Fredholm theory for pairs of closed subspaces
of a Banach space

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Abstract

We study the Fredholm theory for pairs of closed subspaces of a Banach space developed by Kato. We define the strictly singular and the strictly cosingular pairs of subspaces, and we show that some of the results of operator theory can be extended to this context. However, there are some notable differences. On the one hand, the perturbation classes problem has a positive answer in this context: the upper and lower semi-Fredholm pairs are stable under strictly singular and strictly cosingular perturbations, respectively, and this stability characterizes the strictly singular and the strictly cosingular pairs. Note that it has been proved recently that the perturbation classes problem for continuous semi-Fredholm operators has a negative answer. On the other hand, unlike in the case of operators, the Fredholm pairs are not stable under perturbation by strictly singular or strictly cosingular pairs. We also show the stability under composition of the compact, the strictly singular and the strictly cosingular pairs of subspaces.

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1. Introduction

In [5, Chapter IV], Kato develops a Fredholm theory for pairs of closed subspaces of a Banach space, from which he derives some stability theorems under small perturbations for closed semi-Fredholm operators.

Let $M, N$ be a pair of closed subspaces of a Banach space $Z$. The nullity $\text{nul}(M, N)$, the deficiency $\text{def}(M, N)$, and the index $\text{ind}(M, N)$ of the pair $M, N$ are defined by

\[
\text{nul}(M, N) := \dim(M \cap N),
\]
\[
\text{def}(M, N) := \text{codim}(M + N) = \dim(Z/(M + N)),
\]
and
\[
\text{ind}(M, N) := \text{nul}(M, N) - \text{def}(M, N).
\]

The pair $(M, N)$ is said to be upper semi-Fredholm, and we write $(M, N) \in \Phi_+$, if $M + N$ is closed and $\text{nul}(M, N)$ is finite. It is said to be lower semi-Fredholm, and we write $(M, N) \in \Phi_-$, if $M + N$ is closed and $\text{def}(M, N)$ is finite. It is said to be semi-Fredholm if $(M, N) \in \Phi_+ \cup \Phi_-).

Let $g(M, N)$ denote the gap between $M$ and $N$. It was proved in [5, IV Theorem 4.30] that given a semi-Fredholm pair $(M, N)$ there exists a number $\delta > 0$ so that if $L$ is a closed subspace of $Z$ and $g(L, M) < \delta$, then $(L, N)$ is a semi-Fredholm pair with $\text{nul}(L, N) \leq \text{nul}(M, N)$, $\text{def}(L, N) \leq \text{def}(M, N)$, and $\text{ind}(L, N) = \text{ind}(M, N)$.

From the previously mentioned result, identifying a closed operator $T : D(T) \subset X \rightarrow Y$ with the pair $(G(T), X)$ of closed subspaces of $X \times Y$, where $G(T)$ denotes the graph of $T$, we get a result of stability under small perturbations for closed operators [5, IV Theorem 5.17]: if $S$ and $T$ are closed operators, $T$ is semi-Fredholm, and $g(G(S), G(T)) < \delta$, then $S$ is semi-Fredholm with $\text{nul}(S) \leq \text{nul}(T)$, $\text{def}(S) \leq \text{def}(T)$, and $\text{ind}(S) = \text{ind}(T)$. This result extends a well-known result of stability under small norm perturbations for semi-Fredholm operators. See, for example, [2, Theorem V.1.6].

Here we introduce the classes of compact, strictly singular, and strictly cosingular pairs of subspaces. Then we show that the classes of upper semi-Fredholm and lower semi-Fredholm pairs of subspaces are stable under strictly singular and strictly cosingular perturbations, respectively, and that the perturbation classes problem has a positive answer in this context. More precisely, let $L, M,$ and $N$ be closed subspaces of $Z$. We prove the following result:

1. The pair $(L, M)$ is strictly singular if and only $(L, N) \in \Phi_+$ for every closed subspace $N$ such that $(M, N) \in \Phi_+$.
2. The pair $(M, N)$ is strictly cosingular if and only $(L, N) \in \Phi_-$ for every closed subspace $L$ such that $(L, M) \in \Phi_-).

