A new interpretation of Cauchy type singular integrals with an application to singular integral equations

N.I. IOAKIMIDIS
School of Engineering, University of Patras, GR-261 10 Patras, Greece

Received 18 May 1982
Revised 25 September 1984

Abstract: Cauchy type integrals were given the interpretation of the principal value for points inside the integration interval. Here this interpretation is modified and generalized in a very simple manner. The new interpretation in general is not equivalent to the classical one. The relationship between the new interpretation and the classical one is investigated and various applications of the new interpretation (to the Plemelj formulas, the Riemann–Hilbert boundary value problem, singular integral equations, the inversion formula, quadrature rules and interface crack problems) are presented.

Keywords: Cauchy type integrals, crack problems, inversion formula, numerical integration, Plemelj formulas, principal value integrals, Riemann–Hilbert problem, singular integral equations.

AMS(MOS) Subject Classification: 30E20, 30E25, 45E05, 65D32, 65R20, 73M05.

1. Introduction

The concept of the principal value of a Cauchy type singular integral is well known. It is based on the definition [3,6]

\[ F(x) = \int_a^b \frac{f(t)}{t-x} \, dt = \lim_{\epsilon \to 0^+} \left( \int_a^{x-\epsilon} + \int_{x+\epsilon}^b \right) \frac{f(t)}{t-x} \, dt, \quad a < x < b. \quad (1) \]

The function \( f(t) \) is assumed to be Hölder-continuous in a neighbourhood of the point \( x \). A more general definition of the same integral could be [6]

\[ F(x) = \lim_{\epsilon^0 \to 0^+} \left( \int_a^{x-\epsilon^0} + \int_{x+\epsilon^0}^b \right) \frac{f(t)}{t-x} \, dt, \quad a < x < b. \quad (2) \]

This definition does not give a unique value to \( F(x) \). This value depends on the ratio \( \epsilon^0/\epsilon^0 \) or, better, on the limit [6]

\[ k = \lim_{\epsilon^0 \to 0^+} \left( \epsilon'/\epsilon'' \right), \quad k > 0. \quad (3) \]
If $k = 1$, we obtain the principal value of the integral, $I(x)$. If not, we obtain some 'secondary' value of the same integral. Here we assume that $k$ takes any positive value. As far as we know, such an interpretation of Cauchy type singular integrals is new.

2. Definition

We use the definition (2) (under the condition (3)) for Cauchy type singular integrals. Further, evidently, we can write

$$k = e^c,$$

where $c$ is an arbitrary real constant. From (2) and (4) we obtain

$$F(x) = \int_a^b \frac{f(t) - f(x)}{t - x} \, dt + f(x) \left( \ln \frac{b - x}{x - a} + c \right), \quad a < x < b.$$  \hspace{1cm} (5)

For $k = 1$, that is, $c = 0$, (5) gives the principal value of $F(x)$. Otherwise, it does not. For any value of $c$, we suggest the symbol $\int f^{(c)}$ to denote this generalized interpretation of Cauchy type singular integrals. Therefore, we can write (2) as

$$\int_a^b \frac{f(t)}{t - x} \, dt = \lim_{\epsilon \to 0} \left( \int_a^{x - \epsilon} + \int_{x + \epsilon}^b \right) \frac{f(t)}{t - x} \, dt.$$  \hspace{1cm} (6)

Taking into account (5), we obtain also

$$\int_a^b \frac{f(t)}{t - x} \, dt = \int_a^b \frac{f(t)}{t - x} \, dt + cf(x).$$  \hspace{1cm} (7)

We can use (7), instead of (6), for the definition of the generalized interpretation of Cauchy type singular integrals. Of course, this interpretation depends on the value of $c$. Up to now we let $c$ take only real values. From now on we let $c$ take also complex values. Of course, in such cases we cannot use (6) (or (2)); we have to use (7) (or (5)) for the definition of our integral.

