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The Descriptive Complexity of Brownian Motion

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A continuous function x on the unit interval is a *generic Brownian motion* when every probabilistic event which holds almost surely with respect to the Wiener measure is reflected in x, provided that the event has a suitably effective description. We show that a generic one-dimensional Brownian motion can be computed from an infinite binary string which is complex in the sense of Kolmogorov–Chaitin. Conversely, one can construct a Kolmogorov–Chaitin random string from the values at the rational numbers of a generic Brownian motion. In this way, we construct a recursive isomorphism between encoded versions of generic Brownian motions and Kolmogorov–Chaitin random reals.

1. INTRODUCTION

In this paper, we explore the two notions of a "generic" point on the unit interval and a "generic" Brownian motion. We show how the process of identifying a generic point of the unit interval can be unfolded, as it were, or blown up, to a generic Brownian motion and, conversely, how a generic Brownian motion can be enfolded, or contracted, to a generic point of the unit interval. We shall make this dual process very explicit and it will be depicted in an algorithmic manner.

With a point of [0, 1] one can associate an infinite binary string α in the standard manner. The string α encodes the process of locating the point through repeated bisections. We call the point generic if the binary string α, when it is considered as a sequence of outcomes of a fair coin-tossing experiment, meets all the recursive statistical tests in the sense of Martin-Löf [16]. In a similar vein, a Brownian motion is said to be generic if it meets all recursive tests, now expressed in terms of the statistical events associated with Brownian motion on the unit interval. (See Section 3 for precise definitions.) We shall refer to a generic Brownian motion as a *complex oscillation*. This terminology is suggested by the following fact [2, 7]: One can characterise a Brownian motion which is generic (in the

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sense just stated) as an effective and uniform limit of a sequence (x_n) of "finite random walks," where, moreover, each x_n can be encoded by a finite binary string s_n of length n, such that the Kolmogorov complexity, $K(s_n)$, of s_n satisfies, for some constant $d > 0$, the inequality $K(s_n) > n - d$ for all values of n. In the following, we shall denote the set of complex oscillations by $\mathscr{C}$. It was shown by Asarin and Prokovskiy [2] that the set $\mathscr{C}$ has Wiener measure 1. In [7] one finds a recursive characterisation of the almost sure events, with respect to Wiener measure, which are reflected in every complex oscillation. (For details, see Section 3.)

It is well-known that that the notion of an infinite binary string meeting all the recursive statistical tests of Martin-Löf can also be expressed in terms of Kolmogorov–Chaitin complexity. Indeed, for such a string α, there is a constant $d > 0$, such that, writing $\bar{\alpha}(n)$ for the first n bits of α, it is the case that $K(\bar{\alpha}(n)) > n - d$ for all n, and conversely. We call a string α with this property a KC-string (KC for Kolmogorov–Chaitin). (See, for example, [3] and [14] for early work on this notion. Extensive treatments of this theory can be found in [4] and [22]. A philosophical analysis of descriptive complexity versus randomness appears in [21].)

In this paper, we shall show that, in a recursion-theoretic sense, the class of complex oscillations can be identified with the class KC consisting of the infinite binary strings which are complex in the sense of Kolmogorov–Chaitin. We construct a bijection $\Phi\colon \mathrm{KC} \to \mathscr{C}$ which is effective in the following sense: If $\alpha \in \mathrm{KC}$ and $m < \omega$, one can effectively construct from the first m bits of α a function p_m, where p_m is a finite linear combination of piecewise linear functions such that, for some absolute positive constant C, the complex oscillation $\Phi(\alpha)$ is approximated by the sequence (p_m) as

$$\sup_{t \in [0,1]} |\Phi(\alpha)(t) - p_m(t)| \leqslant C \log m/\sqrt{m},$$

for all $m > M$, where M is a constant that depends on α only. Conversely, if $x \in \mathscr{C}$, then one can compute, relative to an infinite binary string which encodes the values of x at the dyadic rational numbers in the unit interval, the KC-string α such that $\Phi(\alpha) = x$.

Consider any effective enumeration $t_0, t_1, \ldots$, without repetition, of the dyadic rationals in the unit interval. To every $x \in C[0,1]$ one can associate the $\omega \times \omega$ array having the dyadic expansion of $x(t_i)$ as its ith row. By using any recursive bijection between ω^2 and ω, one can represent the array associated to the continuous function x as a single binary string, $E(x)$, say. Set $\mathscr{E} = \{E(x)\colon x \in \mathscr{C}\}$. It will be shown that the map Φ induces a partial recursive $\phi\colon \{0,1\}^\omega \to \{0,1\}^\omega$ which maps KC bijectively to $\mathscr{E}$ in such a way that for some partial recursive $\psi\colon \{0,1\}^\omega \to \{0,1\}^\omega$, the restriction of ψ to $\mathscr{E}$ is the inverse to ϕ. In this sense the sets KC and $\mathscr{E}$ are recursively isomorphic. This has the immediate implication that ϕ defines a homeomorphism between KC and $\mathscr{E}$, when both these spaces are viewed as subspaces of the Baire space $\{0, 1\}^{\omega}$. The mapping Φ is also a measure-theoretic isomorphism in the following sense: Write λ for the Lebesgue measure on the space $\{0, 1\}^{\omega}$ and write W for the Wiener measure on $C[0, 1]$. Then, for any Borel subset A of $C[0, 1]$ with the uniform norm topology, we have

$$\lambda(\Phi^{-1}(A)) = W(A).$$

It follows from the theory in [7] that many of the local properties of Brownian motion, such as almost sure nowhere differentiability and Levy's modulus of continuity theorem, are reflected in every complex oscillation. Moreover, each complex oscillation has interesting recursive properties. For example, it is shown in [7] that if $x \in \mathscr{C}$ and $t \in (0, 1)$ is a recursive real number, then $x(t)$ is not recursive. In [8] it is shown that for each $x \in \mathscr{C}$ there is a dense subset D of $[0, 1]$ which consists of the so-called rapid points of x, i.e., if $t \in D$ then

$$\overline{\lim_{h \to 0}} \frac{|x(t+h) - x(t)|}{\sqrt{|h| \log 1/|h|}} > 0.$$

Moreover, all the points in D are *not* recursive real numbers.

The large extent to which each complex oscillation is representative of Brownian motion opens the way towards introducing descriptive complexity as an additional explanatory metaphor to situations where the theory of Brownian motion has proven successful in leading to an understanding of the phenomena involved. This means that one can explore the implications of the assumption that some basic processes in physics, for example, have descriptions which are intrinsically complex. In particular, one can study the stochastic dynamical approach to quantum theory by Fènyes [5] and Nelson [17] from this point of view. This leads to the problem of representing Gaussian white noise in terms of Kolmogorov–Chaitin complexity. The main result of this paper can be viewed as a first step towards constructing such a theory. This line of thought will be elaborated upon in a sequel to this paper. It would also be interesting to study the functional integration approach to quantum theory as exposed in [10], for example, from this point of view.

The author has made every effort to make this paper reasonably self-contained. In particular, all the results from its predecessor [7] are stated in such a way that this paper can be read independently from [7] in a coherent manner.

2. PRELIMINARIES

The set of non-negative integers is denoted by ω and we write $\mathcal{N}$ for the product space $\{0, 1\}^{\omega}$. The set of words over the alphabet $\{0, 1\}$ is denoted by $\{0, 1\}^*$. If $a \in \{0, 1\}^*$, we write $|a|$ for the length of a. If $\alpha = \alpha_0 \alpha_1 \cdots$ is in $\mathcal{N}$, we write $\bar{\alpha}(n)$ for the word $\prod_{j<n} \alpha_j$. We use the usual recursion-theoretic terminology Σ_r^0 and Π_r^0 for the arithmetical subsets of $\omega^k \times \mathcal{N}^l$, $k, l \geqslant 0$. (See, for example, [12].) We write λ for the Lebesgue probability measure on $\mathcal{N}$. For a binary word s of length n, say, we write $[s]$ for the "interval" $\{\alpha \in \mathcal{N} : \bar{\alpha}(n) = s\}$. A sequence (a_n) of real numbers converges *effectively* to 0 as $n \to \infty$ if for some total recursive $f: \omega \to \omega$ it is the case that $|a_n| \leqslant (m+1)^{-1}$ when $n \geqslant f(m)$. A subset A of $\mathcal{N}$ is of *constructive* measure 0 if there is a total recursive $\phi: \omega^2 \to \{0, 1\}^*$ such that $A \subset \bigcap_n \bigcup_m [\phi(n, m)]$, where $\lambda(\bigcup_m [\phi(n, m)])$ converges effectively to 0 as $n \to \infty$.

For any binary word a we denote its Kolmogorov complexity by $K(a)$. One can think of $K(a)$ as the shortest self-delimiting program for a universal Turing machine U which will output a from an empty input. We assume that U accepts self-delimiting programs only. It is well-known that if K_1, K_2 corresponds to universal Turing machines U_1, U_2, then $K_1(a) = K_2(a) + O(1)$ for all a. (See [3] or [21] for a discussion.) In the following, we shall regard our choice of K as fixed. An infinite binary string α is Kolmogorov–Chaitin complex if

$$\exists_d \forall_n \, K(\bar{\alpha}(n)) \geqslant n - d.$$

In the following, we shall denote this set by KC and refer to its elements as KC-strings.