Note that perturbation classes problem for continuous operators has a negative answer [3]: there are operators $S \in \mathcal{L}(X, Y)$ which are not strictly singular, but $T + S$ is upper semi-Fredholm for every upper semi-Fredholm $T \in \mathcal{L}(X, Y)$, and there are operators $S \in \mathcal{L}(X, Y)$ which are not strictly cosingular, but $T + S$ is lower semi-Fredholm for every lower semi-Fredholm $T \in \mathcal{L}(X, Y)$.
Moreover, we show in Examples 3.2–3.5 that the positions of $SS$ and $\Phi_+$ (respectively the positions of $SC$ and $\Phi_-)$ in the previously stated perturbation result cannot be reversed: $(L, M) \in \Phi_+$ and $(M, N) \in \Phi_-$ strictly singular does not imply $(L, N) \in \Phi_+$, and $(L, M)$ strictly cosingular and $(M, N) \in \Phi_-$ does not imply $(L, N) \in \Phi_-$. We also study the stability of Fredholm pairs under perturbation. We show that the class of Fredholm pairs is not stable under compact perturbations; so it is not stable under strictly singular or strictly cosingular perturbations.

Finally, we consider the stability under composition of some classes of pairs of subspaces in the following sense. Let $L, M,$ and $N$ be closed subspaces of $Z$. We say that a class $\mathcal{A}$ of pairs is transitive if $(L, M) \in \mathcal{A}$ and $(M, N) \in \mathcal{A}$ implies $(L, N) \in \mathcal{A}$. This is a property that it is reasonable to expect from a perturbation class. We show that the classes of compact, strictly singular, and strictly cosingular pairs are transitive.

We observe that this Fredholm theory for pairs of subspaces provides a natural context for the study of nonnecessarily bounded linear operators, or more generally, for the study of linear relations. Indeed, a Fredholm theory for linear relations has been developed, in which precompact, strictly singular, strictly cosingular, upper and lower semi-Fredholm, and other classes of linear relations are studied. We refer to [1, Chapter V] for a detailed exposition of this theory. The definitions of these classes of linear relations are similar to the corresponding definitions for operators in $\mathcal{L}(X, Y)$. Since a linear relation from $X$ to $Y$ can be identified with a subspace of $X \times Y$, the study of linear relations could be embedded in a general theory of pairs of subspaces of a normed space.

Along the paper, $X, Y,$ and $Z$ are Banach spaces, and we denote by $X^*$ the dual space of $X$. For a closed subspace $M$ of $Z$, $J_M$ is the inclusion of $M$ into $Z$, and $Q_M$ is the quotient map from $Z$ onto $Z/M$. Given subsets $A \subset X$ and $B \subset X^*$, the annihilators $A^\perp := \{f \in X^*: f(a) = 0 \text{ for all } a \in A\}$ and $^\perp B := \{x \in X: g(x) = 0 \text{ for all } g \in B\}$ are closed subspaces of $X^*$ and $X$, respectively.

We denote by $\mathcal{L}(X, Y)$ the class of all (continuous linear) operators from $X$ into $Y$. For $T \in \mathcal{L}(X, Y)$, we denote by $N(T)$ and $R(T)$ the kernel and the range of $T$, respectively. The graph $G(T)$ of $T$ is the closed subspace

$$G(T) = \{(x, Tx): x \in X\}.$$ 

An operator $T \in \mathcal{L}(X, Y)$ is upper semi-Fredholm, denoted $T \in \Phi_+(X, Y)$, if $R(T)$ is closed and $N(T)$ is finite dimensional. The operator $T$ is lower semi-Fredholm, denoted $T \in \Phi_-(X, Y)$, if $R(T)$ is finite codimensional (hence closed).

2. Classes of pairs of closed subspaces

For a pair of closed subspaces $(M, N)$ of $Z$, $N(Q_N J_M) = M \cap N$ and $R(Q_N J_M) = M + N$. Moreover, it is not difficult to see that $M + N$ is closed if and only if $R(Q_N J_M)$ is closed. Therefore, $(M, N)$ is upper semi-Fredholm if and only if the operator $Q_N J_M$ is upper semi-Fredholm, and $(M, N)$ is lower semi-Fredholm if and only if $Q_N J_M$ is lower semi-Fredholm. These facts suggest the following definition.
Definition 2.1. Let $\mathcal{A}$ be a class of operators. Let $M$ and $N$ be closed subspaces of $Z$. We say that $(M, N)$ belongs to $\mathcal{A}$ if $Q_N J_M \in \mathcal{A}$.