3. The generalized Plemelj formulas

For Cauchy type integrals of the form

$$F(z) = \int_a^b \frac{f(t)}{t - z} \, dt, \quad z = x + iy \notin [a, b].$$  \hspace{1cm} (8)

the Plemelj formulas [3.6]

$$F^\pm(x) = \pm \pi i f(x) + \int_a^b \frac{f(t)}{t - x} \, dt$$  \hspace{1cm} (9)

or, equivalently,

$$F^+(x) - F^-(x) = 2\pi i f(x), \quad F^+(x) + F^-(x) = 2\int_a^b \frac{f(t)}{t - x} \, dt$$  \hspace{1cm} (10)

play a most important role permitting the introduction of principal value integrals to physical
and engineering boundary value problems.

Now, by taking into account (7), we can rewrite (9) as

$$F^\pm(x) = \pm (\pi i \mp c)f(x) + \frac{c}{t - x} \int_a^b f(t) \, dt,$$

(11)

whereas the first of (10) remains, obviously, unchanged. As regards the second of (10), it takes the following form

$$(1 + c/\pi i)F^+(x) + (1 - c/\pi i)F^-(x) = 2 \frac{c}{t - x} \int_a^b f(t) \, dt.$$

(12)

4. The Riemann–Hilbert boundary value problem

In this fundamental boundary value problem [3], we seek a function $F(z)$ of the form (8) under the boundary condition

$$F^+(x) + mF^-(x) = h(x), \quad a < x < b.$$  

(13)

For $m = 1$, the second of (10) yields

$$\int_a^b \frac{f(t)}{t - x} \, dt = \frac{1}{2}h(x).$$  

(14)

For $m \neq 1$ the use of (9) gives

$$\frac{1 - m}{1 + m} \pi i f(x) + \int_a^b \frac{f(t)}{t - x} \, dt = \frac{1}{1 + m}h(x).$$  

(15)

This equation can be simplified if we use (12) with

$$m = (\pi i - c)/(\pi i + c)$$  

(16)

or, equivalently,

$$c = [(1 - m)/(1 + m)]\pi i.$$  

(17)

Then we obtain because of (7) and (15)

$$\frac{c}{t - x} \int_a^b f(t) \, dt = \frac{1}{1 + m}h(x).$$  

(18)

an equation completely analogous to (14) and simpler than (15).

5. Singular integral equations

As is well known, the Riemann–Hilbert boundary value problem is closely related to singular integral equations [3,6]. Consider such an equation of the second kind along $[-1, 1]$ with constant coefficients and index equal to 1 of the form [5]

$$Af(x) + \frac{B}{\pi} \int_{-1}^1 \frac{f(t)}{t - x} \, dt + \int_{-1}^1 k(t, x)f(t) \, dt = h(x), \quad -1 < x < 1.$$  

(19)
supplemented by the condition [5]
\[ \int_{-1}^{1} f(t) \, dt = 0. \]  
(20)
If \( k(t, x) \) is a continuous function and \( h(x) \) a Holder-continuous function along \([-1, 1]\), the solution \( f(t) \) of (19) behaves near the endpoints \( t \to \pm 1 \) of the integration interval \([-1, 1]\) like [5]
\[ f(t) = w(t) g(t). \]  
(21)
where \( g(t) \) remains bounded for \( t \to \pm 1 \) and \( w(t) \) is given by
\[ w(t) = (1 - t)^{\alpha} (1 + t)^{\beta}. \]  
(22)
The constants \( \alpha \) and \( \beta \) in (22) are determined by [5]
\[ -\cot \pi \alpha = \cot \pi \beta = A/B, \quad \alpha + \beta = -1, \quad -1 < \alpha, \beta < 0. \]  
(23)
Taking into account the previous results, we easily observe that the singular integral equation (19) can also be written as
\[ \frac{B^{(c)}}{\pi} \int_{-1}^{1} \frac{f(t)}{t-x} \, dt + \int_{-1}^{1} k(t, x) f(t) \, dt = h(x), \quad -1 < x < 1, \]  
(24)
having now the form of a singular integral equation of the first kind. The appropriate value for the constant \( c \) is
\[ c = \pi A/\beta \]  
(25)
as is clear from (7). Of course, in the previous developments, \( A \) and \( B \) may be also functions of \( x \). In this case, \( c \) will be a function of \( x \) too. Moreover, \( A \) and \( B \) may take also complex values. Then the same will happen for \( c \).