Remark. Algorithmic descriptive complexity is now usually referred to as *Kolmogorov complexity*. With the early developments from the 1960s in mind, *Solomonoff–Chaitin–Kolmogorov* complexity is perhaps the most appropriate term for the notion. Our terminology is closer to what is found in the standard references [4, 22] for the subject.

The core result of the theory of Kolmogorov–Chaitin complexity is the following:

THEOREM 1. *If $\alpha \in \mathcal{N}$, then $\alpha \in$ KC iff α is in the complement of every subset of $\mathcal{N}$ which is of constructive measure* 0.

For a discussion, see [3, 15, 16, 20, 21].

The mean, or expected, value of a random variable X will be denoted by $E(X)$. Two random variables X and Y on possibly different probability spaces are said to be similar when they have the same probability distributions. We write $X \sim Y$ in this case. A random variable X with mean μ and variance σ^2 is normal if it has a density function

$$\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(t-\mu)^2/2\sigma^2}.$$

We say in this case that X has a $N(\mu, \sigma^2)$ distribution. A sequence (ξ_n) of random variables is called a *normal sequence* when the sequence is statistically independent and if each ξ_n has an $N(0, 1)$ distribution. A random variable X is called *Gaussian* when $X \sim \lambda\xi$ where ξ is an $N(0, 1)$ variable and $\lambda \geqslant 0$. A random vector $X = (X_1, ..., X_n)$ is said to be a Gaussian vector if every linear combination of the X_i is a Gaussian random variable. One can show that a random vector $X \in \mathbf{R}^n$ is Gaussian iff there is a finite normal sequence $\xi_1, ..., \xi_m$ such that each of the components X_i of X is similar to a linear combination of the components ξ_j of the normal sequence. (See, for example, pp. 169–170 of [13].) If X is Gaussian, its characteristic function is given by

$$E(e^{iu.X}) = e^{-\psi(u)/2},$$

where ψ is the quadratic form

$$\psi(u) = \sum_j E(X_j^2)\, u_j^2 + 2 \sum_{i>j} E(X_i X_j)\, u_i u_j.$$

(See p. 170 of [13].) This has the implication that the distribution of a Gaussian vector X is fully and uniquely determined by its *correlation matrix* which is, by definition, the matrix $(E(X_i X_j)\colon 1 \leqslant i, j \leqslant n)$.

A Brownian motion on the unit interval is a real-valued function $(\omega, t) \mapsto X_\omega(t)$ on $\Omega \times [0, 1]$, where Ω is the underlying space of some probability space, such that $X_\omega(0) = 0$ a.s., and for $t_1 < \cdots < t_n$ in the unit interval the random variables $X_\omega(t_1), X_\omega(t_2) - X_\omega(t_1), ..., X_\omega(t_n) - X_\omega(t_{n-1})$ are statistically independent and normally distributed with means all 0 and variances $t_1, t_2 - t_1, ..., t_n - t_{n-1}$, respectively. We say in this case that the Brownian motion is *parametrised* by Ω. Alternatively, a map $X\colon \Omega \times [0, 1] \to \mathbf{R}$ defines a Brownian motion iff for $t_1 < \cdots < t_n$ in the unit interval the random vector $(X_\omega(t_1), ..., X_\omega(t_n))$ is Gaussian with correlation matrix $(\min(t_i, t_j)\colon 1 \leqslant i, j \leqslant n)$.

It is a fundamental fact that any Brownian motion has a "continuous version." This means the following: Write Σ for the σ-algebra of Borel sets of $C[0, 1]$ where the latter is topologised by the uniform norm topology.

There is a unique probability measure W on Σ such that for $0 \leqslant t_1 < \cdots < t_n \leqslant 1$ and for a Borel subset B of $\mathbf{R}^n$ we have

$$P(\{\omega \in \Omega : (X_\omega(t_1), ..., X_\omega(t_n)) \in B\})$$
$$= W(\{x \in C[0, 1] : (x(t_1), ..., x(t_n)) \in B\}).$$

(See, for example, pp. 46–50 of [11].) The measure W is known as the *Wiener measure*. We shall usually write $X(t)$ instead of $X_\omega(t)$.

For the purposes of this paper, it will be useful to think of Brownian motion on the unit interval as a continuous curve in some real Gaussian Hilbert space H. This approach to the theory was probably inspired by Kolmogorov's notion of what he called "Wienersche Spiralen" in [14]. For a classic overview of later developments, see [18]. All the background for the purposes of this paper can be found in Chapter 16 of the book [13].

In order to construct a Gaussian Hilbert space, one requires a probability space $(\Omega, P, \mathscr{A})$ on which a normal sequence (ξ_n) of random variables can be defined. It is then clear that, with respect to the (real) Hilbert space $L^2(\Omega)$, the sequence (ξ_n) is an orthonormal sequence. The real Hilbert subspace H of $L^2(\Omega)$ spanned by the orthonormal sequence (ξ_n) is an example of a Gaussian Hilbert space. In fact, any Gaussian Hilbert space can be obtained in this manner. If $X \in H$, it has an expansion in H of the form $X = \sum_n a_n \xi_n$. In this case $X \sim a\xi$ where a is the ℓ^2 norm of the sequence (a_n) and ξ is an $N(0, 1)$ random variable. This follows immediately from computing the characteristic function of X. Note that $a = \|X\|$. An important consequence of this remark is that for any $X_1, ..., X_n \in H$, the vector $(X_1, ..., X_n)$ is Gaussian.

The basic idea of the Hilbert space approach to Brownian motion is to start with a helical curve (or spiral) in $L^2[0, 1]$ and to transform this curve to the real Gaussian Hilbert space $H \subset L^2(\Omega)$ by means of a linear isometry from $L^2[0, 1]$ to H. In this way, one obtains a Brownian motion parametrised by Ω. To be more precise, let $T: L^2[0, 1] \to H$ be any linear isometry and set, writing χ_t for the characteristic function of the interval $[0, t]$, for each t in the unit interval, $X(t) = T(\chi_t)$. Note that for $t, s \in [0, 1]$, we have that $E(X(t)\, X(s)) = (\chi_t, \chi_s) = \min(t, s)$. It follows that the stochastic process $(X(t): t \in [0, 1])$ is a Brownian motion. For future reference we note that for a Borel set A in $C[0, 1]$, the measure $W(A)$ is related to the probability measure P on Ω as follows:

$$W(A) = P(\{\omega \in \Omega : \text{the function } t \mapsto X_\omega(t) \text{ is in } A\}). \qquad (1)$$

This approach to Brownian motion results in very explicit analytic representations of $X(t)$ which are most suitable for the recursive analysis of Brownian motion in this paper. Indeed, let (e_n) be an orthonormal basis of

$L^2[0, 1]$ and set $\xi_n = T(e_n)$, where T is the linear isometry from $L^2[0, 1]$ to the real Gaussian space H. Then writing

$$\chi_t = \sum_n a_n(t)\, e_n$$

(the sum in $L^2[0, 1]$), it follows that we have the explicit representation

$$X(t) = \sum_n a_n(t)\, \xi_n$$

(the sum now in H), where

$$a_n(t) = (\chi_t, e_n) = \int_0^t e_n(s)\, ds.$$

Note that the sequence (ξ_n) is orthonormal.

3. COMPLEX OSCILLATIONS

We next survey the results from [2] and [7] which will play an important role in this paper. The key idea in these papers is that of a so-called complex oscillation, which is a limit of a sequence of finitary random walks of growing Kolmogorov complexity, which is in a definite sense also a generic Brownian motion. We first introduce some notation. For $n \geqslant 1$, we write C_n for the class of continuous functions on the unit interval that vanish at 0 and are linear with slopes $\pm\sqrt{n}$ on the intervals $[(i-1)/n, i/n]$, $i = 1, ..., n$. With every $x \in C_n$, one can associate a binary string $a = a_1 \cdots a_n$ by setting $a_i = 1$ or $a_i = 0$ according to whether x increases or decreases on the interval $[(i-1)/n, i/n]$. We call the sequence a the code of x and denote it by $c(x)$. The following notion was introduced by Asarin and Prokovskiy in [2].

Definition 1. A sequence (x_n) in $C[0, 1]$ is complex if $x_n \in C_n$ for each n and there is a constant $d > 0$ such that $K(c(x_n)) \geqslant n - d$ for all n. A function $x \in C[0, 1]$ is a complex oscillation if there is a complex sequence (x_n) such that $\|x - x_n\|$ converges effectively to 0 as $n \to \infty$.

The class of complex oscillations is denoted by $\mathscr{C}$. It was shown by Asarin and Prokovskiy [2] that the class $\mathscr{C}$ has Wiener measure 1.

