$ on boundary Γ. In other words, there are two ways to continue a solution from surface Γ to the far field. The first way is to match p on Γ. The second way is to match v_n on Γ.

It is worthwhile to note that the solution to initial boundary value problem defined by Eqs. (2) to (6) consists of two parts. One part is associated with the initial conditions and zero boundary values on Γ. The other part is zero initial conditions and non-zero boundary value on Γ. In most problems, interest is on the second part of the solution at large time. At large time, the transient solution (the first part) would propagate away. So for long time solution, it is possible to ignore the initial condition altogether. In the rest of this chapter, attention is focused only on continuing the solution from surface Γ without reference to initial conditions.

3. Surface Green's Function: Pressure as Matching Variable

Eqs. (2) and (3) are linear. The solution of these equations satisfying boundary condition $p = p_{Euler}$ on surface Γ can be found by means of a Green's function. For clarity, a superscript 'g' is used to denote the variables of the Green's function. Let (ξ,η,ς) be a set of body fitted curvilinear coordinates. The surface Γ corresponds to $\varsigma = \varsigma_0$. The Green's function $\left(\boldsymbol{v}^{(g)}, p^{(g)}\right)$ satisfies the following boundary value problem.

$$\rho_0 \frac{\partial \boldsymbol{v}^{(g)}}{\partial t} + \nabla p^{(g)} = 0 \tag{9}$$

$$\frac{\partial p^{(g)}}{\partial t} + \gamma p_0 \nabla \bullet \boldsymbol{v}^{(g)} = 0 \tag{10}$$

On Γ, i.e. $\varsigma = \varsigma_0$

$$p^{(g)}\left(\boldsymbol{x},t;\xi_0,\eta_0,\varsigma_0,t_0\right) = \delta\left(\xi-\xi_0\right)\delta\left(\eta-\eta_0\right)\delta\left(t-t_0\right) \tag{11}$$

[Note: The notation that the first set of spatial and time argument of the Green's function represents the location and time of the observer while the second set represents that of the source is adopted here].

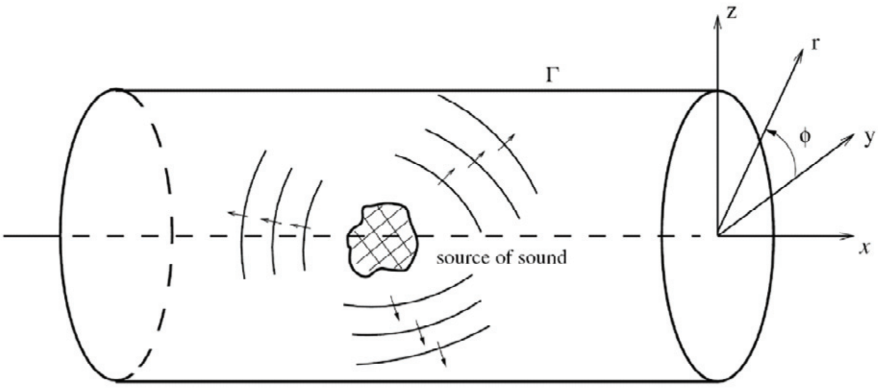

Figure 5. Source of sound enclosed by an infinite cylindrical surface of diameter D.

When the Green's function is found, the far field pressure is given by,

$$p(\boldsymbol{x},t) = \int_{-\infty}^{\infty} \iint_{\Gamma} p^{(g)}\left(\boldsymbol{x},t;\xi_0,\eta_0,\varsigma_0,t_0\right) p_{Euler}\left(\xi_0,\eta_0,\varsigma_0,t_0\right) d\xi_0 d\eta_0 dt_0 \tag{12}$$

As an example on the use of surface Green's function, the case of Γ in the form of an infinite circular cylindrical surface of diameter D (see Fig. 5) is considered. The cylindrical coordinates (r,ϕ,x) are the natural body fitted coordinates of this problem. By eliminating $\boldsymbol{v}^{(g)}$ from Eqs. (9) and (10) in favor of $p^{(g)}$ and upon applying Fourier transform to t, the time variable, the governing equation for $\tilde{p}^{g}$ (the Fourier transform of $p^{(g)}$), is found to be,

$$\frac{\partial^2 \tilde{p}^{(g)}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{p}^{(g)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{p}^{(g)}}{\partial \phi^2} + \frac{\partial^2 \tilde{p}^{(g)}}{\partial x^2} + \frac{\omega^2}{a_0^2}\tilde{p}^{(g)} = 0 \tag{13}$$