One advantage of the singular integral equation (24) over the original equation (19) is that (24) is simpler than (19). Another advantage of (24) is that it remains valid at the endpoints \( x = \pm 1 \) of the integration interval, provided that the generalized interpretation of the Cauchy type singular integral in (24) is changed to its interpretation as a finite-part integral [7]. Therefore, we can write (24) for \( x = \pm 1 \)
\[ \frac{B}{\pi} \int_{-1}^{1} \frac{f(t)}{t-x} \, dt + \int_{1}^{1} k(t, x) f(t) \, dt = h(x), \quad x = \pm 1. \]  
(26)

6. The inversion formula

We consider now the dominant singular integral equation of (19)
\[ Aw(x) g(x) + \frac{B}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt = h(x), \]  
(27)
where (21) was also taken into account. The solution of this equation has the form [3,5]
\[ (A^2 + B^2) g(t) = \frac{A}{w(t)} h(t) - \frac{B}{\pi} \int_{-1}^{1} \frac{1}{w(x)} \frac{h(x)}{x-t} \, dx. \]  
(28)
This formula is frequently called the inversion formula. Here, taking into account (7), (24) and (25), we rewrite (27) and (28) as

\[
\frac{B}{\pi} \int_{-1}^{1} w(t) \frac{g(t)}{t-x} \, dt = h(x)
\]

(29)

and

\[
(A^2 + B^2) g(t) = -\frac{B}{\pi} \int_{-1}^{1} \frac{h(x)}{x-t} \, dx.
\]

(30)

respectively. These equations are similar to the corresponding equations for Cauchy type singular integral equations of the first kind (with \(A = 0, \alpha = \beta = -\frac{1}{2}\)) [3].

7. Quadrature rules

We proceed now to the generalization of the interpolatory quadrature rules (including Gaussian quadrature rules) to the numerical evaluation of Cauchy type singular integrals interpreted as was suggested by (5) or (7). Such a quadrature rule for Cauchy type integrals of the type (8) has the form [1]

\[
\int_{a}^{b} w(t) \frac{g(t)}{t-z} \, dt = \sum_{i=1}^{n} A_{in} \frac{g(t_{i})}{t_{i}-z} + \frac{q_{n}(z)}{\sigma_{n}(z)} g(z) + E_{n}(g), \quad z \notin [a, b].
\]

(33)

where \(w(t)\) is the weight function (in general not given by (22)). \(A_{in}\) the corresponding weights, \(\sigma_{n}(z)\) the polynomial

\[
\sigma_{n}(z) = \prod_{i=1}^{n} (z-t_{i})
\]

(34)

\(q_{n}(z)\) the transcendental function [1]

\[
q_{n}(z) = \int_{a}^{b} \frac{\sigma_{n}(t)}{t-z} \, dt
\]

(35)

and \(E_{n}(g)\) the error term. The weights \(A_{in}\) are determined by [1]

\[
A_{in} = q_{n}(t_{i})/\sigma_{n}'(t_{i}), \quad i = 1(1)n.
\]

(36)
By taking into account (21) and (12), we obtain from (33)

\[
\frac{1}{2} \int_a^b w(t) \frac{g(t)}{t-x} \, dt = \sum_{i=1}^n A_{in} \frac{g(t_{in})}{t_{in}-x} + q_n^{(c)}(x) \frac{g(x)}{\sigma_n(x)} + E_n(g),
\]

where

\[
q_n^{(c)}(x) = \frac{1}{2} \left[ \left( 1 + c/\pi i \right) q_n^+(x) + \left( 1 - c/\pi i \right) q_n^-(x) \right].
\]

Because of (12) and (35), it is more convenient to define \( q_n^{(c)}(x) \) by

\[
q_n^{(c)}(x) = \frac{1}{2} \int_a^b w(t) \frac{\sigma_n(t)}{t-x} \, dt.
\]

We proceed now to an alternative derivation of the quadrature rule (37). To this end, we take into account that

\[
\frac{1}{2} \int_a^b w(t) \frac{g(t)}{t-x} \, dt = \int_a^b w(t) \frac{g(t) - g(x)}{t-x} \, dt + g(x) \int_a^b w(t) \frac{1}{t-x} \, dt.
\]