For the results in this paper, we shall require a recursive characterisation of the almost sure events, with respect to Wiener measure, which are reflected in each complex oscillation. In order to describe this characterisation, we use, as in [7], an analogue of a Π_2^0 subset of $C[0, 1]$ which is of constructive measure 0. We introduce some notation. If F is a subset of $C[0, 1]$, we denote by $\bar{F}$ the topological closure of F in $C[0, 1]$. For $\varepsilon > 0$, we let $O_\varepsilon(F)$ be the set $\{f \in C[0, 1] : \exists_{g \in F} \|f - g\| < \varepsilon\}$. For convenience sake, we write F^0 for the complement of F and F^1 for F.

DEFINITION 2. A sequence $\mathscr{F}_0 = (F_i : i < \omega)$ in Σ is an effective generating sequence if

1. for $F \in \mathscr{F}_0$, $\varepsilon > 0$, and $\delta \in \{0, 1\}$, we have, for $G = O_\varepsilon(F^\delta)$ or $G = F^\delta$, that $W(\bar{G}) = W(G)$;
2. there is an efffective procedure that yields, for each sequence $0 \leqslant i_1 < \cdots < i_n < \omega$ and $k < \omega$, a binary rational number β_k such that

$$|W(F_{i_1} \cap \cdots \cap F_{i_n}) - \beta_k| < 2^{-k};$$

3. for $n, i < \omega$, a strictly positive rational number ε and $x \in C_n$, both the relations $x \in O_\varepsilon(F_i)$ and $x \in O_\varepsilon(F_i^0)$ are recursive in x, ε, i, and n.

If $\mathscr{F}_0 = (F_i : i < \omega)$ is an effective generating sequence and $\mathscr{F}$ is the algebra generated by $\mathscr{F}_0$, then there is an enumeration $(T_i : i < \omega)$ of the elements of $\mathscr{F}$ (with possible repetition) in such a way, for a given i, that one can effectively describe T_i as a finite union of sets of the form

$$F = F_{i_1}^{\delta_1} \cap \cdots \cap F_{i_n}^{\delta_n}$$

where $0 \leqslant i_1 < \cdots < i_n$ and $\delta_i \in \{0, 1\}$ for each $i \leqslant n$. We call any such sequence $(T_i : i < \omega)$ a *recursive enumeration* of $\mathscr{F}$. We say in this case that $\mathscr{F}$ is *effectively generated* by $\mathscr{F}_0$ and refer to $\mathscr{F}$ as an *effectively generated algebra* of sets. A sequence (A_n) of sets in $\mathscr{F}$ is said to be $\mathscr{F}$-*semi-recursive* if it is of the form $(T_{\phi(n)})$ for some total recursive function $\phi : \omega \to \omega$ and some effective enumeration (T_i) of $\mathscr{F}$. (Note that the sequence (A_n^c), where A_n^c is the complement of A_n, is also an $\mathscr{F}$-semirecursive sequence.) In this case, we call the union $\bigcup_n A_n$ a $\Sigma_1^0(\mathscr{F})$ set. A set is a $\Pi_1^0(\mathscr{F})$ set if it is the complement of a $\Sigma_1^0(\mathscr{F})$ set. It is of the form $\bigcap_n A_n$ for some $\mathscr{F}$-semirecursive sequence (A_n). A sequence (B_n) in $\mathscr{F}$ is a *uniform* sequence of $\Sigma_1^0(\mathscr{F})$ sets if, for some total recursive function $\phi : \omega^2 \to \omega$ and some effective enumeration (T_i) of $\mathscr{F}$, each B_n is of the form

$$B_n = \bigcup_m T_{\phi(n, m)}.$$

In this case, we call the intersection $\bigcap_n B_n$ a $\Pi_1^0(\mathscr{F})$ set. If, moreover, the W-measure of B_n converges *effectively* to 0 as $n \to \infty$, we say that the set given by $\bigcap_n B_n$ is a $\Pi_1^0(\mathscr{F})$ set of constructive measure 0.

The proof of the following theorem appears in [7].

THEOREM 2. *Let $\mathscr{F}$ be an effectively generated algebra of sets. If x is a complex oscillation, then x is in the complement of every $\Pi_1^0(\mathscr{F})$ set of constructive measure* 0.

We shall also make frequent use of the following result from [7].

THEOREM 3. *If B is a $\Sigma_1^0(\mathscr{F})$ set and $W(B)=1$, then $\mathscr{C}$, the set of complex oscillations, is contained in B.*

The analogue of this result, for Lebesgue measure, appears in [6]. The following result from [7] is an effective version of the Borel–Cantelli lemma for the Wiener measure. The analogue, for Lebesgue measure, appears in [19].

THEOREM 4. *If (A_k) is a uniform sequence of $\Sigma_1^0(\mathscr{F})$ sets with $\sum_k W(A_k) < \infty$, then, for each complex oscillation x, it is the case that $x \notin A_k$ for all large values of k.*

We introduce a class of effective generating sequences which is most useful for reflecting local properties of one-dimensional Brownian motion into complex oscillations. Let $\mathscr{G}_0$ be a family of sets in Σ each having a description of the form

$$a_1 X(t_1) + \cdots + a_n X(t_n) \leqslant L \tag{2}$$

or of the form (2) with $\leqslant$ replaced by $<$, where all the a_j, t_j $(0 \leqslant t_j \leqslant 1)$ are rational numbers, L is a recursive real number, and X is one-dimensional Brownian motion. If $\varepsilon > 0$ and $G \in \Sigma$ is described by (2), we have that $O_\varepsilon(G)$ is described by the inequality

$$a_1 X(t_1) + \cdots + a_n X(t_n) < L + \varepsilon \sum_j |a_j| \tag{3}$$

while $O_\varepsilon(G^0)$ is given by

$$a_1 X(t_1) + \cdots + a_n X(t_n) > L - \varepsilon \sum_j |a_j|. \tag{4}$$

We require that it be possible to find an enumeration $(G_i\colon i < \omega)$ of $\mathscr{G}_0$ such that, for given i, if G_i is given by (2), we can effectively compute the sign, the denominators, and the numerators of the rational numbers a_j, t_j and, moreover, that recursive real L can be computed up to arbitrary accuracy. This has the implication that there is an effective procedure, Π, such that, for given i, ε, m with $i, m < \omega$ and ε a positive rational, the validity of (3) and (4) can be decided by Π when G_i is given by (2) and $X \in C_m$.

We now show that under these conditions $\mathcal{G}_0 = (G_i \colon i < \omega)$ is an effective generating sequence in the sense of Definition 2. It is clear that condition (3) of Definition 2 is met. The topological closures of events having descriptions of the form (3) or (4) have similar descriptions with the strict inequality signs replaced by weak ($\leqslant$ and $\geqslant$) inequality signs. Now we can effectively rewrite the left-hand side Y, say, of (2) as a linear combination of $X(t_1)$ and the differences $X(t_j) - X(t_{j-1})$ for $j = 2, ..., n$. Indeed,

$$\sum_{i=1}^{n} a_i X(t_i) = \left(\sum_{i=1}^{n} a_i \right) X(t_1) + \sum_{j=1}^{n-1} \left(\sum_{j < i \leqslant n} a_i \right) (X(t_{j+1}) - X(t_j)).$$

Hence Y has the same distribution as a sum of the form $\tau_1 \xi_1 + \cdots + \tau_n \xi_n$ where the τ_i are reals and $\xi_1, ..., \xi_n$ are independent $N(0, 1)$ random variables. It follows that the distribution of Y is absolutely continuous with respect to Lebesgue measure. Consequently, for $\varepsilon \geqslant 0$ and $G \in \mathcal{G}_0$, it is the case that for $\delta \in \{0, 1\}$ and $H = G^\delta$ the W-measure of $O_\varepsilon(H)$ and $\overline{O_\varepsilon(H)}$ are the same. It follows that condition (1) of Definition 2 is met. Finally, for G of the form $G_{i_1} \cap \cdots \cap G_{i_m}$, we can write $W(G)$ as the probability of an event of the form $A\xi \in H$ where A is a finite matrix and ξ is a vector whose components are independent $N(0, 1)$ random variables; moreover H is an intersection of halfspaces in some finite dimensional Euclidean space. The entries in the matrix A and the description of the halfspaces can be effectively retrieved from the description of G. If ξ has n components, say, then ξ has a density function given by

$$(2\pi)^{-n/2}\, e^{-(y_1^2 + \cdots + y_n^2)/2}.$$

It follows that $W(G)$ can be computed up to arbitrary accuracy. This takes care of the second condition of Definition 2. This concludes the proof that $\mathcal{G}_0$ is an effective generating sequence. In the following, we call a sequence $\mathcal{G}_0$ meeting with these requirements a *Gaussian sequence* and the algebra $\mathcal{G}$ generated by $\mathcal{G}_0$, a *Gaussian algebra.*

As an illustration of how one can reflect almost sure properties of Brownian motion in each complex oscillation, we now prove the following

PROPOSITION 1. *If $x \in \mathcal{C}$ and if $C > 1$, then*

$$|x(t+h) - x(t)| \leqslant \sqrt{2C\, |h| \log(1/|h|)}$$

for all t in the unit interval and all sufficiently small values of h.

