Linearized Oscillation Theorems for Certain Nonlinear Delay Partial Difference Equations

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Abstract—This paper is concerned with the nonlinear delay partial difference equation

\[ A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}f(A_{m-k,n-l}) = 0, \]

where \( k \) and \( l \) are nonnegative integers and \( m, n = 0, 1, \ldots \). Some linearized oscillation theorems for this equation are obtained, i.e., we will prove that under some conditions on \( f \), this equation has the same oscillatory character as an associated linear equation. An existence result for positive solutions of this equation is obtained also.

Keywords—Partial difference equations, Linearized oscillation theorems, Nonlinear.

1. INTRODUCTION

Partial difference equations are difference equations that involve functions of two or more independent integer variables. Such equations arise from considerations of random walk problems, the study of molecular orbits [1], mathematical physics problems [2], and the numerical difference approximation problems [3]. In this paper, we consider a nonlinear delay partial difference equation

\[ A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}f(A_{m-k,n-l}) = 0, \quad m, n = 0, 1, \ldots, \]

where \( f \in C(R, R) \), \( p_{m,n} \geq 0 \) on \( N_0^2 \) and \( k, l \in N_1 \). The notation \( N_i \) is used to denote the ray \( \{i, i + 1, \ldots\} \) of integers. Difference equation (1.1) can be obtained from the difference approximation of a class of the delay partial differential equation of the form [3]

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + g(x, y, u(x, y), u(x, y - \sigma_1), u(x - \tau_1, y), u(x - \tau_2, y - \sigma_2)) = 0. \]

The oscillation of the delay partial differential equation (1.2) has been investigated by Tramov [4]. Recently, the oscillation problem of (1.1) has been investigated in [5-8].

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A solution \( \{A_{i,j}\} \) of (1.1) is said to be eventually positive if \( A_{i,j} > 0 \) for all large \( i \) and \( j \). It is said to be oscillatory if it is neither eventually positive nor eventually negative.

In this paper, we will show some linearized oscillation theorems for (1.1), i.e., we will show that under some assumptions on \( f(1.1) \) has the same oscillatory character as an associated linear equation. Next, an existence result for positive solutions of (1.1) is obtained also. To the best of our knowledge, there are no results for the linearized oscillation and the existence of positive solution of (1.1) up to now.

2. MAIN RESULTS

We consider (1.1) together with the linear equation

\[
A_{m+1,n} + A_{m,n+1} - A_{m,n} + pA_{m-k,n-l} = 0, \quad (2.1)
\]

where \( k \) and \( l \) are positive integers and \( p > 0 \).

A solution \( \{A_{i,j}\} \) of (2.1) is said to be proper, if there exist positive constants \( M, \alpha, \) and \( \beta \) such that

\[
|A_{m,n}| \leq M\alpha^m\beta^n, \quad \text{for all large } m \text{ and } n.
\]

The first lemma is taken from [7].

**Lemma 2.1.** The following statements are equivalent.

(a) Every proper solution of (2.1) oscillates.

(b) The characteristic equation

\[
\lambda + \mu - 1 + p\lambda^{-k}\mu^{-l} = 0 \quad (2.2)
\]

has no positive roots.

(c) \( p > \frac{k^{k+l}}{(k+l+1)^{k+l+1}} \).

(\( \lambda, \mu \)) is said to be a positive root of (2.2) if it satisfies (2.2) and \( \lambda > 0, \mu > 0 \).

**Lemma 2.2.** Assume that every proper solution of (2.1) oscillates. Then there exists \( \epsilon_0 \in (0, p) \), such that for each \( \epsilon \in [0, \epsilon_0] \), every proper solution of the equation

\[
A_{m+1,n} + A_{m,n+1} - A_{m,n} + (p - \epsilon)A_{m-k,n-l} = 0 \quad (2.4)
\]

also oscillates.

