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On the Representation Theory of the Symmetric Groups and Associated Hecke Algebras

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1. INTRODUCTION

In [1, 2], Dipper and James studied the representation theory of a Hecke algebra for the finite general linear group, and derived a close analogue of the representation theory of the symmetric groups. Their derivation, however, involves considerable computational difficulties; in this article we shall present a rather different approach which largely avoids these. The symmetric group algebra can be considered as a special case of the Hecke algebra, and this the simplifications can be extended to that case also. We shall be primarily interested in what may be termed the ordinary representations, that is the case where the algebra is semisimple; however, this restriction is not applied until Section 5, until which stage the treatment is "characteristic-free."

In order to set the stage, it is useful to review the methods used for the symmetric group W on n letters over a domain K. The Specht module is defined, either as a polynomial module [5] or as a left ideal in the group algebra [3]; this is a cyclic module generated by a single element, corresponding to a fixed Young tableau, whose annihilator ideal is generated by the "Garnir elements" [11] together with elements of the form $1+\tau$, where τ is a transposition in the column stabiliser of the tableau. Using this, the standard basis for the module, indexed by standard Young tableaux, can be derived. Finally, this basis can be appropriately orthogonalised to give the "seminormal basis," with respect to which the matrices representing transpositions take a particularly simple form ("Young's seminormal representation"). In [9, 10], this last step was simplified by the demonstration that the seminormal basis constitutes a complete set of eigenfunctions of a certain set of commuting elements of the group algebra, denoted $\{L_m\}$; moreover, the corresponding projection operators comprise a complete set of primitive idempotents. These idempotents were not new, they coincide

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with those constructed by Thrall (see, for example, [13]), and were cast by Jucys [6–8] into a form equivalent to that of [10].

In [1, 2], this prescription is applied largely unchanged to the Hecke algebra $\mathscr{H}$. However, the computation is much more difficult, primarily due to the increased complexity of algebraic operations, and the Garnir relations take a particularly awkward form. We shall show that these difficulties can be avoided by working with the algebra itself, rather than with an ideal; the Specht module only appears at the end of Section 3 and in Section 6. This has the further advantage that it is not necessary to prove additionally the completeness of the representations obtained. We rely on material from [1,2], particularly in the combinatoric properties of permutations and tableaux; this is described in Section 2 and in the final part of Section 3. In Section 3 we derive a basis for $\mathscr{H}$, indexed by pairs of standard tableaux; its importance derives from the fact that, suitably ordered, it defines a sequence of subspaces which constitute a flag of $\mathscr{H}$, which we shall later show to be stabilised by $\{L_m\}$. The Garnir elements do have an analogue here, in Lemma 3.5, and indeed Lemma 3.8 shows how they may be constructed. In Section 4 $\{L_n\}$ is introduced, the basis remodelled in terms of this set, and its action on the basis is demonstrated. In Sections 5 and 6, a seminormal basis is derived for $\mathscr{H}$, and its properties explored; this analysis may be compared to the derivation of the "seminormal units" in [13]. Finally, we shall derive expressions for the dimensions of the irreducible representations of $\mathscr{H}$, the well-known "Hook Theorem," and the corresponding expressions for the dimensions of the unipotent irreducible representations of $GL_n(q)$.

2. Basic Combinatorics

Our notation is essentially that of [1, 2], with some modifications. Let n be a positive integer. W is the symmetric group acting on $\{1, 2, ..., n\}$ on the right; W_{ij} is the subgroup which permutes $\{i, i+1, ..., j\}$, and leaves all other elements of $\{1, 2, ..., n\}$ fixed. The set of basic transpositions, that is permutations of the form $(i, i+1)$, $1 \leqslant i < n$, is denoted by $\mathscr{B}$. Each $w \in W$ can be expressed in the form $w = v_1 v_2 \cdots v_k$, $v_i \in \mathscr{B}$; if k is minimal, then this is a reduced expression for w, whose length $l(w)$ is k.