At $r = D/2$,

$$\tilde{p}^{(g)} = \frac{\delta(x-x_0)\delta(\phi-\phi_0)e^{i\omega t_0}}{\pi D} \tag{14}$$

Now, by applying Fourier transform to x (denoting the transform variable by k) in Eqs. (13) and (14) and then expanding the solution in a Fourier series in ϕ, the Green's function $\tilde{p}^{(g)}$ can be found in terms of the Hankel function. To obtain the Green's function in physical space and time, the inverse Fourier transform is performed. For radiation to the far field, it is advantageous to switch from cylindrical coordinates (r,ϕ,x) to spherical polar coordinates (R,θ,ϕ). The coordinates are related by,

$$x = R\cos\theta, \quad r = R\sin\theta$$

For large R, one may use the asymptotic form of the Hankel function. In addition, the inverse Fourier transform may be evaluated by the method of stationary phase. In this way, it is straightforward to obtain,

$$\underset{R\to\infty}{p^{(g)}}(R,\theta,\phi,t;x_0,\phi_0,t_0) = \frac{1}{2\pi^3 DR}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{e^{i\frac{\omega}{a_0}(R-x_0\cos\theta)-i(n+1)\frac{\pi}{2}+in(\phi-\phi_0)-i\omega(t-t_0)}}{H_n^{(1)}\left(\frac{\omega D}{2a_0}\sin\theta\right)}d\omega \tag{15}$$

where $H_n^{(1)}(z)$ is the n^{th} Hankel function of the first kind.

As a simple demonstration that Eq. (15) is the correct surface Green's function, the case of a sound field produced by a time periodic monopole of angular frequency Ω located at the origin of coordinates is considered. The pressure field is,

$$p(R,t) = \frac{e^{i\left(\frac{R}{a_0}-t\right)\Omega}}{4\pi R} \tag{16}$$

where R is the radial distance. On the cylindrical surface of diameter D, the polar distance is $R = \left(\frac{D^2}{4}+x_0^2\right)^{1/2}$. Thus the fluctuating surface pressure is given by,

$$p(x_0,\phi_0,t_0) = \frac{e^{i\left[\frac{\left(D^2/4+x_0^2\right)^{1/2}}{a_0}-t_0\right]\Omega}}{4\pi\left(D^2/4+x_0^2\right)^{1/2}} \tag{17}$$

By inserting $p^{(g)}$ from Eq. (15) and $p(x_0,\phi_0,t_0)$ from Eq. (17) into Eq. (12), the far field pressure due to surface pressure on the cylindrical surface is,

$$p(R,\theta,\phi,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{2\pi}\frac{1}{8\pi^4 DR}\sum_{n=-\infty}^{\infty}\frac{e^{i\frac{\omega}{a_0}(R-x_0\cos\theta)-i(n+1)\frac{\pi}{2}+in(\phi-\phi_0)-i\omega(t-t_0)+i\left(D^2/4+x_0^2\right)^{1/2}\frac{\Omega}{a_0}-it_0\Omega}}{\left(D^2/4+x_0^2\right)^{1/2}H_n^{(1)}\left[\frac{\omega D}{2a_0}\sin\theta\right]}\frac{D}{2}d\phi_0 dt_0 d\omega dx_0 \tag{18}$$

The right side of Eq. (18) may be integrated step by step as follows. Integration over $d\phi_0$ is zero except for $n = 0$. In this case, the integral is 2π. Integration over dt_0 gives $2\pi\delta(\omega-\Omega)$. Therefore, on integrating over $d\omega$, the right side simplifies to,

$$p(R,\theta,\phi,t) = \frac{1}{4\pi^2 R}\int_{-\infty}^{\infty}\frac{e^{i\frac{\Omega}{a_0}(R-x_0\cos\theta)+i\left(D^2/4+x_0^2\right)^{\frac{1}{2}}\frac{\Omega}{a_0}-i\frac{\pi}{2}-i\Omega t}}{\left(\frac{D^2}{4}+x_0^2\right)^{1/2}H_0^{(1)}\left(\frac{\Omega D}{2a_0}\sin\theta\right)}dx_0 \tag{19}$$

Now, from the extensive compilation of Fourier integrals of Erdelyi *et al* [22], the integral of Eq. (19) can be evaluated to yield,

$$p(R,\theta,\phi,t) = \frac{e^{i\Omega\left(\frac{R}{a_0}-t\right)}}{4\pi R} \tag{20}$$

Thus, the monopole sound field at large R is recovered.