**Proof.** By Lemma 2.1, every proper solution of (2.1) oscillates if and only if (2.2) has no positive roots. We claim that

\[
\lambda + \mu - 1 + (p - \epsilon)\lambda^{-k}\mu^{-l} - 0 \quad (2.5)
\]

has no positive roots. Clearly, (2.5) has no positive roots on the region \( \lambda + \mu \geq 1 \). In the following, we only need to prove that (2.5) has no positive roots in the region \( 0 < \lambda + \mu < 1 \). Let

\[
F(\lambda, \mu) = \lambda + \mu - 1 + p\lambda^{-k}\mu^{-l}.
\]

Since (2.2) has no positive roots, it follows that \( F(\lambda, \mu) > 0 \) for \( (\lambda, \mu) \in (0, \infty) \times (0, \infty) \). By direct calculation, \( F(\lambda, \mu) \) reaches its minimum value \( m \) at

\[
\lambda^* = \left( (l^{-1})^{kp^{l+1}} \right)^{1/(k+l+1)}, \quad \mu^* = \left( kp^{(\lambda^*)^{-k-1}} \right)^{1/l},
\]

i.e.,

\[
\min_{\lambda > 0, \mu > 0} F(\lambda, \mu) = F(\lambda^*, \mu^*) = m.
\]
where
\[ m = \left( \frac{k+1}{k} \right) \left( \frac{(t^{i/(t+k+1)}p_{l/(k+l+1)}k(t+1)/(k+l+1)}{1} \right) + \frac{1}{(k+l+1)}k^{-k/(k+l+1)}t^{(k+1)/(k+l+1)} - 1. \]

Due to (2.3), it is not difficult to see that \( m > 0 \). Let
\[ G(s,t) = s + t - 1 + \frac{1}{2}ps^{-k}t^{-l} \]
and \( \alpha = \min[(p/2)^{1/k}, (p/2)^{1/l}, (p/2)^{1/(k+l)}] \). Choose \( \epsilon_0 \in (0, p/2) \), such that \( \epsilon_0 \alpha^{-(k+l)} \leq m/2 \).

Then,
\[ \lambda + \mu - 1 + (p - \epsilon)\lambda^{-k}mu^{-l} > \lambda + \mu - 1 + \frac{p}{2} \lambda^{-k}mu^{-l} > -1 + \frac{p}{2} \alpha^{-(k+l)} \geq 0, \]
for \( 0 < \lambda \leq \alpha, \quad 0 < \mu \leq \alpha. \)

For \( \lambda > \alpha, \mu > \alpha \), we have
\[ \lambda + \mu - 1 + (p - \epsilon)\lambda^{-k}mu^{-l} \geq F(\lambda, \mu) - \epsilon \lambda^{-k}mu^{-l} \geq F(\lambda, \mu) - \epsilon \alpha^{-(k+l)} \geq \frac{m}{2} > 0. \]

For \( 0 < \lambda \leq \alpha, \quad 1 > \mu \geq \alpha \), we have
\[ \lambda + \mu - 1 + (p - \epsilon)\lambda^{-k}mu^{-l} > \lambda + \mu - 1 + \frac{p}{2} \lambda^{-k}mu^{-l} > \lambda - 1 + \frac{p}{2} \alpha^{-k} > 0. \]

Similarly, \( \lambda + \mu - 1 + (p - \epsilon)\lambda^{-k}mu^{-l} > 0 \), for \( 1 > \lambda \geq \alpha, \quad 0 < \mu \leq \alpha \), i.e., we have proved that (2.5) has no positive roots. By Lemma 2.1, every proper solution of (2.4) is oscillatory. The proof is complete.

**Theorem 2.1.** Assume that

(i) \( \liminf_{m,n \to \infty} p_{m,n} = p > 0 \),
(ii) \( (f(x))/x > 0 \) for \( x \neq 0 \) and \( \lim_{z \to 0} (f(z))/x = 1. \)

Then every proper solution of (2.1) oscillates implies that every solution of (1.1) oscillates.

**Proof.** Suppose to the contrary that \( \{A_{m,n}\} \) is an eventually positive solution of (1.1). Then there exist \( m_0 \) and \( n_0 \), such that \( A_{m,n} > 0 \), for \( m \geq m_0, n \geq n_0 \). We can show that \( \lim_{m,n \to \infty} A_{m,n} = 0, \lim_{m \to \infty} A_{m,n} = 0, \) and \( \lim_{n \to \infty} A_{m,n} = 0. \) Otherwise, let \( \lim_{m,n \to \infty} A_{m,n} = d > 0. \)

Taking the limit in (1.1), we obtain \( d + pf(d) \leq 0 \), which is a contradiction. Similarly, we can prove that \( \lim_{m \to \infty} A_{m,n} = 0 \) and \( \lim_{n \to \infty} A_{m,n} = 0. \) Let
\[ \bar{p}_{m,n} = \frac{f(A_{m-k,n-l})}{A_{m-k,n-l}}. \]