Remark 2.2. It follows from the relations of duality for pairs of closed subspaces [5, Chapter IV] that
\[(M, N) \in \Phi_+ \iff (N, M) \in \Phi_+ \iff (N \perp, M \perp) \in \Phi_- \quad \text{and} \quad (M, N) \in \Phi_- \iff (N, M) \in \Phi_- \iff (N \perp, M \perp) \in \Phi_+.
\]

The proof of the following result is similar to the proof of the first proposition in [4]. We include a part of it for completeness.

Lemma 2.3 [4, Proposition]. Let $M$ and $N$ be closed subspaces of $Z$. Suppose that $M + N$ is not closed.

(1) There exist sequences $(x_n)$ in $M$, $(y_n)$ in $N$, and $(f_n)$ in $Z^*$ so that $\|x_n\| = \|y_n\| = 1$, $f_m(x_n) = \delta_{mn}$, and $\|f_n\| \|x_n - y_n\| < 2^{-n}$ for every $n$.

(2) There exist sequences $(f_n)$ in $M^\perp$, $(g_n)$ in $N^\perp$, and $(x_n)$ in $Z$ so that $\|f_n\| = \|g_n\| = 1$, $f_m(x_n) = \delta_{mn}$, and $\|x_n\| \|f_n - g_n\| < 2^{-n}$ for every $n$.

Proof. (1) Since $M + N$ is not closed,
\[
\inf \{ \|m - n\| : m \in M, n \in N, \|m\| = \|n\| = 1 \} = 0.
\]

Thus, denoting $M_0 := M$, we can choose $x_1 \in M_0$ and $y_1 \in N$ such that $\|x_1\| = \|y_1\| = 1$ and $\|x_1 - y_1\| < 2^{-1}$, and $f_1 \in Z^*$ such that $\|f_1\| = f_1(x_1) = 1$.

The codimension of $M_1 := M \cap N(f_1)$ in $M$ is equal to 1. Thus $M_1 + N$ is not closed. Let $P_1$ denote the projection on $M$ with $R(P_1) = M_1$ and $N(P_1) = [x_1]$.

Proceeding in this way, we can find sequences $(x_n)$ in $M$, $(y_n)$ in $N$ and $(f_n)$ in $Z^*$ such that
\[
x_n \in M_{n-1} := M \cap N(f_1) \cap \cdots \cap N(f_{n-1}),
\]
\[
\|x_n\| = \|y_n\| = 1 \quad \text{and} \quad f_n(x) = \delta_{ni} \quad \text{for all } n \leq i.
\]

Moreover, denoting by $P_n$ the projection on $M$ with $R(P_n) = M_n$ and $N(P_n) = [x_1, \ldots, x_n]$, we can assume that
\[
\|x_n - y_n\| < \frac{1}{2^n \|P_{n-1}\|} \quad \text{and} \quad \|f_n\| < \|P_{n-1}\|.
\]

The sequences $(x_n)$, $(y_n)$, and $(f_n)$ satisfy the required conditions.

(2) Suppose that $M + N$ is not closed. Then $M^\perp + N^\perp$ is not closed, and an argument similar to the one given in the proof of part (1) provides the sequences we need. \qed

Proposition 2.4. Let $M$ and $N$ be closed subspaces of $Z$.

(1) The pair $(M, N)$ fails to be upper semi-Fredholm if and only if there exist a compact operator $K : Z \to Z$ and an infinite dimensional closed subspace $L$ of $M$ such that $\|K\| < 1$ and $(I - K)L \subset N$. 

(2) The pair \((M, N)\) fails to be lower semi-Fredholm if and only if there exist a compact operator \(K : Z \to Z\) and an infinite codimensional closed subspace \(L\) containing \(N\) such that \(\|K\| < 1\) and \((I - K)M \subset L\).