4. Surface Green's Function: Normal Velocity as the Matching Variable

Instead of using surface pressure as the variable for continuation to the far field, an equally good variable to use is the velocity component normal to the surface or the normal pressure gradient. If this choice is made, the appropriate surface Green's function is given by solution of the following problem (a superscript 'G' will be used to denote this Green's function),

$$\rho_0 \frac{\partial \boldsymbol{v}^{(G)}}{\partial t} + \nabla p^{(G)} = 0 \tag{21}$$

$$\frac{\partial p^{(G)}}{\partial t} + \gamma p_0 \nabla \bullet \boldsymbol{v}^{(G)} = 0 \tag{22}$$

On surface Γ, i.e. $\varsigma = \varsigma_0$, the boundary condition is,

$$v_n^{(G)}\left(\boldsymbol{x},t;\xi_0,\eta_0,\varsigma_0,t_0\right) = \delta(\xi-\xi_0)\delta(\eta-\eta_0)\delta(t-t_0) \tag{23}$$

To find $v_n^{(G)}\left(\boldsymbol{x},t;\xi_0,\eta_0,\varsigma_0,t\right)$, it is advantageous to introduce a velocity potential $\Phi^{(G)}$ defined by,

$$\boldsymbol{v}^{(G)} = \nabla\Phi^{(G)}, \qquad p^{(G)} = -\rho_0 \frac{\partial \Phi^{(G)}}{\partial t} \tag{24}$$

Eq. (24) satisfies Eq. (21) identically. Substitution of Eq. (24) into Eq. (22), the governing equation for $\Phi^{(G)}$ is found to be,

$$\frac{1}{a_0^2}\frac{\partial^2 \Phi^{(G)}}{\partial t^2} - \nabla^2 \Phi^{(G)} = 0 \tag{25}$$

The surface boundary condition on $\varsigma = \varsigma_0$ is found by inserting expression (24) into Eq. (23). This yields,

$$\frac{\partial \Phi^{(G)}}{\partial n} = \delta(\xi-\xi_0)\delta(\eta-\eta_0)\delta(t-t_0) \tag{26}$$

where $\hat{\boldsymbol{n}}$ is the unit outward pointing normal of surface Γ.

Suppose $\Phi^{(G)}\left(\boldsymbol{x},t;\xi_0,\eta_0,\varsigma_0,t_0\right)$ is found. The pressure in the far field is given by,

$$p(\boldsymbol{x},t) = \int\int\int p^{(G)}\left(\boldsymbol{x},t;\xi_0,\eta_0,t_0\right) v_n\left(\xi_0,\eta_0,t_0\right) d\xi_0 d\eta_0 dt_0 \tag{27}$$

where $p^{(G)}$ is to be computed by Eq. (24).

To illustrate the construction and use of surface Green's function $p^{(G)}$, consider again the case of Γ in the form of an infinite circular cylindrical surface as shown in Fig. (5). The governing equation for $\Phi^{(G)}$ is Eq. (25). This equation and surface boundary condition (26) may be solved by using the same method as in the previous section. The surface Green's function may be written out as,

$$\underset{R\to\infty}{p}^{(G)}\left(R,\theta,\phi,t;\phi_0,x_0,t_0\right) = \frac{i\rho_0 a_0}{2\pi^3 RD} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\left[\frac{\omega}{a_0}(R-x_0\cos\theta)-\omega(t-t_0)+m(\phi-\phi_0)-(m+1)\frac{\pi}{2}\right]}}{H_m'^{(1)}\left(\frac{\omega D}{2a_0}\sin\theta\right)\cdot\sin\theta} d\omega \tag{28}$$

where $H_m'^{(1)}(z)$ is the derivative of the m^{th} order Hankel function of the first kind.

By following the steps of the previous section, it is possible to demonstrate for the case of a monopole source that the far pressure field can be recovered by integrating the normal pressure gradient on the cylindrical surface weighted by Green's function of Formula (29).