Then \( \liminf_{m,n \to \infty} \bar{p}_{m,n} = p. \) For each \( \epsilon \in (0, \epsilon_0) \), there exist \( M > m_0 \) and \( N > n_0 \), such that \( \bar{p}_{m,n}p - \epsilon, \) for \( m \geq M, \quad n \geq N. \) Therefore,
\[ A_{m+1,n} + A_{m,n+1} - A_{m,n} + (p - \epsilon)A_{m-k,n-l} \leq 0, \quad \text{(2.6)} \]
for \( m \geq M, \quad n \geq N. \)

Summing (2.6) in \( n \) from \( n \geq N \) to \( \infty \), we have
\[ \sum_{i=n}^{\infty} A_{m+1,i} - A_{m,n} + (p - \epsilon) \sum_{i=n}^{\infty} A_{m-k,i-l} \leq 0. \]
We rewrite the above inequality in the form

$$A_{m+1,n} - A_{m,n} + \sum_{i=n+1}^{\infty} A_{m+1,i} + (p - \varepsilon) \sum_{i=n}^{\infty} A_{m-k,i-l} \leq 0. \quad (2.7)$$

Summing (2.7) in $m$ from $m(\geq M)$ to $\infty$, we get

$$-A_{m,n} + \sum_{j=m}^{\infty} \sum_{i=n+1}^{\infty} A_{j+1,i} + (p - \varepsilon) \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} A_{j-k,i-l} \leq 0.$$ 

Thus,

$$A_{m,n} \geq \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} + (p - \varepsilon) \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} A_{j-k,i-l}, \quad \text{for } m \geq M, \ n \geq N. \quad (2.8)$$

Define the set of real double sequences

$$X = \{ \{B_{m,n}\} : 0 \leq B_{m,n} \leq 1, m \geq M - k, n \geq N - l \}$$

and an operator $T$ on $X$ by

$$(TB)_{m,n} = \begin{cases} \frac{1}{A_{m,n}} \left[ \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} B_{j,i} + (p - \varepsilon) \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} A_{j-k,i-l} B_{j-k,i-l} \right], & \text{for } m \geq M, n \geq N, \\ 1, & \text{otherwise}. \end{cases} \quad (2.9)$$

In view of (2.8), we see that $TX \subset X$. Define $\{B_{m,n}^{(r)}\}, i = 0, 1, \ldots$, as follows:

$$B_{m,n}^{(0)} = 1, \quad B_{m,n}^{(r)} = (TB)_{m,n}^{(r-1)}, \quad r = 1, 2, \ldots.$$ 

By induction and (2.8), we can prove that

$$B_{m,n}^{(0)} \geq B_{m,n}^{(1)} \geq \cdots \geq B_{m,n}^{(r)} \geq B_{m,n}^{(r+1)} \geq \cdots,$$

for $(m, n) \in [M-k, \infty) \times [N-l, \infty)$. Thus, the limit $B_{m,n} = \lim_{r \to \infty} B_{m,n}^{(r)}$ exists and

$$B_{m,n} = \begin{cases} \frac{1}{A_{m,n}} \left[ \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} B_{j,i} + (p - \varepsilon) \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} A_{j-k,i-l} B_{j-k,i-l} \right], & \text{for } m \geq M, n \geq N, \\ 1, & \text{otherwise}. \end{cases}$$

Clearly, $B_{m,n} > 0$ for $m \geq M-k, n \geq N-l$. Set $x_{m,n} = A_{m,n} B_{m,n}$. Then, $x_{m,n} > 0, m \geq M-k, n \geq N-l$, and

$$x_{m,n} = \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} x_{j,i} + (p - \varepsilon) \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} x_{j-k,i-l}, \quad \text{for } m \geq M, \ n \geq N.$$ 

From the last equation, we get

$$x_{m+1,n} - x_{m,n} = -\sum_{i=n+1}^{\infty} x_{m+1,i} + (p - \varepsilon) \sum_{i=n}^{\infty} x_{m-k,i-l}$$

or

$$x_{m,n} = \sum_{i=n}^{\infty} x_{m+1,i} + (p - \varepsilon) \sum_{i=n}^{\infty} x_{m-k,i-l}. $$
Summing the above equation in \( n(\geq N) \), we have

\[
x_{m,n+1} - x_{m,n} = -x_{m+1,n} - (p - \epsilon)x_{m-k,n-i},
\]

i.e., (2.4) has a positive solution. In view of \( x_{i,j} \leq A_{i,j} \) for all large \( i \) and \( j \), \( \{x_{i,j}\} \) is a proper solution. By Lemma 2.2, (2.1) has a positive proper solution, which is a contradiction. The proof is complete.

