Asymptotic behavior of the singular values of a generalization of the operator fractional integration✩

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In this paper we find the first term in the asymptotics of singular values of the generalized fractional integration operator.
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1. Introduction and notation

In this paper we consider integral operators of the form

\[ Af(x) = \int_0^x K(x, y) f(y) \, dy, \]

\[ Bf(x) = \int_x^1 K(y, x) f(y) \, dy, \]

which act on the space \( L^2(0, 1) \) with kernels that satisfy the following condition

\[ K(x, y)K(y, t) \equiv \varphi(y)K(x, t) \quad \text{on } [0, 1] \times [0, 1]. \]

(\( \varphi(y) = K(y, y). \))

Operators \( A \) and \( B \) represent a wide class of integral operators which play important role in the theory of fractional integration and differentiation. In the paper [9] fractional power of the operators \( A \) and \( B \) were introduced in the following way:

\[ A^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x K(x, y)\varphi_{\alpha-1}(x, y) f(y) \, dy. \]

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Formula of fractional integration by parts, the inversion formula and semigroup property for operators $A^\alpha$ and $B^\alpha$ are proved in [9].

If $K \equiv 1$, the operator $A^\alpha$ is reduced to the Riemann–Liouville operator; if $K(x, y) = \sigma x^{-\sigma} y^{\sigma n+\sigma - 1}$ the operator $A^\alpha$ (up to the factor $x^{-\sigma}$) is reduced to the Erdélyi–Kober operator [10].

If $K(x, y) = \frac{1}{y}$ the operator $A^\alpha$ is reduced to Hadamard fractional integral operator [10].

If $K(x, y) = g'(y)$ the operator $A^\alpha$ is reduced to fractional integral of functions with respect to function $g$ [10].

If $K \equiv 1$ then

$$A^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y) \, dy \quad (\equiv I^\alpha f, \text{ Riemann–Liouville fractional integral operator}).$$

In [4,11] it was proved that

$$\lim_{n \to \infty} (n\pi)^\alpha s_n(I^\alpha) = 1 \quad (\alpha > 0)$$

where $s_n(I^\alpha)$ denotes the singular values of the operator $I^\alpha$.

In [1] a more precise result is given i.e.

$$s_n(I^\alpha) = (n\pi)^{-\alpha} \left( 1 + O \left( \frac{\ln^2 n}{n} \right) \right), \quad n \to \infty.$$  

In [3], P. Burman gave two-sided estimates for $s_n(I^\alpha)$ and as a consequence he obtained a sharp result

$$s_n(I^\alpha) = (n\pi)^{-\alpha} \left( 1 + O \left( \frac{1}{n} \right) \right), \quad n \to \infty.$$  

D.E. Edmunds and V.D. Stepanov in [5] gave some criteria of belonging to Schatten–von Neumann ideals for a certain class of integral operators whose kernels satisfy the condition that is close to additive version of the condition ($\ast$).

In [8] asymptotic behavior of the singular values of some integral operators of Riemann–Liouville type was also found. In this paper we determine asymptotic behavior of singular values of the operators $A^\alpha$, $B^\alpha$ (when $A^\alpha$ and $B^\alpha$ are compact).

For example, as a consequence of the main result of this paper, we obtain:

a) If $K \equiv 1$ then $s_n(A^\alpha) \sim (n\pi)^{-\alpha}, \quad n \to \infty$.

b) If $K(x, y) = g'(y)$ then $s_n(A^\alpha) \sim (n\pi)^{-\alpha} \left( \int_0^1 |g'(t)| \, dt \right)^\alpha, \quad n \to \infty$.

c) If $K(x, y) = a(x) b(y)$ and $F$ is a function such that $F'(x) = a(x) b(x)$, then

$$A^\alpha f(x) = \int_0^x a(x) \left( F(x) - F(y) \right)^{\alpha-1} b(y) f(y) \, dy$$

and we have

$$s_n(A^\alpha) \sim (n\pi)^{-\alpha} \left( \int_0^1 |a(x)| b(x) \, dx \right)^\alpha.$$  

(We refer to [6] for basic facts on singular values of compact operators.) In this paper, $z^\alpha = e^{\alpha \ln z}$, $\ln z = \ln |z| + i \arg z$, $-\pi < \arg z \leq \pi$.