Permutations are partially ordered by the strong Bruhat order. Let $v_1 v_2 \cdots v_k$ be a reduced form for $w \in W$; then for any $u \in W$, $u \trianglerighteq w$ if and only if $u = v_{i_1} v_{i_2} \cdots v_{i_j}$ for some sequence $1 < i_1 < i_2 < \cdots < i_j \leqslant k$, $j \leqslant k$.

A composition λ of n, written $\lambda \models n$, is a sequence $\lambda_1, \lambda_2, ...$ of non-negative integers whose sum is n; if the sequence is non-increasing, then λ is a partition, $\lambda \vdash n$. We shall think of the sequence as infinite, but if the last non-zero element is λ_i, we shall say that λ has i parts. The result of sorting the parts of λ into non-increasing order is of course always a partition, which we shall denote by $\bar{\lambda}$.

Let $\lambda \models n$; the Young diagram $[\lambda]$ is the set of ordered pairs $\{(i, j) : 1 \leqslant j \leqslant \lambda_i, i = 1, 2, ...\}$; a λ-tableau is a bijection $t: [\lambda] \leftrightarrow \{1, 2, ..., n\}$. If λ is a partition, then the diagram conjugate to $[\lambda]$ is $[\lambda]' = \{(i, j) : (j, i) \in [\lambda]\}$; similarly, the conjugate of a λ-tableau t is t', where $t'(i, j) = t(j, i)$. Note that $[\lambda]' = [\lambda']$ is a Young diagram for a partition λ', the partition conjugate to λ, and t' is a λ'-tableau. It is possible to define the conjugate when λ is not a partition, but we shall not need this. A λ-tableau t is row-standard if the sequence $t(i, 1)$, $t(i, 2)$, ... is strictly increasing for each i; it is standard if λ is a partition and both t and t' are row-standard.

Compositions are partially ordered by the dominance order. For $\lambda \models n$, let $\lambda_i^+ = \lambda_1 + \lambda_2 + \cdots + \lambda_i$; then $\lambda \trianglelefteq \mu$, $\mu \models n$, if and only if, for each i, $\lambda_i^+ \leqslant \mu_i^+$. It is useful to embed this in a total order; we define $\lambda < \mu$ if, for some i, $\lambda_i < \mu_i$ and for each $j < i$, $\lambda_j = \mu_j$. We shall extend both orderings to row-standard tableaux. If t is a row-standard λ-tableau, $\lambda \models n$, then the restriction of t to $\{1, 2, ..., m\}$, $m \leqslant n$ is a row-standard tableau for some composition t_m of m. We now order tableaux s and t, of arbitrary shapes, as follows: $s \trianglelefteq t$ if, for each i, $s_i \trianglelefteq t_i$. If s, t are respectively a λ- and a μ-tableau, then $s < t$ if $s \lhd t$ or if $\lambda < \mu$; $<$ is not total here, of course. We shall extend both orders to pairs of tableaux in a natural way; thus, we set $(s, t) \trianglelefteq (u, v)$ if $s \trianglelefteq u$ and $t \trianglelefteq v$, and $(s, t) \leqslant (u, v)$ if $s \leqslant u$ and $t \leqslant v$. It should be noted that while our definition of $\trianglelefteq$ accords with that of [1] for tableaux, it differs for compositions; in that article, $\lambda \trianglelefteq \mu$ means, in our notation, $\bar{\lambda} \trianglelefteq \bar{\mu}$. However, if μ is a partition then $\bar{\lambda} \trianglelefteq \bar{\mu} \Rightarrow \lambda \trianglelefteq \mu$, which is all that we shall need.

The nodes of a Young diagram can be ordered lexicographically, (i, j) preceding (k, m) if $i < k$ or if $i = k$ and $j < m$. We shall say that numbers i, $i + 1, ..., j$ occur by rows in a tableau t if they are the images of a lexicographically increasing sequence of nodes. For each $\lambda \models n$ there is a unique tableau t^λ in which $1, 2, ..., n$ occur by rows; t^λ dominates all λ-tableaux.