**Theorem 2.2.** Assume that

(i) \( 0 \leq p_{m,n} \leq p \),

(ii) there exists a positive number \( h \) such that \( f(x) \) is nondecreasing in \( x \in [-h, h] \) and

\[
0 \leq \frac{f(x)}{x} \leq 1, \quad \text{for } 0 < |x| \leq h. \quad (2.10)
\]

Suppose (2.1) has a positive proper solution, then (1.1) also has a positive proper solution.

**Proof.** If (2.1) has a positive proper solution, by Lemma 1.1, its characteristic equation

\[
\lambda + \mu - 1 + p \lambda^{-k} \mu^{-l} = 0
\]

has a positive root \((\lambda, \mu)\) with \( 0 < \lambda < 1, 0 < \mu < 1 \), and \( \{\lambda^m \mu^n\} \) is a positive solution of (2.1). Choose \( a > 0 \) such that

\[
A_{m,n} = a \lambda^m \mu^n < h, \quad \text{for all } m \geq -k, \quad n \geq -l.
\]

Clearly, \( \{A_{m,n}\} \) is a positive solution of (2.1) and satisfies \( f(A_{m,n}) \leq A_{m,n} \). Similar to (2.8), summing (2.1), we can get

\[
A_{m,n} = \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} + p \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} A_{j-k,i-l}, \quad m \geq 0, \quad n \geq 0,
\]

and hence,

\[
A_{m,n} \geq \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} p_{j,i} f(A_{j-k,i-l}), \quad m \geq 0, \quad n \geq 0.
\]

Define a set by

\[
X = \{ \{B_{m,n}\} : 0 \leq B_{m,n} \leq 1, \quad m \geq -k, \quad n \geq -l \}
\]

and a mapping \( T \) on \( X \) by

\[
(TB)_{m,n} = \left\{ \frac{1}{A_{m,n}} \left[ \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} B_{j,i} + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} p_{j,i} f(A_{j-k,i-l} B_{j-k,i-l}) \right] \right\}, \quad \text{for } m \geq 0, n \geq 0,
\]

otherwise.

Clearly, \( TX \subset X \). Similar to (2.9), it is easy to prove that there exists \( \{B_{m,n}\} \), such that \( (TB)_{m,n} = B_{m,n} \) for \( m \geq 0, n \geq 0 \), i.e.,

\[
B_{m,n} = \left\{ \frac{1}{A_{m,n}} \left[ \sum_{j=m+1}^{\infty} \sum_{i=n+1}^{\infty} A_{j,i} B_{j,i} + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} p_{j,i} f(A_{j-k,i-l} B_{j-k,i-l}) \right] \right\}, \quad \text{for } m \geq 0, n \geq 0,
\]

otherwise.
It is easy to see that $B_{m,n} > 0$. Set $x_{m,n} = A_{m,n} B_{m,n}$, then

$$x_{m,n} = \sum_{j=m+1}^{\infty} \sum_{i=m+1}^{\infty} x_{j,i} + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} p_{j,i} f(x_{j-k,i-l}), \quad m \geq 0, \quad n \geq 0.$$  

As before, the above equation implies that $\{x_{m,n}\}$ is a positive proper solution of (1.1). The proof is complete.

Combining Theorems 2.1 and 2.2, we obtain the following corollary.

**COROLLARY 2.1.** Assume that $p_{m,n} \equiv p > 0$, (ii) of Theorem 2.1 and (ii) of Theorem 2.2 hold. Then every solution of (1.1) oscillates if and only if every proper solution of (2.1) oscillates.

**COROLLARY 2.2.** Assume that (ii) of Theorem 2.2 holds and

$$0 \leq p_{m,n} \leq \frac{k^{k+1}}{(k+1)^{k+1}}.$$  

Then, (1.1) has a positive solution.

**REMARK.** The above results can be extended to the more general equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_i(m,n)f_i(A_{m-k_i,n-l_i}) = 0.$$  

**REFERENCES**