Also, we use notation $\int_0^x H(x, y) \, dy$ for the integral operator which acts on $L^2(0, 1)$ and has kernel $H(\cdot, \cdot)$.
2. Main result

**Theorem 1.** Let $K \in C^{1+|\alpha|}([0, 1]^2)$ ($\alpha > 0$), satisfy the condition $(\ast)$, $\varphi(x) = K(x, x) \neq 0$ on $[0, 1]$ and $\int_I K(x, x) \, dx \neq 0$ for every interval $I \subseteq [0, 1]$. Then formula

$$
\lim_{n \to \infty} (n\pi)^\alpha s_n(A^\alpha) = \lim_{n \to \infty} (n\pi)^\alpha s_n(B^\alpha) = \left( \int_0^1 |K(x, x)| \, dx \right)^\alpha
$$

holds ([$\alpha$] denotes integral part of $\alpha$).

For the proof of the previous theorem, we need some lemmas.

**Lemma 2.** Let $\alpha > 0$ and $s > 1 + [\alpha]$. Then

$$
\lim_{\lambda \to \infty} \lambda^{\alpha} \int_0^1 (1 - t^2)^s t^{\alpha-1} e^{-i\lambda t} \, dt = I'(\alpha)e^{-\frac{\pi \alpha}{2}}. \tag{1}
$$

**Proof.** Let $\alpha = 1$. Integrating by parts we obtain

$$
\lambda \int_0^1 (1 - t^2)^s e^{-i\lambda t} \, dt = -i - \frac{2s}{i} \int_0^1 (1 - t^2)^s t^{\alpha-1} e^{-i\lambda t} \, dt
$$

and, by Riemann lemma, we have

$$
\lim_{\lambda \to \infty} \lambda \int_0^1 (1 - t^2)^s e^{-i\lambda t} \, dt = -i.
$$

If $\alpha \in \mathbb{N}$ and $\alpha > 2$ the statement of Lemma 1 follows by induction.

Let now $0 < \alpha < 1$ (then $s > 1$).

The function $t \mapsto r(t) = \frac{(1 - t^2)^{s-1}}{t}$ is differentiable on $[0, 1]$ and hence

$$
\lambda^{\alpha} \int_0^1 r(t)t^{\alpha-1}e^{-i\lambda t} \, dt = 0 \left( \frac{1}{\lambda^{1-\alpha}} \right), \quad \lambda \to \infty.
$$

Using equality

$$
\lambda^{\alpha} \int_0^1 (1 - t^2)^s t^{\alpha-1} e^{-i\lambda t} \, dt = \lambda^{\alpha} \int_0^1 r(t)t^{\alpha-1}e^{-i\lambda t} \, dt + \int_0^\lambda t^{\alpha-1}e^{-it} \, dt \tag{2}
$$

and

$$
\lim_{\lambda \to \infty} \int_0^\lambda t^{\alpha-1}e^{-it} \, dt = \frac{\Gamma(\alpha)e^{-\frac{\pi \alpha}{2}}}{\lambda^{\alpha-1}}.
$$

(1) follows from (2).

Let now $\alpha > 1$ and $\alpha \notin \mathbb{N}$ (and $s > [\alpha] + 1$).

The function $t \mapsto t^\alpha (1 - t^2)^{s-1}$ is $[\alpha]$-times differentiable and

$$(t^\alpha (1 - t^2)^{s-1})^{(n)} = 0$$

for $t = 0$ and $t = 1$ for all $n \in [0, 1, \ldots, [\alpha]]$.