**Proof.** First observe that \(\|K\| < 1\) implies that \((I - K)\) is an isomorphism on \(Z\).

(1) Suppose that \((M, N)\) is not upper semi-Fredholm. In the case \(\dim M \cap N = \infty\), we take \(K = 0\) and \(L = M \cap N\). Otherwise, \(M + N\) is not closed. Let \((x_n) \subset M\), \((y_n) \subset N\) and \((f_n) \subset Z^*\) be the sequences provided by part (1) of Lemma 2.3. We define the operator \(K : Z \to Z\) by

\[
K(z) := \sum_{n=1}^{\infty} f_n(z)(x_n - y_n).
\]

Clearly \(K\) is compact and \(\|K\| \leq \sum_{n=1}^{\infty} \|f_n\| \|x_n - y_n\| < 1\). Moreover, \((I - K)x_n = y_n\) for each \(n\). Thus we can take as \(L\) the closed subspace generated by the sequence \((x_n)\).

Conversely, if there exist a compact operator \(K : Z \to Z\) and an infinite dimensional closed subspace \(L\) of \(M\) such that \(\|K\| < 1\) and \((I - K)L \subset N\), then \(Q_N (I - K)J_L = 0\). Thus \(Q_N J_M\) is a compact operator. Hence \(Q_N J_M\) is not upper semi-Fredholm.

(2) Suppose that \((M, N)\) is not lower semi-Fredholm. In the case \(M + N\) closed, we take \(K = 0\) and \(L = M + N\). Otherwise, let \((f_n) \subset M^\perp\), \((g_n) \subset N^\perp\) and \((x_n) \subset Z\) be the sequences provided by part (2) of Lemma 2.3. We define the operator \(K : Z \to Z\) by

\[
K(z) := \sum_{n=1}^{\infty} (g_n - f_n)(z)x_n.
\]

As in the proof of part (1), \(K\) is compact and \(\|K\| < 1\). We take \(L = \perp \{g_n\}\).

Clearly, \(N\) is infinite codimensional and \(N \subset L\). Moreover, \((I - K)^*g_n = f_n\) for each \(n\). Thus \((I - K)^*L^\perp \subset M^\perp\). Hence \(M \subset \perp ((I - K)^*L) = (I - K)^{-1}L\).

Conversely, if there exist a compact operator \(K : Z \to Z\) and an infinite dimensional closed subspace \(L\) of \(Z\) such that \(\|K\| < 1\), \(N \subset L\) and \((I - K)L \subset M\), then \(Q_L (I - K)J_M = 0\). Thus \(Q_L J_M\) is a compact operator. Hence \(Q_N J_M\) is not lower semi-Fredholm. □

The study of continuous operators in \(L(X, Y)\) can be reduced to the study of pairs of closed subspaces. Indeed, let \(Z := X \times Y\). We write \(X\) for \(X \times \{0\}\) and \(Y\) for \(\{0\} \times Y\).

The graph of \(T \in L(X, Y)\) is isomorphic to \(X\). Indeed,

\[
\|x\| \leq \|x\| + \|Tx\| \leq (1 + \|T\|)\|x\|.
\]

Moreover, for \(S, T \in L(X, Y)\), we can identify \(S - T\) with the pair \((G(S), G(T))\) as follows.
Proposition 2.5. For each pair of operators $S, T \in \mathcal{L}(X, Y)$ there exist two isomorphisms $U \in \mathcal{L}(X, G(S))$ and $V \in \mathcal{L}(Z / G(T), Y)$ such that $S - T = V Q_{G(T)} J G(S) U$.

Proof. The equality $(x, y) = (x, Tx) + (0, y - Tx)$ gives the decomposition $Z = G(T) \oplus Y$.
Thus we have two natural isomorphisms

\[ U : X \rightarrow G(S) \quad \text{and} \quad V : Z \rightarrow Y \]

defined by $U(x) := (x, Sx)$ and $V((x, y) + G(T)) := y - Tx$. It is easy to check that

\[ V Q_{G(T)} J G(S) U(x) = (S - T)x, \]
for every $x \in X$. \qed

It follows from the previous result that there is some symmetry in the description of operators in terms of pairs of closed subspaces.

Corollary 2.6. Let $\mathcal{A}$ be a class of operators stable under products by isomorphisms and let $T \in \mathcal{L}(X, Y)$.
Then the following assertions are equivalent:

1. $T \in \mathcal{A}$;
2. $(G(T), X) \in \mathcal{A}$;
3. $(X, G(T)) \in \mathcal{A}$.