5. The Adjoint Green's Function

The Fourier transform of Eqs. (9), (10) and boundary condition (11) in time *t* are,

$$-i\omega\rho_0\tilde{\boldsymbol{v}}^{(g)} + \nabla\tilde{p}^{(g)} = 0\,, \tag{29}$$

$$-i\omega\tilde{p}^{(g)} + \gamma p_0 \nabla\cdot\tilde{\boldsymbol{v}}^{(g)} = 0\,, \tag{30}$$

On Γ or $\varsigma = \varsigma_0$,

$$\tilde{p}^{(g)}\left(\boldsymbol{x},\omega;\xi_0,\eta_0,\varsigma_0,\tau\right) = \frac{1}{2\pi}\delta\left(\xi-\xi_0\right)\delta\left(\eta-\eta_0\right)e^{i\omega\tau}\,. \tag{31}$$

To find the adjoint system of equations to Eq. (29) and (30), multiply Eq. (29) by $\boldsymbol{v}^{(a)}$ and Eq. (35) by $p^{(a)}$ (the superscript '*a*' denotes the adjoint) and integrate over all space outside Γ. This yields, after rearranging the terms,

$$\iiint\limits_{\substack{space\\ outside\ \Gamma}} [-i\omega\rho_0\boldsymbol{v}^{(a)}\cdot\tilde{\boldsymbol{v}}^{(g)} + \nabla\cdot\left(\tilde{p}^{(g)}\boldsymbol{v}^{(a)}\right) - \tilde{p}^{(g)}\nabla\cdot\boldsymbol{v}^{(a)} - i\omega\tilde{p}^{(g)}p^{(a)} + \gamma p_0\nabla\cdot\left(p^{(a)}\tilde{\boldsymbol{v}}^{(g)}\right)$$
$$-\gamma p_0\tilde{\boldsymbol{v}}^{(g)}\cdot\nabla p^{(a)}]\; dxdydz = 0 \tag{32}$$

The two divergence terms in Eq. (32) may be integrated by means of the Divergence theorem to become surface integrals over Γ. This leads to,

$$\iiint\limits_{outside\ \Gamma}\left[-i\omega\rho_0\boldsymbol{v}^{(a)} - \gamma p_0\nabla p^{(a)}\right]\cdot\tilde{\boldsymbol{v}}^{(g)}\; dxdydz + \iiint\limits_{outside\ \Gamma}\left[-i\omega p^{(a)} - \nabla\cdot\boldsymbol{v}^{(a)}\right]\tilde{p}^{(g)}\; dxdydz$$
$$-\iint\limits_{surface\ \Gamma}\left[\tilde{p}^{(g)}\boldsymbol{v}^{(a)} + \gamma p_0 p^{(a)}\tilde{\boldsymbol{v}}^{(g)}\right]\cdot\hat{\boldsymbol{n}}\; dS = 0 \tag{33}$$

where $\hat{\boldsymbol{n}}$ is the unit outward pointing normal of surface Γ.

Now, the adjoint system is chosen to satisfy the following equations and boundary conditions,

$$-i\omega\rho_0\boldsymbol{v}^{(a)} - \gamma p_0\nabla p^{(a)} = 0 \tag{34}$$

$$-i\omega p^{(a)} - \nabla\cdot\boldsymbol{v}^{(a)} = \frac{1}{2\pi}\delta\left(\boldsymbol{x}-\boldsymbol{x}_1\right)e^{i\omega\tau} \tag{35}$$

On Γ or $\varsigma = \varsigma_0$,

$$p^{(a)} = 0 \tag{36}$$

By means of the above choice of the adjoint system, the integrals of Eq. (33) may be easily evaluated. This gives the reciprocity relation,

$$\tilde{p}^{(g)}\left(\boldsymbol{x}_1;\xi_0,\eta_0,\varsigma_0;\omega,\tau\right) = v_n^{(a)}\left(\xi_0,\eta_0,\varsigma_0;\boldsymbol{x}_1;\omega,\tau\right) \tag{37}$$

where $v_n^{(a)} = \boldsymbol{v}^{(a)}\cdot\hat{\boldsymbol{n}}$ is the component of the adjoint velocity in the direction of outward pointing unit normal $\hat{\boldsymbol{n}}$. Thus, once the adjoint problem is solved, the direct surface Green's function on the entire surface is found.

6. Summary and Conclusion

In this paper, a different method for extending the near acoustic field of a jet to the far field is briefly discussed. The method is based on the use of an adjoint Green's function. The adjoint Green's function and the direct surface Geen's function are related by a reciprocity relation.

The present method differs from the popular Kirchhoff integral method and the method of Ffowcs-Williams and Hawkings equation in one important way. The present method requires the prescription of either the surface pressure or the normal derivative of the pressure on the matching surface but not both. The Kirchhoff as well as the Ffowcs-

Williams and Hawkings method require the prescription of both variables on the matching surface. Because of this important difference, the present method has a definite advantage if experimentally measured near acoustic field data is to be extended to the far field. Further details and applications of the proposed method can be found in Ref. [20] and [21].

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