Using integration by parts we get

$$
\lim_{\lambda \to +\infty} \lambda^{\alpha} \int_0^1 (1 - t^2)^s t^{\alpha-1} e^{-i\lambda t} \, dt = \frac{\alpha - 1}{i} \lim_{\lambda \to +\infty} \lambda^{\alpha-1} \int_0^1 (1 - t^2)^s t^{\alpha - 2} e^{-i\lambda t} \, dt.
$$
Applying the previous equality \( [\alpha] \) times, we obtain

\[
\lim_{\lambda \to +\infty} \lambda^\alpha \frac{1}{0} \int (1 - t^2) \frac{e^{-i\alpha t}}{t} dt = \left( \prod_{\nu=1}^{[\alpha]} \frac{\alpha - \nu}{i} \right) \lim_{\lambda \to +\infty} \frac{1}{0} \int (1 - t^2) \frac{e^{-i[\alpha] - 1} e^{-i\lambda t}}{t} dt. \tag{3}
\]

Since \( \alpha - [\alpha] \in (0, 1) \), according to already demonstrated, it follows from (3) that

\[
\lim_{\lambda \to +\infty} \lambda^\alpha \frac{1}{0} \int (1 - t^2) \frac{e^{-i\alpha t}}{t} dt = \left( \prod_{\nu=1}^{[\alpha]} \frac{\alpha - \nu}{i} \right) \frac{\Gamma(\alpha)}{\Gamma(\alpha - [\alpha])} e^{-i\pi \frac{\alpha}{2}} e^{-i\pi \frac{[\alpha]}{2}}. \tag{2}
\]

Lemma 3. Let \( \alpha > 0 \) and \( k(\xi) = \theta(\xi)|\xi|^{\alpha - 1} \) where

\[
\theta(\xi) = \begin{cases} 
1; & \xi \geq 0, \\
0; & \xi < 0.
\end{cases}
\]

Let an operator \( T_\alpha : L^2(0, 1) \to L^2(0, 1) \) be defined by

\[
T_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \varphi(x)^\alpha k(x - y) f(y) dy.
\]

Then:

\[
\lim_{n \to \infty} (n\pi)^\alpha s_n(T_\alpha) = \left( \frac{1}{0} \int |\varphi(x)|^\alpha dx \right)^\alpha.
\]

Proof. Let

\[
\tilde{k}(\xi) = \lim_{r \to \infty} \int_{|\xi| < r} (1 - \xi^2)^\frac{s}{r^2} k(x)e^{-ix\xi} dx
\]

be Riesz–Fourier transform of the function \( k \). If \( \xi > 0 \) we obtain (by a change of variable \( x = rt \)):

\[
\xi^\alpha \tilde{k}(\xi) = \lim_{r \to +\infty} \frac{1}{0} \int (1 - t^2)^\frac{s}{r^2} t^{\alpha - 1} e^{-i\pi t r \xi} dt \cdot (r\xi)^\alpha
\]

\[
= \lim_{\lambda \to +\infty} \lambda^\alpha \frac{1}{0} \int (1 - t^2)^\frac{s}{r^2} t^{\alpha - 1} e^{-i\lambda t} dt = \Gamma(\alpha) e^{-i\pi \frac{\alpha}{2}} \quad \text{(Lemma 1)}.
\]

In a similar way, if \( \xi < 0 \), we obtain

\[
(-\xi)^\alpha \tilde{k}(\xi) = \Gamma(\alpha)e^{i\pi \frac{\alpha}{2}}.
\]

So, if \( \xi \neq 0 \), we have

\[
|\xi|^\alpha |\tilde{k}(\xi)| = \Gamma(\alpha).
\tag{4}
\]

Let

\[
N_t(T_\alpha) = \sum_{s_n(T_\alpha) \geq t} 1, \quad t > 0.
\]

According to the Birman–Solomyak theorem [2, pp. 75–76] or [7, p. 81], we get

\[
\lim_{t \to +0} t^\delta N_t(T_\alpha) = \gamma \cdot \frac{1}{\Gamma(\alpha)} \left( \frac{1}{0} \int |\varphi(x)|^\alpha dx \right)^\delta
\]

where \( \delta^{-1} = \alpha \) and
\[ \gamma = \frac{1}{2\pi} \text{measure} \{ \xi : |\hat{k}(\xi)| > 1 \} \]

\[ \gamma = \frac{1}{\pi} \left( \Gamma(\alpha) \right)^{1/2} \frac{1}{\pi} \] (it follows from (4)).