3. The perturbation results

Recall that an operator $T \in \mathcal{L}(X, Y)$ is strictly singular if there exists no infinite dimensional subspace $M$ of $X$ such that the restriction $T J_M$ is an isomorphism. Moreover, $T$ is strictly cosingular if there exists no infinite codimensional subspace $N$ of $Y$ such that $Q_N T : X \rightarrow Y / N$ is surjective. It is well known that $T$ is strictly singular if and only if for every infinite dimensional subspace $M_1 \subset X$ there exists an infinite dimensional subspace $M_2 \subset M_1$ such that $T J_{M_2}$ is compact; and $T$ is strictly cosingular if and only if for every infinite codimensional subspace $N_1 \subset Y$ there exists an infinite codimensional subspace $N_2 \supset N_1$ such that $Q_{N_2} T$ is compact [6].

We denote by $K$, $SS$, and $SC$ the compact, the strictly singular, and the strictly cosingular operators, respectively.

Next we give the perturbation results for the classes $\Phi_+$ and $\Phi_-$ of semi-Fredholm pairs.
Theorem 3.1. Let $L$, $M$, and $N$ be closed subspaces of $Z$.

(1) If $(L, M) \in SS$ and $(M, N) \in \Phi_+$, then $(L, N) \in \Phi_+$.

(2) If $(L, M) \in \Phi_-$ and $(M, N) \in SC$, then $(L, N) \in \Phi_-$.

Proof. (1) Suppose that $(L, M) \in SS$ and $(L, N) \notin \Phi_+$. By Proposition 2.4, there exist a compact operator $K_0$ with $\|K_0\|$ arbitrarily small and an infinite dimensional closed subspace $L_0$ of $L$ so that $(I - K_0)L_0 \subset N$.

Note that $(L_0, M) \in SS$. Therefore, there exist a compact operator $K_1$ with $\|K_1\|$ arbitrarily small and a closed, infinite dimensional $L_1$ of $L_0$ so that $(I - K_1)L_1 \subset M$.

Let us denote $M_1 = (I - K_1)L_1$. Note that for $\|K_1\|$ and $\|K_2\|$ small enough, we can write and $(I - K_0)(I - K_1)^{-1} = (I - K)$, where $K$ is a compact operator with $\|K\| < 1$.

Now, since $(I - K)M_1 = (I - K_0)L_1 \subset N$, applying again Proposition 2.4, we get $(M, N) \notin \Phi_+$.

(2) Suppose that $(M, N) \in SC$ and $(L, N) \notin \Phi_-$. By Proposition 2.4, there exist a compact operator $K_0$ with $\|K_0\|$ arbitrarily small and an infinite codimensional closed subspace $N_0 \supset N$ so that $(I - K_0)L \subset N_0$.

Note that $(M, N_0) \in SC$. Therefore, there exist a compact operator $K_1$ with $\|K_1\|$ arbitrarily small and a closed, infinite codimensional $N_1 \supset N_0$ so that $(I - K_1)M \subset N_1$.

Let us denote $M_1 = (I - K_1)^{-1}N_1$. Note that $M_1$ is infinite codimensional and $M_1 \supset M$. Moreover, for $\|K_1\|$ and $\|K_2\|$ small enough, we can write and $(I - K_1)^{-1}(I - K_0) = (I - K)$, where $K$ is a compact operator with $\|K\| < 1$.

Now, since $(I - K)L \subset (I - K_1)^{-1}N_0 \subset M$, applying again Proposition 2.4, we get $(L, M) \notin \Phi_-$. \qed

The following two examples show that in the first part of Theorem 3.1, the positions of $SS$ and $\Phi_+$ cannot be reversed. The first one is a general example. The second one involves concrete spaces, but it is stronger and shows that the condition $\text{codim } L + M = \infty$ that appears in the first example is not necessary.

Example 3.2. Let $M$ and $N$ be closed subspaces of $Z$. Suppose that $(L, M) \in \Phi_+$ with $\text{codim } L + M = \infty$. We take $N = L + M$. Then $Q_NJ_M = Q_NJ_L = 0$. Thus $(M, N) \in SS$, but $(L, N) \notin \Phi_+$.