Proof. For $n \geqslant 1$ and a fixed rational number $C > 1$, write K_n^C for the event over $C[0, 1]$ defined by

$$\forall_{|h| < 1/n}\, \forall_{t \in [0, 1]}\, |x(t+h) - x(t)| \leqslant \sqrt{2CF(h)},$$

where $F(h) = |h| \log(1/|h|)$. It is quite easily shown that $W(K_n^C) \geqslant s(n)$, where

$$s(n) = \exp\left(-2 \left(\frac{1}{n}\right)^{C-1} \right),$$

provided that $C > 1$. (See pp. 97–98 of [1], for example.) Write D for the set of dyadic rationals in the unit interval. Let L_n^C be the set in $C[0, 1]$ defined by the predicate

$$\exists_h \exists_{t \in D} (|h| \in D) \wedge \left(|h| < \frac{1}{n} \right) \wedge (|x(t+h) - x(t)| > \sqrt{2CF(h)}).$$

Note that (L_n^C) is a uniform sequence of $\Sigma_1^0(\mathcal{G})$ sets for a suitable Gaussian algebra $\mathcal{G}$. Moreover, a simple continuity argument shows that the complement of L_n^C in $C[0, 1]$ is given by K_n^C. It follows that

$$W(L_n^C) \leqslant 1 - s(n).$$

Consequently, $\bigcap_n L_n^C$ is a $\Pi_2^0(\mathcal{G})$ set of constructive measure 0. We conclude from Theorem 2 that, if $x \in \mathscr{C}$, then $x \in K_n^C$ for some n, which concludes the proof of the proposition.

Remark. It follows that if $x \in \mathscr{C}$, then the modulus of continuity around a point t is $\ll \sqrt{|h| \log(1/|h|)}$. This result is the best possible. Indeed, one can show that there is a dense set of points t in the unit interval such that

$$\overline{\lim_{h \to 0}} \frac{|x(t+h) - x(t)|}{\sqrt{|h| \log(1/|h|)}} > 0.$$

These are the so-called rapid points of x. The rapid points of a complex oscillation are all non-recursive reals, for one can show that if t is a recursive real, then

$$\overline{\lim_{h \to 0}} \frac{|x(t+h) - x(t)|}{\sqrt{2 |h| \log \log(1/|h|)}} = 1.$$

These results are discussed in [8].

4. KC STRINGS VIEWED AS COMPLEX OSCILLATIONS

In this section we show how one compute a complex oscillation from any KC-string. Our argument is based on the so-called Franklin–Wiener representation of Brownian motion. Such a representation is obtained by mapping the Haar functions in $L^2([0, 1])$ to a normal sequence in a real Gaussian Hilbert space by means of a linear isometry.

For a subset I of the unit interval, we write $\chi(I)$ for the characteristic function of I. The Haar system in $L^2([0, 1])$ is defined by

$$e_0 = 1, \qquad e_1 = \chi([0, 1/2)) - \chi([1/2, 1))$$

and

$$e_{jn} = \{\chi([n2^{-j}, n2^{-j} + 2^{-(j+1)})) - \chi([n2^{-j} + 2^{-(j+1)}, (n+1)\, 2^{-j}))\}\; 2^{j/2},$$

where $0 \leqslant n < 2^j$ and $j \geqslant 1$. Let $\Delta_0(t)$, $\Delta_1(t)$, $\Delta_{jn}(t)$ be the (zig-zag) functions obtained by integrating e_0, e_1, e_{jn} from 0 to t. It follows that each Δ_{jn} has support on the dyadic interval $(n2^{-j}, (n+1)\, 2^{-j})$ and satisfies

$$\|\Delta_{jn}\| \leqslant 2^{j/2} \times 2^{-(j+1)}.$$

Let $(\Omega, P, \mathscr{A})$ be a probability space on which a normal sequence $(\xi_0, \xi_1, \xi_{jn}\colon j \geqslant 1, 0 \leqslant n < 2^j)$ is defined. The curve $X\colon [0, 1] \to L^2(\Omega)$ given by

$$X(t) \sim \xi_0 \Delta_0(t) + \xi_1 \Delta_1(t) + \sum_{j \geqslant 1} \sum_{n < 2^j} \xi_{jn} \Delta_{jn}(t)$$

(the expansion in $L^2(\Omega)$) defines a version of Brownian motion. The series on the right is called the *Franklin–Wiener* series.

The idea that a version of Brownian motion can be defined which is parametrised by $\mathscr{N}$ goes back to Wiener. (See [23], for a discussion.) This idea will play an important role in what follows. We first introduce some notation. We define a function $b\colon (0, 1) \to \mathscr{N}$ by requiring that for $\alpha \in (0, 1)$ and $n \geqslant 1$, if s is the dyadic rational encoded by the first n bits of $b(\alpha)$, then $s \leqslant \alpha < s + 2^{-n}$. In the sequel we shall identify $\alpha \in (0, 1)$ with its binary representation.

Let $g\colon [0, 1] \to \mathbf{R}$ be the function defined by

$$\alpha = \int_{-\infty}^{g(\alpha)} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt, \qquad \alpha \in (0, 1). \tag{5}$$

Note that g is a recursive function, i.e., there is a uniform computation that outputs $g(\alpha)$ up to arbitrary accuracy using only a finite number of bits of α. Since the function $e^{-t^2/2}$ is invariant under the transformation $t \mapsto -t$, we have

$$g(\alpha) = -g(1 - \alpha)$$

for all $0 < \alpha < 1$.

We fix a recursive bijection $\langle , \rangle$ from ω^2 to ω. To any $\alpha \in \mathcal{N}$, we associate a sequence $B = (\beta_0, \beta_1, \beta_{jn}\colon j \geqslant 1,\ 0 \leqslant n < 2^j)$, where the sequence (β_{jn}) is lexicographically ordered with respect to the double indices jn, in such a way that the kth term of the sequence B is given by

$$\alpha_{k0}\alpha_{k1}\cdots.$$

Here we have written kl instead of $\langle k, l \rangle$. (We point out that we shall later impose further restrictions on the recursive bijection $\langle , \rangle$). For $1 \leqslant j < \omega$, $0 \leqslant n < 2^j$, set $\xi_{jn} = g(\beta_{jn})$; in addition, set $\xi_k = g(\beta_k)$, for $k = 0, 1$. We can view ξ_0, ξ_1, ξ_{jn} as functions defined on $\mathcal{N}$ and, with respect to the Lebesgue measure λ on the latter, they are independent $N(0, 1)$-random variables. In this way one can associate a Franklin–Wiener series x_α to each infinite binary string α, namely

$$x_\alpha(t) = \xi_0\, \Delta_0(t) + \xi_1\, \Delta_1(t) + \sum_{j<\omega} \sum_{n<2^n} \xi_{jn}\, \Delta_{jn}(t). \tag{6}$$

At this stage, x_α is a formal series. However, the association $\alpha \mapsto x_\alpha$ induces a map from $\mathcal{N}$ into $L^2[0, 1]$ and defines a version of Brownian motion parametrised by $\mathcal{N}$. Moreover, we shall soon see that for almost all α, the series converges and defines a continuous function in t on the unit interval.

Note that it follows from equation (1) that for any Borel subset A of $C[0, 1]$, we have:

$$W(A) = \lambda(\alpha \in \mathcal{N} : x_\alpha \in A),$$

where λ is the Lebesgue measure on $\mathcal{N}$.

Our first aim is to show that if $\alpha \in \mathrm{KC}$, then x_α will be a complex oscillation. Thereafter, we shall show precisely how one can compute finitary approximations to x_α from finite initial segments of α when $\alpha \in \mathrm{KC}$. In the following section we shall show that $\alpha \mapsto x_\alpha$ defines a bijection from KC onto $\mathscr{C}$ in such a way that we obtain a recursive isomorphism between KC and a set $\mathscr{E}$, which is an "encoded version" of $\mathscr{C}$. As a first step, we prove the following lemma.

Lemma 1. *If $\alpha \in \mathrm{KC}$, then the associated Franklin–Wiener series x_α converges and represents a continuous function on the unit interval.*

Proof. Associate with α the sequence ξ_0, ξ_1, ξ_{jn} and set, for $j \geqslant 1$,

$$a_j = \sup_{t \in [0, 1]} \left| \sum_{n < 2^j} \xi_{jn}\, \Delta_{jn}(t) \right|.$$

Since for each fixed j, the functions Δ_{jn} have disjoint supports, we have

$$a_j \leqslant 2^{-(j/2)-1} \sup\{|\xi_{jn}|: n < 2^j\}. \tag{7}$$

We claim that, if $\alpha \in \mathrm{KC}$, then, for all $j \geqslant j_\alpha$, we have

$$\sup\{|\xi_{jn}|: n < 2^j\} \leqslant 2\sqrt{j}. \tag{8}$$

This will suffice for the proof of the lemma, for it will follow from (8) that

$$a_j \leqslant \sqrt{j}\, 2^{-j/2}$$

for all large values of j and hence that the series x_α converges uniformly in $t \in [0, 1]$.