From (5) it follows that

\[ \lim_{t \to 0^+} t^{\frac{1}{\alpha}} N_t(T_\alpha) = \frac{1}{\pi} \int_0^1 |\varphi(x)| \, dx. \tag{6} \]

Substitution \( t = s_n(T_\alpha) \) into (6) gives

\[ \lim_{n \to \infty} s_n(T_\alpha)^{\frac{1}{\alpha}} \cdot n = \frac{1}{\pi} \int_0^1 |\varphi(x)| \, dx. \]

**Lemma 4.** Let function \( K \) satisfy the conditions of Theorem 1, \( \varphi(x) = K(x, x) \) and \( D_\alpha \) be operator defined by

\[ D_\alpha f(x) = \int_0^1 \varphi(x)\varphi_{\alpha-1}(x, y) f(y) \, dy. \]

Then we have

\[ \lim_{n \to \infty} (n\pi)^{\alpha} s_n(D_\alpha) = \left( \int_0^1 |K(x, x)| \, dx \right)^{\alpha}. \]

**Proof.** Let \( G_\alpha : L^2(0, 1) \to L^2(0, 1) \) be the operator defined by

\[ G_\alpha f(x) = \int_0^x H(x, y) f(y) \, dy \]

where

\[ H(x, y) = \varphi_{\alpha-1}(x, y) - \varphi(x)^{\alpha-1}(x - y)^{\alpha-1}. \]

Then we have

\[ D_\alpha = T_\alpha + M_\varphi G_\alpha \tag{7} \]

where

\[ M_\varphi f(x) = \varphi(x) f(x). \]

If we prove that

\[ \lim_{n \to \infty} n^{\alpha} s_n(G_\alpha) = 0 \tag{8} \]

then from (7), Lemma 3 and Ky Fan theorem [6, p. 52] the statement of Lemma 4 follows.

Let now \( 0 < \alpha < 1 \). Then we have

\[ \int_y^x \varphi(s) \, ds = \varphi(x)(x - y)(1 + O(x - y)), \quad y \to x_-, \]

and so

\[ \varphi_{\alpha-1}(x, y) = \varphi(x)^{\alpha-1}(x - y)^{\alpha-1}(1 + O(x - y)) \]

i.e.

\[ H(x, y) = O(x - y)^{\alpha}, \quad y \to x - . \tag{9} \]
Let $I : L^2(0, 1) \to L^2(0, 1)$, $I(x) = \int_0^x f(s) \, ds$. Applying integration by parts (using a fact that $H(x, x) = 0$, which follows from (9)) we obtain

\[ G_\alpha = -G_\alpha' I \]  \hspace{1cm} (10)

where

\[ G_\alpha' f(x) = \int_0^x \frac{\partial H}{\partial y} f(y) \, dy. \]

By a direct verification we get

\[ \left| \frac{\partial H}{\partial y} \right| \leq \text{const} (x - y)^{\alpha - 1}; \hspace{1cm} 0 \leq y < x < 1; \]

hence the operator $G_\alpha'$ is bounded on $L^2(0, 1)$.

It follows from (10) that

\[ s_n(G_\alpha) = O \left( \frac{1}{n} \right). \]

Hence we have proved (8).

Let now $\alpha \geq 1$.

Then we have

\[ \varphi_{\alpha - 1}(x, y) = \varphi(x)^{\alpha - 1}(x - y)^{\alpha - 1} \left( 1 - \frac{\varphi'(x)}{2 \varphi(x)} (x - y) + O(x - y)^2 \right), \hspace{1cm} y \to x- \]

and, for a function $\Omega$, defined by

\[ \Omega(x, y) = \varphi_{\alpha - 1}(x, y) - \varphi(x)^{\alpha - 1}(x - y)^{\alpha - 1} + \frac{\varphi(x)^{\alpha - 2} \varphi'(x)}{2} (x - y)^{\alpha - 1} \]

the equality

\[ \Omega(x, y) = O((x - y)^{\alpha + 1}), \hspace{1cm} y \to x- \]  \hspace{1cm} (11)

holds.