Example 3.3. Let $Z = \ell_1 \times \ell_1$. We take $L = \ell_1 \times \{0\}$ and $M = \{0\} \times \ell_1$. It is well known that there exists a closed subspace $A$ of $\ell_1$ such that $\ell_1/A$ is isomorphic to $\ell_2$. We take $N = L \times A$. Then $Q_N$ is strictly singular; thus $(L, M) \in \Phi$ and $(M, N) \in SS$. However $(L, M) \notin \Phi$ because $Q_NJ_L = 0$.

The following two examples show that in the second part of Theorem 3.1, the positions of $\Phi_-$ and $SC$ cannot be reversed. The first one is a general example. The second one involves concrete spaces, but it is stronger and shows that the condition $\text{dim } M \cap N = \infty$ that appears in the first example is not necessary.
Example 3.4. Let $M$ and $N$ be closed subspaces of $Z$. Suppose that $(M, N) \in \Phi_-$ with $\dim M \cap N = \infty$. We take $L = M \cap N$. Then $QM_{|L} = QN_{|L} = 0$. Thus $(L, M) \in SC$ but $(L, N) \notin \Phi_-.

Example 3.5. Let $Z = \ell_\infty \times \ell_\infty$. We take $M = \ell_\infty \times \{0\}$ and $N = \{0\} \times \ell_\infty$. It is well known that there exists a closed subspace $A$ of $\ell_2$ which is isomorphic to $\ell_2$. We take $L = \{0\} \times A$. Then $J_L$ is strictly cosingular, because it is weakly compact and any surjective operator from $\ell_\infty$ into a separable space takes weakly convergent sequences into convergent ones. Thus $(L, M) \in SC$ and $(M, N) \in \Phi$. However $(L, N) \notin \Phi_-$ because $Q_{N|J_L} = 0$.

Let us see that the perturbation class of $\Phi_+$ is $SS$ and the perturbation class of $\Phi_-$ is $SC$.

**Theorem 3.6** (The perturbation classes of $\Phi_+$ and $\Phi_-).$ Let $L$, $M$, and $N$ be closed subspaces of $Z$.

1. $(L, M) \in SS$ if and only if $(L, N) \in \Phi_+$ for every closed subspace $N$ of $Z$ such that $(M, N) \in \Phi_+$.
2. $(M, N) \in SC$ if and only if $(L, N) \in \Phi_-$ for every closed subspace $L$ of $Z$ such that $(L, M) \in \Phi_-.$

**Proof.** The direct implications are contained in Theorem 3.1. Let us prove the converse implications.

Suppose that $(L, M) \notin SS$. Then there exists a closed, infinite dimensional subspace $N$ of $L$ such that $(N, M) \in \Phi_+$; thus $(M, N) \in \Phi_+$. However, $(L, N) \notin \Phi_+$, because $N(Q_{N|J_L}) = N$ is infinite dimensional.

Suppose that $(M, N) \notin SC$. Then there exists a closed, infinite codimensional subspace $L$ such that $N \subset L$ and $(M, L) \in \Phi_-;$ thus $(L, M) \in \Phi_-.$ However, $(L, N) \notin \Phi_-$, because $L + N$ is infinite codimensional. □

Recall that an operator $T \in L(X, Y)$ is Fredholm if and only if it is upper semi-Fredholm and lower semi-Fredholm. We denote by $\Phi$ the Fredholm operators. The following result shows that the perturbation class of the Fredholm is quite small.

**Theorem 3.7.**

1. There exist pairs $(L, M) \in K$ and $(M, N) \in \Phi$ such that $(L, N) \notin \Phi.$
2. There exist pairs $(L, M) \in \Phi$ and $(M, N) \in K$ such that $(L, N) \notin \Phi.$

**Proof.** Here $X$ is a Banach space and $Y$ is a closed subspace of $X$ such that $\dim Y = \dim X/Y = \infty$.

1. We take $Z = X \times X, L = X \times \{0\}, M = \{0\} \times X,$ and $N = Y \times X$. Then $(L, M) \in \Phi, Q_{N|J_M} = 0$ and $(L, N) \notin \Phi_-. $
We take $Z = X \times X$, $L = Y \times \{0\}$, $M = X \times \{0\}$ and $N = \{0\} \times X$. Then $Q_M J_L = 0$, $(M, N) \in \Phi$ and $(L, N) \in \Phi_+ \setminus \Phi$. □

Finally, we consider the stability under composition of some classes of pairs of subspaces associated to operator ideals. This is a property that it is reasonable to expect from a perturbation class. We show that the classes of compact, strictly singular and strictly cosingular pairs are stable under composition.