Define the event X_j by

$$X_j = [\sup_{n < 2^j} |\xi_{jn}| > 2\sqrt{j}].$$

Here we view each ξ_{jn} as a function on $\mathcal{N}$. This event holds with a probability (as measured by the Lebesgue measure) which is at most

$$2^j \int_{2\sqrt{j}}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt \leqslant \frac{2^j}{\sqrt{2\pi}} \int_{2\sqrt{j}}^{\infty} \frac{t}{2\sqrt{j}}\, e^{-t^2/2}\, dt.$$

It follows that

$$\mathrm{Prob}[a_j > \sqrt{j}\, 2^{-j/2}] \leqslant \mathrm{Prob}(X_j) \leqslant \frac{1}{2\sqrt{2\pi j}} (2/e^2)^j. \tag{9}$$

Fix again $\alpha \in \mathrm{KC}$ and assume that the inequality (8) does not hold for all but finitely many j. Then $\alpha \in X_j$ infinitely often, i.e.,

$$\alpha \in \bigcap_{m < \omega} \bigcup_{j > m} X_j.$$

Since the relation $\beta \in X_j$ is Σ_1^0 in β and j, it follows from the previous estimates that α is in a set of constructive measure 0—which is impossible. This concludes the proof of the lemma.

We note that it follows from the proof of the lemma that for some absolute positive constant C, if $\alpha \in \mathrm{KC}$, there is some $n_0 = n_0(\alpha)$, such that for all $m \geqslant n_0$, we have

$$\left| x_\alpha(t) - \left(\xi_0 \Delta_0(t) + \xi_1 \Delta_1(t) + \sum_{j < m} \sum_{n < 2^j} \xi_{jn} \Delta_{jn}(t) \right) \right| \leqslant C \frac{\sqrt{m}}{2^{m/2}} \tag{10}$$

uniformly in t. Indeed, in the notation in the proof of the previous lemma, for m sufficiently large, the left-hand side is bounded from above by

$$\sum_{j\geqslant m} a_j \leqslant \sum_{j\geqslant m} \frac{\sqrt{j}}{2^{j/2}}.$$

PROPOSITION 2. *If* $\alpha\in \mathrm{KC}$, *then* x_α *is a complex oscillation.*

Proof. Suppose there is some $\alpha\in\mathrm{KC}$ such that x_α is not a complex oscillation. It follows from the proof of Theorem 5.3 in [7] that

$$x_\alpha \in \bigcap_n \bigcup_m T_{n,m} \tag{11}$$

where each $T_{n,m}$ can be written as

$$\bigcap_{i<k} [A_i < X(t_i) < B_i], \tag{12}$$

the numbers A_i, B_i and t_i being dyadic rationals, in such a way that one can retrieve an effective description of the parameters k, A_i, B_i and t_i from n and m. In addition, $W(\bigcup_m T_{n,m})$ converges effectively to 0 as $n\to\infty$. We may assume, without loss of generality, that

$$W\left(\bigcup_m T_{n,m}\right)\leqslant 2^{-n}.$$

Write, for $\beta\in\mathcal{N}$ and $N<\omega$,

$$x^N_\beta(t)=\xi_0(\beta)\,\Delta_0(t)+\xi_1(\beta)\,\Delta_1(t)+\sum_{j<N}\sum_{n<2^j}\xi_{jn}(\beta)\,\Delta_{jn}(t).$$

For given n, m, we define the set $S_{n,m}\subset\mathcal{N}$ as follows: If $T_{n,m}$ is given by (12), then $S_{n,m}$ is defined by

$$\beta\in S_{n,m} \leftrightarrow \forall_{i<k}\,\exists_{N>n+m+1}(A_i+2^{-N/3}<x^N_\beta(t_i)<B_i-2^{-N/3}). \tag{13}$$

Note that the relation $\beta\in S_{n,m}$ is Σ^0_1 (i.e. semirecursive) in β, n and m.

For a given $\alpha\in\mathrm{KC}$, choose $n_0=n_0(\alpha)$, such that (10) holds for all $m\geqslant n_0$. We also require that $n_0\geqslant n_1$, where n_1 is some absolute constant to be determined below. Our first aim is to show that

$$\alpha\in\bigcap_{n\geqslant n_0}\bigcup_m S_{n,m}.$$

For given $n \geqslant n_0$, choose m such that $x_\alpha \in T_{n,m}$. If $T_{n,m}$ is given by (12), say, then, for some $L>0$ and all $i<k$, we have

$$A_i + \frac{1}{L} < x_\alpha(t_i) < B_i - \frac{1}{L}.$$

Note that $n+m+1>n_0$. Choose $N>n+m+1$ such that $C\sqrt{N}\,2^{-N/2} + 2^{-N/3} < L^{-1}$, where C is the constant that appears in (10). Then, since $N>n_0$, we have, by (10), that

$$x_\alpha(t_i) - C\frac{\sqrt{N}}{2^{N/2}} \leqslant x_\alpha^N(t_i) \leqslant x_\alpha(t_i) + C\frac{\sqrt{N}}{2^{N/2}}.$$

It now follows from our choice of N that

$$A_i + 2^{-N/3} < x_\alpha^N(t_i) < B_i - 2^{-N/3}$$

for all $k<i$. We conclude that $\alpha \in S_{n,m}$, as required.

We now claim that, if $\beta \in S_{n,m}$ where $n \geqslant n_0$ and, in the notation used in the proof of Lemma 1, if $a_j \leqslant \sqrt{j}\,2^{-j}$ for all $j \geqslant n+m+1$, then $x_\beta \in T_{n,m}$. Suppose $S_{n,m}$ is given by (13). For given i, choose $N>n+m+1$ such that $A_i + 2^{-N/3} < x_\beta^N(t_i) < B_i - 2^{-N/3}$. Since $a_j \leqslant \sqrt{j}\,2^{-j/2}$ for all $j \geqslant N$, we have

$$|x_\beta(t_i) - x_\alpha^N(t_i)| \leqslant \sum_{j \geqslant N} \frac{\sqrt{j}}{2^{j/2}} \leqslant C\frac{\sqrt{N}}{2^{N/2}}$$

and, consequently, writing

$$F(N) = 2^{-N/3} - C\sqrt{N}\,2^{-N/2},$$

we have

$$A_i + F(N) \leqslant x_\beta(t_i) \leqslant B_i - F(N).$$

For n_1 sufficiently large, $F(N)>0$ for all $n \geqslant n_1$. If we have chosen $n_0 \geqslant n_1$ for such a choice of n_1, we find that $x_\beta \in T_{n,m}$, as required. Consequently, if $n \geqslant n_0$, then

$$\left(\beta \in \bigcup_m S_{n,m}\right) \to \exists_m[(x_\beta \in T_{n,m}) \vee \beta \in U_{n,m}],$$

where

$$\beta \in U_{n,m} \leftrightarrow \exists_{j>n+m+1}(a_j(\beta) > \sqrt{j}\,2^{-j/2}).$$

But by (9), for each m,

$$\lambda(U_{n,m}) \leqslant \frac{1}{\sqrt{2\pi}} \sum_{j>n+m+1} \left(\frac{2}{e}\right)^j \frac{1}{e^j \sqrt{j}} \ll \left(\frac{2}{e}\right)^{n+m+1}.$$

Finally, since

$$\lambda\left(\bigcup_m S_{n,m}\right) \leqslant \lambda\left(\bigcup_m [\alpha : x_\alpha \in T_{n,m}]\right) + \sum_m \lambda(U_{n,m})$$

for $n \geqslant n_0$ and

$$\lambda\left(\bigcup_m [\alpha : x_\alpha \in T_{n,m}]\right) = \lambda\left(\left[\alpha : x_\alpha \in \bigcup_m T_{n,m}\right]\right) = W\left(\bigcup_m T_{n,m}\right),$$

we can conclude that

$$\lambda\left(\bigcup_m S_{n,m}\right) \ll \frac{1}{2^n} + \left(\frac{2}{e}\right)^n$$

which converges effectively to 0 as $n \to \infty$. We have shown that α is in a Π_2^0 set of constructive measure 0 –which contradicts the fact that $\alpha \in \mathrm{KC}$. We conclude that x_α is a complex oscillation, as required.

In the following theorem, we show how one can compute finitary approximations to the complex oscillation x_α from finite initial segments of α when $\alpha \in \mathrm{KC}$.

THEOREM 5. *Let α be an infinite binary string which is complex in the sense of Kolmogorov-Chaitin. Then one can effectively associate to α a complex oscillation x_α. Indeed, there is an oracle computation that computes, for given $m > 0$, by using the first m bits of α only, a natural number N and a sequence g_0, g_1, g_{jn}, where $j < N$, $n < 2^j$, of dyadic rationals such that, if we set*

$$p_m = g_0 \Delta_0 + g_1 \Delta_1 + \sum_{j<N} \sum_{n<2^j} g_{jn} \Delta_{jn},$$

then

$$\|x_\alpha - p_m\| \ll \frac{\log m}{\sqrt{m}},$$

for all $m \geqslant M_\alpha$.