Let $R_\alpha : L^2(0, 1) \to L^2(0, 1)$ be the operator defined by

\[ R_\alpha f(x) = \int_0^x \Omega(x, y) f(y) \, dy. \]

The function $\Omega$ has derivatives with respect to $y$ up to the order $[\alpha] + 1$ and from (11) it follows that

\[ \frac{\partial^{[\alpha]+1} \Omega}{\partial y^{[\alpha]+1}} = O((x - y)^{[\alpha]-[\alpha]}), \hspace{1cm} y \to x-. \]

Therefore, according to Theorem 4 [6, p. 157] we have

\[ s_n(R_\alpha) = O \left( \frac{1}{n^{\frac{1}{2}+\alpha}} \right), \hspace{1cm} n \to \infty. \]  \hspace{1cm} (12)

Since

\[ R_\alpha = G_\alpha + \frac{\varphi(x)^{\alpha - 2} \varphi'(x)}{2} \int_0^x (x - y)^{\alpha} \, dy, \]

from (12) and the fact that

\[ s_n \left( \int_0^x (x - y)^{\alpha} \varphi(x)^{\alpha - 2} \varphi'(x) \, dy \right) = O \left( \frac{1}{n^{\alpha+1}} \right) \]

holds, and from the properties of singular values of the sum of two operators, we obtain (8). \hfill \Box
Lemma 5. Let 

\[ L(x, y) = (K(x, y) - K(x, x))\varphi_{\alpha - 1}(x, y) \]

then 

\[ \lim_{n \to \infty} n^\alpha s_n \left( \int_0^x L(x, y) \cdot dy \right) = 0. \]

Proof. The function \( L \) has all derivatives with respect to \( y \) up to order \([\alpha]+1\). Since 

\[ \varphi_{\alpha - 1}(x, y) = \varphi(x)^{\alpha - 1}(x - y)^{\alpha - 1}(1 + O(x - y)), \quad y \to x^-, \]

we have 

\[ L(x, y) = \frac{\partial K}{\partial y} \bigg|_{y=x} \varphi(x)^{\alpha - 1}(x - y)^\alpha (1 + O(x - y)), \quad y \to x^- . \] (13)

Let 

\[ L_1(x, y) = L(x, y) - \frac{\partial K}{\partial y} \bigg|_{y=x} \varphi(x)^{\alpha - 1}(x - y)^\alpha. \]

From (13) it follows that 

\[ \frac{\partial^{[\alpha]+1} L_1}{\partial y^{[\alpha]+1}} = O((x - y)^{\alpha - [\alpha]}), \quad y \to x^- . \] (14)

From (14), and Theorem 4 [6, p. 157], we obtain 

\[ s_n \left( \int_0^x L(x, y) \cdot dy \right) = O \left( \frac{1}{n^{[\alpha]+\frac{1}{2}}} \right), \quad n \to \infty. \] (15)

Note that 

\[ s_n \left( \int_0^x \frac{\partial K}{\partial y} \bigg|_{y=x} \cdot \varphi(x)^{\alpha - 1}(x - y)^\alpha \cdot dy \right) = o \left( \frac{1}{n^{[\alpha]+1}} \right), \quad n \to \infty. \]

Now Lemma 5 follows from the above asymptotics and the properties of the singular values of the sum of two operators. \( \square \)

3. Proof of Theorem 1

Since 

\[ A^\alpha = D_\alpha + \int_0^x L(x, y) \cdot dy \]

from Lemma 4, Lemma 5 and Ky Fan theorem, the statement of Theorem 1 follows.

For operator \( B^\alpha \) the statement follows from equality 

\[ B^\alpha = (A^\alpha)^\ast. \]

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