**Definition 3.8.** We say that a class of pairs $A$ is **transitive** if given closed subspaces $L$, $M$, and $N$ of $Z$,

$$(L, M) \in A, \quad (M, N) \in A \implies (L, N) \in A.$$

**Theorem 3.9.** The classes of pairs $K$, $SS$, and $SC$ are transitive.

**Proof.** Let $L$, $M$, and $N$ be closed subspaces of $Z$. First we consider the class $K$. We have to show that

$$Q_M J_L \in K, \quad Q_N J_M \in K \implies Q_N J_L \in K.$$

Suppose that $Q_N J_L \notin K$ and $Q_M J_L \in K$. We take a bounded sequence $(x_n)$ in $L$ such that $(Q_N x_n)$ has no convergent subsequence, and we select a subsequence $(x_{n_k})$ such that $(Q_M x_{n_k})$ converges to some $z \in X/M$. Then we choose $x \in X$ such that $Q_M x = z$, and take a sequence $(y_k)$ in $M$ such that

$$\lim \|x_{n_k} - y_k - x\| = 0.$$

Since $(Q_N x_{n_k})$ has no convergent subsequences, the same happens to $(Q_N y_k)$. Thus $Q_N J_M$ is not compact.

For the class $SS$ it is enough to show that $Q_N J_M \notin SS$ when $Q_N J_L \notin SS$ and $Q_M J_L \in SS$. Since $Q_N J_L \notin SS$, there exists an infinite dimensional closed subspace $L_1 \subset L$ such that $Q_N J_{L_1}$ is an isomorphism into a closed subspace of $Z/N$. Moreover, since $Q_M J_{L_1} \in SS$, there exists an infinite dimensional closed subspace $L_2 \subset L_1$ and a compact operator $K: Z \to Z$ such that $I - K$ is an isomorphism and $(I - K)L_2 \subset M$. We denote $M_2 := (I - K)L_2$.

Let us see that $Q_N J_{M_2}$ is an isomorphism. In order to do that we take $\delta > 0$ and a bounded $\delta$-separated sequence $(m_i)$ in $M_2$; i.e., $\|m_i - m_j\| \geq \delta$ for $i \neq j$. Then $l_i := (I - K)^{-1} m_i$ gives a bounded sequence $(l_i)$ in $L_2$ which is $c$-separated for some $c > 0$, and $(Q_M l_i)$ is $d$-separated for some $d > 0$.

Now, since $Q_N m_i = Q_N l_i - Q_N K l_i$, $(Q_N m_i)$ has no convergent subsequences, $Q_N J_{M_2}$ is an isomorphism, hence $Q_N J_M \notin SS$.

For the class $SC$ the proof is similar. We will show that $Q_M J_L \notin SC$ when $Q_N J_L \notin SC$ and $Q_N J_M \in SC$.

Since $Q_N J_L \notin SC$, there exists an infinite codimensional closed subspace $N_1 \supset N$ such that $Q_N J_{L_1}$ is surjective. Moreover, since $Q_N J_M \in SC$, there exists an infinite codimensional closed subspace $N_2 \supset N_1$ and a compact operator $K: Z \to Z$ such that $I - K$ is a compact operator and $(I - K)N_2 \subset N_2$. We denote $M_2 := (I - K)^{-1} N_2$. 

(2)
In order to finish the proof it is enough to show that $Q_{M} J_{L}$ is surjective. We can do it as follows.

Since $I - K$ induces an isomorphism from $\mathbb{Z}/M_{2}$ onto $\mathbb{Z}/N_{2}$, $(I - K)^*$ induces an isomorphism from $N_{2}^{-1}$ onto $N_{1}^{-1}$. Moreover $(Q_{N_{2}} J_{L})^* = Q_{L_{1}} J_{N_{1}}$ is an isomorphism into. So, proceeding as we did for the class $SC$, we get that $(Q_{M_{2}} J_{L})^*$ is an isomorphism; hence $Q_{M_{2}} J_{L}$ is surjective.

References