Proof. Fix $\alpha \in \mathrm{KC}$. For the proof of this theorem we require a more explicit construction of the numbers $\beta_0, \beta_1, \beta_{jn}$ from a given $\gamma \in \mathcal{N}$ than was the case in the preceding results. This means essentially we need to place additional restrictions on the recursive bijection $\langle , \rangle$ from ω^2 to ω. For $k \geqslant 1$, set $L_k = 2^{k+1}$. This is the number of elements in the sequence $B_k := (\beta_0, \beta_1, \beta_{jn}: 1 \leqslant j \leqslant k,\ 0 \leqslant n < 2^j)$. We define the array B from the bits of γ in stages as follows: At stage 1 we use the first $4L_1$ bits of γ to define the first 4 bits of β_0, β_1, β_{10}, β_{11}, respectively, in some fixed systematic manner. At the end of the kth stage the first $4kL_k$ bits of γ have been placed in the first $4k$ positions of each of the elements of the sequence B_k. At stage $k+1$, take the next $4(k+1)\,L_k - 4kL_k$ bits of γ to fill up positions $4k+1, ..., 4k+4$ of each of the elements of B_k together with the first $4k+4$ positions of $(\beta_{k+1,n}: n < 2^{k+1})$. For our purposes, any systematic procedure that leads to a recursive computation of the array B from γ in such a way that, for all $k \geqslant 1$, the initial segment of size $4kL_k$ of γ will be used to determine the first $4k$ positions of all the elements of the sequence B_k, will suffice for the arguments that follow. In the sequel, we sometimes write $\beta_{jn}(\gamma)$ when we wish to emphasise that β_{jn} has been constructed from the binary string γ. For the fixed $\alpha \in \mathrm{KC}$, we shall always write β_{jn} in stead of $\beta_{jn}(\alpha)$. A similar convention will be followed for the other constructions ξ_{jn} from infinite binary strings γ.

We begin by showing that there is a natural number $N_0 = N_0(\alpha)$, such that, for all $N \geqslant N_0$, we have $\bar{\beta}_0(2N) \neq 0$, $\bar{\beta}_1(2N) \neq 0$ and $\bar{\beta}_{jn}(2N) \neq 0$ for all j, n with $1 \leqslant j < N$ and $0 \leqslant n < 2^j$.

To see this, let A_N be the event defined by

$$\gamma \in A_N \leftrightarrow [\overline{\beta_0(\gamma)}(2N) = 0] \vee [\overline{\beta_1(\gamma)}(2N) = 0] \vee \exists_{j<N} \exists_{n<2^j} [\overline{\beta_{jn}(\gamma)}(2N) = 0].$$

This event holds with a probability which satisfies

$$\mathrm{Prob}(A_N) \leqslant \frac{2}{2^{2N}} + \sum_{j<N} \frac{2^j}{2^{2N}} \ll 2^{-N}.$$

It follows that the set $\bigcap_N \bigcup_{j>N} A_j$ is of constructive measure 0 so that, α being in KC, we have $\alpha \in A_N$ for at most finitely many N.

Choose $N_1 = N_1(\alpha)$ such that for all $j \geqslant N_1$, we have $|\xi_{jn}| \leqslant 2\sqrt{j}$, for all $n < 2^j$. The existence from N_1 follows from the proof of Lemma 7. Next choose a natural number $L = L(\alpha)$ such that the function g^2 (where g is given by (5)) assumes values all of which are $\leqslant L$ on the finite set, Y, consisting of the points $\beta_0, \beta_1, \beta_{jn}$, where $j < N_1, n < 2^j$. (Note that none of the elements in Y equals zero or one, since they are all themselves KC-strings for it is well-known that any infinite recursive subsequence of a KC-string is itself a KC-string.)

The approximations p_m to x_α are constructed as follows: For $m \geqslant 16$, let N be the largest natural number such that $4NL_N \leqslant m$. Next compute, relative to α, the first N bits of $g(\bar{\beta}_k(4N))$, $k=0, 1$ and of $g(\bar{\beta}_{jn}(4N))$ $j<N$, $n<2^j$. Denote these dyadic rationals by g_0, g_1, g_{jn}. (It follows from the enumeration scheme introduced in the first paragraph of the proof that we can compute these numbers from $\leqslant m$ bits of α). Set

$$p_m = g_0 \Delta_0 + g_1 \Delta_1 + \sum_{j<N} \sum_{n<2^j} g_{jn} \Delta_{jn}.$$

We shall show that

$$\|x_\alpha - p_m\| \ll \log m/\sqrt{m},$$

for all $m \geqslant M_0$, where M_0 is a natural number depending on α only. As a first step, we show that, for $N \geqslant N_0(\alpha)$, we have, for $\beta=\beta_k$, $k=0, 1$ and $\beta=\beta_{jn}$, where $j<N$, $n<2^j$, that

$$|g(\beta) - g(\bar{\beta}(4N))| \ll_\alpha 2^{-N}. \tag{14}$$

We consider two cases:

Case 1. $e^{-t^2/2} \geqslant e^{-g^2(\beta)/2}$, for all $g(\bar{\beta}(4N)) \leqslant t \leqslant g(\beta)$. This is the same as to say that $g(\beta)>0$ and $g(\bar{\beta}(4N) > -g(\beta)$. In this case

$$|\beta - \bar{\beta}(4N)| = \frac{1}{\sqrt{2\pi}} \int_{g(\bar{\beta}(4N))}^{g(\beta)} e^{-t^2/2}\, dt \geqslant \frac{e^{-g^2(\beta)/2}}{\sqrt{2\pi}} |g(\beta) - g(\bar{\beta}(4N))|.$$

If $\beta=\beta_{jn}$, where $j \geqslant N_1$, then $g(\beta)=\xi_{jn} \leqslant 2\sqrt{j} \leqslant 2\sqrt{N}$ and, therefore,

$$|g(\beta) - g(\bar{\beta}(4N)| \ll e^{2N} |\beta - \bar{\beta}(4N)| \ll (e^2/2^4)^N \ll 2^{-N}$$

since $e^2<2^3$. If $\beta=\beta_0, \beta_1$ or β_{jn} with $j<N_1$, then

$$|g(\beta) - g(\bar{\beta}(4N))| \leqslant e^{L/2}\, |\beta - \bar{\beta}(4N)| \ll_\alpha 2^{-2N}.$$

Case 2. $e^{-t^2/2} \geqslant e^{-g^2(\bar{\beta}(4N))/2}$, for all $g(\bar{\beta}(4N)) \leqslant t \leqslant g(\beta)$. This is the same as to say that either $g(\beta)>0$ but $g(\bar{\beta}(4N)) < -g(\beta)$ or $g(\beta) \leqslant 0$. Note that for $u \geqslant 1$ and ξ a $N(0, 1)$ random variable,

$$\mathrm{Prob}(\xi \geqslant u) \ll e^{-u^2/2}.$$

(For a proof one can use the same argument leading to the final inequality in (9).) It follows that for $0<\gamma<1$, we have, when $g(\gamma) \geqslant 1$,

$$1-\gamma = \mathrm{Prob}(\xi \geqslant g(\gamma)) \ll e^{-g^2(\gamma)/2}$$

and hence, since $g(1-\gamma) = -g(\gamma)$, it follows, when $g(\gamma) \leqslant -1$, that

$$e^{g^2(\gamma)/2} \ll \gamma^{-1}.$$

By using this inequality, we see, whenever $g^2(\bar{\beta}(4N) \geqslant 1$, that

$$|g(\beta) - g(\bar{\beta}(4N)| \ll e^{g^2(\bar{\beta}(4N))/2} \, |\beta - \bar{\beta}(4N)|$$
$$\ll \frac{1}{\bar{\beta}(4N)} 2^{-4N}.$$

But since $N \geqslant N_0$, we have that $\bar{\beta}(2N) \neq 0$ and consequently

$$\bar{\beta}(4N) \geqslant \bar{\beta}(2N) \geqslant 2^{-2N}.$$

We conclude that the left-hand side of (14) is $\ll 2^{-2N}$. This estimate trivially holds when $g^2(\bar{\beta}(4N)) \leqslant 1$. This concludes the proof of (14).

Finally, for $N \geqslant \max(N_0, N_1)$,

$$\|x_\alpha - p_m\| = \|x_\alpha - x_\alpha^N\| + \|x_\alpha^N - p_m\|$$
$$\leqslant \sum_{j \geqslant N} a_j + O_\alpha\left(2^{-N} \sum_{j<N} \frac{2^j}{2^{j/2}}\right)$$
$$\leqslant \sum_{j \geqslant N} \sqrt{j}\, 2^{-j/2} + O_\alpha(2^{-N/2}).$$

The final expression clearly is $\ll \sqrt{N}\, 2^{-N/2}$, for all N sufficiently large (depending on α). But N is the largest integer such that

$$4NL_N = 4N.2^{N+1} \leqslant m.$$

This means that m is large enough that the dyadic rationals g_0, g_1, g_{jn} for $j<N$, $n<2^j$ can be computed from the first m bits of α. Moreover, $m \gg N2^N$ and $N \ll \log m$, so we have:

$$\frac{N}{2^N} \ll \frac{N^2}{m} \ll \frac{\log^2 m}{m}.$$

This concludes the proof of the theorem.