Next, we apply (33) but for regular integrals [1]

\[
\int_a^b w(t) g(t) \, dt = \sum_{i=1}^n A_{in} g(t_{in}) + E_n(g)
\]

to the first term of the right-hand side of (40). Then we obtain

\[
\frac{1}{2} \int_a^b w(t) \frac{g(t)}{t-x} \, dt = \sum_{i=1}^n A_{in} \frac{g(t_{in})}{t_{in}-x} + g(x) \left[ \int_a^b w(t) \frac{1}{t-x} \, dt - \sum_{i=1}^n \frac{A_{in}}{t_{in}-x} \right] + E_n(g),
\]

where \( g(t) \) is assumed to be a differentiable function. Next, we apply (42) with \( g(t) = \sigma_n(t) \). Then we see that

\[
\frac{1}{2} \int_a^b w(t) \frac{1}{t-x} \, dt - \sum_{i=1}^n \frac{A_{in}}{t_{in}-x} = q_n^{(c)}(x) \frac{1}{\sigma_n(x)}. \quad x \neq t_{in}, \quad i = 1(1)n.
\]

Now, by combining (42) and (43), we obtain (37). The previous results were analogous to the corresponding results for Cauchy type principal value integrals [4]. We proceed now to an application of the results of this section.

### 8. The Gauss–Jacobi quadrature rule

We consider now the classical Gauss–Jacobi quadrature rule [1]. This rule is of the form (41) with \([a, b] = [-1, 1] \) and \( w(t) \) defined by (22). Moreover, the polynomial \( \sigma_n(x) \) is the Jacobi polynomial \( P_n^{(\alpha, \beta)}(x) \). In this case, taking into account the formula [5]

\[
Aw(x) P_n^{(\alpha, \beta)}(x) + \frac{B}{\pi} \int_{-1}^1 w(t) P_n^{(\alpha, \beta)}(t) \, dt = -\frac{B}{2 \sin \pi \alpha} P_n^{(-\alpha, -\beta)}(x),
\]


Now it is possible to apply (37) and (41) to the approximate numerical solution of (24). The results that we obtain are obviously identical to those available in the literature [5]. Yet, the formulas derived by the present approach are simpler.

9. A crack along an interface

Now we apply the present results to a simple problem of practical importance: that of a simple crack \([a, b] = [-1, 1]\) along the straight interface between two bonded isotropic elastic half-planes. This problem can be reduced to the following Riemann–Hilbert boundary value problem along the crack \([-1, 1]\) [2]

\[
F^+(x) + mF^-(x) = h(x), \quad -1 < x < 1.
\] (46)

Here, taking into account the results of Sections 4 to 6, we observe that the solution of (46) is reduced to the solution of (18) with the constant \(c\) determined by (17). Evidently, since \(m\) is a real number, \(c\) is an imaginary number and the definitions (5) and (7) should be used. As regards the solution of (18), it is determined by the inversion formula (30). In our case, it yields

\[
f(t) = -\frac{w(t) \sin^2 \pi a^{(e)}}{(1 + m) \pi^2} \int_{-1}^{1} \frac{1}{w(x)} \frac{h(x)}{x - t} dx.
\] (47)

We can also mention that in the case of a simple straight crack inside an isotropic elastic plane we have obviously \(m = 1\) in (46) and \(c = 0\) (because of (17)). Therefore, \(\alpha = \beta = -\frac{1}{2}\) because of (23) and (25). In this special case, (47) takes the form

\[
f(t) = -\frac{w(t)}{2\pi^2} \int_{-1}^{1} \frac{1}{w(x)} \frac{h(x)}{x - t} dx.
\] (48)

well known in the literature [3].

10. Conclusions

From the previous results we conclude that Cauchy type singular integrals constitute a convenient tool for the solution of problems of mathematical physics and engineering and nothing more. The definition of these integrals is simply a matter of convenience for the user of them and is not dictated by the physics of the considered problem. Clearly, other definitions for these integrals, quite different from that suggested here, but, simultaneously, quite useful under appropriate conditions, can be proposed as well.

Acknowledgement

The results reported here belong to a research project supported by the National Hellenic Research Foundation. The author is grateful to this Foundation for this financial support.
References