5. COMPUTING KC STRINGS FROM COMPLEX OSCILLATIONS

We write Φ: KC $\to \mathscr{C}$ for the mapping which associates with α the complex oscillation x_α given by (6). Let $t_0, t_1 \cdots$ be an effective enumeration, without repetition, of the dyadic rationals in the unit interval. With every $x \in C[0, 1]$, we associate the $\omega \times \omega$ array $A(x)$ such that the ith row of $A(x)$ is a binary representation of $x(t_i)$. The binary representation is as follows: If $x(t) = n + \alpha$, where n is an integer and where $0 \leqslant \alpha < 1$, we encode $x(t)$ as $1^m 0 b(\alpha)$ where $m = 2n$ when $n \geqslant 0$ and $m = 2\,|n| - 1$ when $n < 0$; moreover, $b(\alpha)$ is the dyadic representation of α. By using any recursive bijection between ω^2 and ω we can effectively represent $A(x)$ as a single binary string which we denote by $E(x)$. Set

$$\mathscr{E} = \{ E(x) \colon x \in \mathscr{C} \}.$$

The set $\mathscr{E}$ is topologised as a subspace of $\mathscr{N}$. In order to investigate the continuity of maps from and to $\mathscr{E}$, it will be useful to have the following fundamental systems of neighbourhoods of points in the space $\mathscr{E}$ available. For $x \in \mathscr{C}$ and $n, m \geqslant 1$, let E_{nm} be the subword of $E(x)$ corresponding to the first m bits of $x(t_j)$ for $0 \leqslant j < n$. These correspond to the bits in the $n \times m$ matrix at the northwestern corner of the $\omega \times \omega$ array $A(x)$. Set

$$[E_{nm}(x)] = \{ E(y) \in \mathscr{E} : E_{nm}(y) = E_{nm}(x) \}.$$

This is all the codes of all the complex oscillations y such that the binary representations of $y(t_j)$ agree with those of $x(t_j)$ in the first m positions for all $j = 1, ..., n$. It is clear that the family $\{ E_{nm}(x) \colon n, m \geqslant 1 \}$ is a fundamental system of neighbourhoods in $\mathscr{E}$ of x.

Define $v \colon \mathscr{E} \to \mathscr{C}$ by $E(x) \mapsto x$. This is a bijection since a continuous function is uniquely determined by its values at the dyadic rationals. Our next aim is to show that the sets KC and $\mathscr{E}$ are recursively isomorphic.

THEOREM 6. *There are partial recursive functions* $\phi, \psi \colon \mathscr{N} \to \mathscr{N}$ *such that* $\phi\psi = \mathrm{id}$ *on* $\mathscr{E}$ *and* $\psi\phi = \mathrm{id}$ *on KC. In particular,* ϕ *defines a homeomorphism from KC to* $\mathscr{E}$ *where these spaces are viewed as subspaces of* $\mathscr{N}$. *The map* ϕ *induces a bijection* $\Phi \colon \mathrm{KC} \to \mathscr{C}$ *which is nowhere continuous even though its converse* $\Phi^{-1} \colon \mathscr{C} \to \mathrm{KC}$ *is everywhere continuous.*

Proof. In the proof we use the same recursive bijection from ω^2 to ω which was defined in the beginning of the proof of Theorem 5. Let $(\Omega, P, \mathscr{A})$ be a probability space on which an infinite sequence $(\xi_0, \xi_1, \xi_{jn} \colon j \geqslant 1,\ n < 2^j)$ of independent $N(0, 1)$-variables is defined. Then for almost all $\omega \in \Omega$ the associated Wiener–Franklin series defines a Brownian motion $x_\omega(t)$. Moreover, for almost all ω,

$$\xi_0 = x(1), \qquad \xi_1 = 2x(\tfrac{1}{2}) - x(1) \tag{15}$$

and

$$\xi_{jn} = 2^{j/2}(2x(t_{jn}) - x(t_{jn} + \delta_j) - x(t_{jn} - \delta_j)), \tag{16}$$

where $t_{jn} = (2n+1)/2^{j+1}$ and $\delta_j = 2^{-(j+1)}$. These equalities hold whenever the Franklin-Wiener series converges uniformly. In particular, if we let $(\Omega, P, \mathscr{A}) = (\mathscr{N}, \lambda, B)$, then we see that these equalities define $\xi_0(\alpha)$, $\xi_1(\alpha)$, $\xi_{jn}(\alpha)$ uniquely from $x_\alpha \in KC$, where the $\xi(\alpha)$ are defined from α as in the proof of Theorem 5. By construction, the association $\alpha \mapsto (\xi_0, \xi_1, \xi_{jn})$ is a bijection. We conclude that $\Phi\colon \alpha \mapsto x_\alpha$ is an injective mapping from KC to $\mathscr{C}$.

We can look at Φ in a different way: Beginning with α, we construct $\xi_0, \xi_1, \xi_{jn}, j \geqslant 1, 0 \leqslant n < 2^j$ as in Theorem 5, whereafter we recursively find $x(n/2^j)$ from the ξ by solving first (15) and then (16) for increasing values of j. Indeed, we must solve the equations

$$x(1) = \xi_0, \qquad 2x(\tfrac{1}{2}) = \xi_0 + \xi_1,$$

and

$$2x\left(\frac{2n+1}{2^{j+1}}\right) = 2^{-j/2}\xi_{jn} + x\left(\frac{n+1}{2^j}\right) + x\left(\frac{n}{2^j}\right).$$

In this way, one can effectively compute any finite initial segment of $E(x_\alpha)$ from some initial segment of α. The association $\alpha \mapsto E(x_\alpha)$ induces a partial recursive map $\phi\colon \mathscr{N} \to \mathscr{N}$ whose restriction to KC defines a mapping, which we shall denote by ϕ, from KC into $\mathscr{E}$. In particular, ϕ is continuous on KC. Note that we can factorise Φ as

$$\mathrm{KC} \xrightarrow{\phi} \mathscr{E} \xrightarrow{\nu} \mathscr{C}.$$

If we let $(\Omega, P, \mathscr{A}) = (C[0,1], W, \Sigma)$ and if we define ξ_0, ξ_1, ξ_{jn} by (15) and (16) for $x \in C[0,1]$, we can conclude that ξ_0, ξ_1, ξ_{jn} is a sequence of independent $N(0,1)$-variables on $(C[0,1], W, \Sigma)$. As a first step to constructing a KC-string from a complex oscillation, we shall show that, if $x \in \mathscr{C}$, then

$$x = \xi_0 \Delta_0 + \xi_1 \Delta_1 + \sum_{j \geqslant 1} \sum_{n < 2^j} \xi_{jn} \Delta_{jn},$$

for all $t \in [0,1]$, where ξ_0, ξ_1, ξ_{jn} are given by (15) and (16). For this purpose, consider the event X_j' in Σ defined by the condition $[\sup_{n<2^j} |\xi_{jn}|$

$> 2\sqrt{j}]$. By using the fact that the ξ_{jn} are $N(0, 1)$ on $C[0, 1]$, we see, as in (9), that

$$W(X'_j) \ll \frac{1}{\sqrt{j}} \left(\frac{2}{e^2}\right)^j.$$

By using (16), we see that (X'_j) is a uniform sequence of $\Sigma_1^0(\mathscr{G})$ sets in a suitable Gaussian algebra $\mathscr{G}$. It follows that the set

$$\bigcap_m \bigcup_{j > m} X'_j$$

is a $\Pi_2^0(\mathscr{G})$ set of constructive Wiener measure 0 and, therefore, does not contain any complex oscillation x. We conclude that, if $x \in \mathscr{C}$, then, for some absolute constant C,

$$\left\| \sum_{j > m} \sum_{n < 2^j} \xi_{jn} \Delta_{jn} \right\| \leqslant C \frac{\sqrt{m}}{2^{m/2}}. \tag{17}$$

Write $h(m)$ for the right-hand side of the inequality and write x^m for the partial sum of the Franklin-Wiener series associated with x with the summation restricred to $j \leqslant m$. It follows from the preceding that for almost all $x \in C[0, 1]$, it is the case that $\|x - x^m\| < h(m)$, for m sufficiently large. In particular, for a fixed dyadic rational t, the event A_L defined by

$$\exists_{m > L} \, |x(t) - x^m(t)| < h(m)$$

has Wiener measure 1 for all values of L. It is clear that for each dyadic t and $L < \omega$, the event is $\Sigma_1^0(\mathscr{G})$ for a suitable Gaussian algebra $\mathscr{G}$. It follows from Theorem 3 that $x \in A_L$, for all $x \in \mathscr{C}$. In particular, if $x \in \mathscr{C}$, and t is a dyadic rational in the unit interval, then $x^{m_j}(t)$ converges to $x(t)$, for some unbounded subsequence m_j of natural numbers. It follows from (17) that x^m is a Cauchy sequence in $C[0, 1]$ when $x \in \mathscr{C}$. We conclude that x is the limit of x^m when $x \in \mathscr{C}$, as required.

It is now an easy matter to find, for a given complex oscillation x, some $\alpha \in \mathscr{N}$, such that $x = x_\alpha$. We first construct the ξ from x by equations (15) and (16), whereafter we define the β by $\beta = F(\xi)$, where F is the inverse function of g, i.e.,

$$F(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A} e^{-t^2/2} \, dt, \qquad A \in \mathbf{R}.$$

We will see in the next paragraph that for none of the ξ constructed from the complex oscillation x, will $F(\xi)$ be a dyadic rational. Hence we can compute, relative to the values of x at dyadic rationals, for each $n \geqslant 1$ and each associated ξ, a dyadic rational β_n of length n such that $\beta_n < \beta < \beta_n + 2^{-n}$ when $\beta = F(\xi)$. Hereafter, the string α corresponding to the β can be retrieved. If we reverse the construction of α from x, we find that x and α are related by $x = x_\alpha$, where x_α is the Franklin–Wiener series associated with α. In this sense the association $x \mapsto \alpha$ such that $x = x_\alpha$ is recursive. In particular it defines a continuous map from $\mathscr{C}$ to $\mathscr{N}$. All that remains to be shown is that the α thus obtained is a KC-string.

We first show that if x is a complex oscillation, then for each of the associated ξ, it is the case that $\beta = F(\xi)$ is a nonrecursive real. For otherwise, ξ itself will be recursive so that there is a recursive sequence ξ_k of dyadic rationals such that $|\xi_k - \xi| < 2^{-k}$ for all k. Thus, for a suitable Gaussian algebra $\mathscr{G}$, the complex oscillation will be in the $\Pi^0_1(\mathscr{G})$ set, A, described by

$$\alpha \in A \leftrightarrow \forall_k (|\xi_k - \xi| < 2^{-k}).$$

This set is obviously of W-measure 0, since ξ is a normal (hence non-atomic) variable on $C[0, 1]$. This is a contradiction for, on the one hand, $x \in A$, but, on the other hand, the complement B of A is a $\Sigma^0_1(\mathscr{G})$ set of W-measure 1 which, by Theorem 2 must contain all the complex oscillations.

We must show that if $\alpha \in \mathscr{N}$ is such that, within our coding scheme, x_α is a complex oscillation, then $\alpha \in KC$. If not, there is a total recursive function $(n, m) \mapsto s_{nm}$ from $\omega \times \omega$ to $\{0, 1\}^*$ such that

$$\alpha \in \bigcap_n \bigcup_m [s_{nm}]$$

and

$$\lambda\left(\bigcup_m [s_{nm}]\right) \leqslant 2^{-n},$$

for all n. Without loss of generality, we may assume that all the s_{nm} have lenghts of the form $4kL_k$ for some $k \geqslant 1$. For if k is the smallest natural number such that $|s_{nm}| \leqslant 4kL_k$, we may replace $[s_{nm}]$ by the union of all $[s_{nm}\gamma]$ with γ exactly the size so that $|s_{nm}\gamma| = 4kL_k$. If $|s_{nm}| = 4kL_k$, then, we can construct a sequence $(\gamma_0, \gamma_1, \gamma_{jp}\colon j \leqslant k, p < 2^j)$ of dyadic rationals each of length $4k$, which are the initial segments of $(\beta_0, \beta_1, \beta_{jp}\colon j \leqslant k, p < 2^j)$ which are associated to any $\gamma \in [s_{nm}]$. For notational convenience, we set $B_k = \{0, 1\} \cup \{jp\colon j \leqslant k, p < 2^j\}$. Writing $\mathscr{O}_{nm}$ for the set of γ in $\mathscr{N}$, such that none of the associated $\beta_i = \beta_i(\gamma)$, with $i \in B_k$, are dyadic rationals, we have

$$\gamma \in [s_{nm}] \cap \mathcal{O}_{nm} \leftrightarrow \forall_{i \in B_k} (\gamma_i < \beta_i(\gamma) < \gamma_i + 2^{-4k}). \tag{18}$$

For n, m with $|s_{nm}| = 4kL_k$, define the subset V_{nm} of $C[0, 1]$ by

$$X \in V_{nm} \leftrightarrow \forall_{i \in B_k} \exists_{A, B} (A < \xi_i(X) < B) \wedge (\gamma_i < F(A) < F(B) < \gamma_i + 2^{-4k}),$$

where A, B ranges over all rational numbers. Note that, by (18), we have $\gamma \in [s_{nm}] \cap \mathcal{O}_{nm}$ iff $x_\gamma \in V_{nm}$. Since $\alpha \in \bigcap_n \bigcup_m [s_{nm}]$ and the associated $\beta_i(\gamma)$ are non-dyadic, it follows from (18) that $x_\alpha \in \bigcap_n \bigcup_m V_{nm}$. The set V_{nm} is $\Sigma^0_1(\mathcal{G})$ for some Gaussian algebra $\mathcal{G}$. Moreover,

$$W\left(\bigcup_m V_{nm}\right) = \lambda\left(\bigcup_m [s_{nm}] \cap \mathcal{O}_{nm}\right) \leqslant \lambda\left(\bigcup_m [s_{nm}]\right) \leqslant 2^{-n},$$

for all n. Hence the set $V = \bigcap_n \bigcup_m V_{nm}$ is a $\Pi^0_2(\mathcal{G})$ set of constructive measure 0. It follows that x_α is in V, which is a set of constructive measure 0 —a contradiction.

For $x \in \mathcal{C}$, the construction of α from x such that $\Phi(\alpha) = x$ requires only the information encoded in $E(x)$. The effective nature of the construction

$$E(x) \mapsto (\xi) \mapsto (\beta) \mapsto \alpha$$

when $x \in \mathcal{C}$ ensures that that the association $E(x) \mapsto \alpha$ defines a partial recursive map $\psi: \mathcal{N} \to \mathcal{N}$ which maps $\mathcal{E}$ to KC. In fact the restriction of ψ to $\mathcal{E}$ is the inverse of ϕ: KC $\to \mathcal{E}$. In this sense KC and $\mathcal{E}$ are recursively isomorhic. In particular, ϕ: KC $\to \mathcal{E}$ is a homeomorphism when we view KC and $\mathcal{E}$ as subspaces of $\mathcal{N}$. Moreover, Φ: KC $\to \mathcal{C}$ is a bijection, since $\Phi = \nu\phi$.

It is readily seen that $\nu^{-1}: \mathcal{C} \to \mathcal{E}$ is continuous, when we view $\mathcal{C}$ as a subspace of $C[0, 1]$. To see this, we note that for given $n, m \geqslant 1$ and $x \in \mathcal{C}$, there is a $\delta > 0$ such that if $\|x - y\| < \delta$, then $x(t_j)$ and $y(t_j)$ will agree in the first m bits for all $j < n$. (Note that it follows from Theorem 4.2 in [7] that a complex oscillation assumes non-dyadic values at dyadic rationals.) This means exactly that all the y in a δ-neighbourhood of x will belong to the neighbourhood $[E_{nm}(x)]$ of $E(x)$. We can conclude that Φ^{-1} is continuous since $\Phi^{-1} = \psi\nu^{-1}$.

However, Φ is nowhere continuous on KC. Since $\Phi = \nu\phi$ and ϕ is a homeomorphism, it suffices to show that $\nu: \mathcal{E} \to \mathcal{C}$ is nowhere continuous. For suppose ν were continuous at $E(x)$ for some $x \in \mathcal{C}$. Then there are $n, m \geqslant 1$ such that, if $z \in \mathcal{C}$ and $E_{nm}(x) = E_{nm}(z)$, then $\|z - x\| < 1$. For $k < n$

one can find a dyadic rational number such that, if $z \in \mathscr{C}$ and $b_k \leqslant z(t_k) < b_k + 2^{-m}$, then the binary representations of $z(t_k)$ and $x(t_k)$ will agree in the first m bits. For a fixed $t^0 \in (0, 1)$ with $t^0 \neq t_k$ for all $k < n$, let y be the piecewise linear function such that $y(0) = 0$, $y(t_k) = b_k + 2^{-(m+1)}$ and $y(t^0) > x(t^0) + 2$. Set

$$O = \{z \in C[0, 1] : z(0) = 0 \text{ and } \|z - y\| < 2^{-(m+1)}\}.$$

It follows from Theorem 38 on p. 30 of [9] that $W(O) > 0$. Since $W(\mathscr{C}) = 1$, there is some $z \in \mathscr{C} \cap O$. For such a z we have that $E_{nm}(z) = E_{nm}(x)$. Moreover, by the construction of the set O,

$$\|z - x\| \geqslant |z(t^0) - x(t^0)| > 1.$$

We conclude that v is not continuous at $E(x)$.

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