W acts naturally on tableaux, the action being defined by $(tw)(i, j) = t(i, j)w$, $w \in W$. The row-stabiliser of t^λ is denoted by W_λ; it is the direct product of the subgroups W_{km}, where $k = \lambda_{i-1}^+ + 1$, $m = \lambda_i^+$, $i = 1, 2, ...$. The element $w \in W$ such that the elements of $t^\lambda w$ are entered by columns is denoted by w_λ; thus if λ is a partition, $t^\lambda w_\lambda$ is the transpose of $t^{\lambda'}$. Let $\mathscr{D}_\lambda = \{w \in W: t^\lambda w \text{ is row-standard}\}$; then $\mathscr{D}_\lambda$ is a set of coset representatives of W_λ in W. If t is row-standard, we shall denote the element $d \in \mathscr{D}_\lambda$ for which $t = t^\lambda d$ by $d(t)$. Lemma (1.5) of [1] is of crucial importance, since it relates the two partial orders; it may be paraphrased thus:

LEMMA 2.1. *Let* $\lambda \models n$, $d_1, d_2 \in \mathcal{D}_\lambda$; *then* $t^\lambda d_1 \lhd t^\lambda d_2 \Leftrightarrow d_1 \lhd d_2$.

LEMMA 2.2. *Let t be a standard* λ*-tableau,* $\lambda \vdash n$*, and s an arbitrary row-standard* λ*-tableau such that* $t \lhd s$; *then* $l(d(s))\, l(d(t')^{-1}) < l(w_\lambda)$.

Proof. Either $t = t^\lambda w_\lambda$ or there is some $v \in \mathcal{B}$ such that tv is standard and $t \lhd tv$, so that any reduced form for $d(t)$ can be extended by a sequence of right multiplications to a reduced form for w_λ; in particular, $d(t)\, d(t')^{-1} = w_\lambda$, and $l(d(t))\, l(d(t')^{-1}) = l(w_\lambda)$. By the previous lemma, $d(t) \lhd d(s)$, whence $l(d(s)) < l(d(t))$, and the result follows. ∎

3. THE HECKE ALGEBRA

Let K be a principal ideal domain and q an invertible element of K. $\mathcal{H} = \mathcal{H}_{K,q}[W]$ is the Hecke algebra defined in [2]. It has K-basis $\{T_w : w \in W\}$; multiplication rules for $w \in W$, $v \in \mathcal{B}$ are given by [2, Lemma 2.1]

$$T_w T_v = \begin{cases} T_{wv} & \text{if } \ l(wv) = l(w) + 1, \\ qT_{wv} + (q-1)\, T_w & \text{otherwise,} \end{cases}$$

$$T_v T_w = \begin{cases} T_{vw} & \text{if } \ l(vw) = l(w) + 1, \\ qT_{vw} + (q-1)\, T_w & \text{otherwise.} \end{cases}$$

The unit element T_1 of $\mathcal{H}$ we shall normally write simply as 1. For each $w \in W$, T_w is invertible; in particular,

$$T_v^{-1} = q^{-1}(T_v - (q-1)) \qquad \text{for all} \quad v \in \mathcal{B}.$$

Observe that, for any pair of elements u and v of W, $T_u T_v$ is a linear combination of elements T_w, $w \in W$, such that $|l(u) - l(v)| \leqslant l(w) \leqslant l(u) + l(v)$. For $X \subseteq W$, define

$$\iota(X) = \sum_{w \in X} T_w, \qquad \varepsilon(X) = \sum_{w \in X} (-q)^{-l(w)} T_w.$$

In particular, $\iota(W)$ and $\varepsilon(W)$ are the elements x and y of [2]. Let $1 \leqslant i < j$; then

$$\begin{aligned} \iota(W_{ij})\, T_w &= q^{l(w)} \iota(W_{ij}), \\ \varepsilon(W_{ij})\, T_w &= (-1)^{l(w)} \varepsilon(W_{ij}), \quad \text{for all } \ w \in W_{ij}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \iota(W_{ij})\, T_{(k,k+1,\ldots,m)} &= T_{(k,k+1,\ldots,m)} \iota(W_{i-1,j-1}), \\ \varepsilon(W_{ij})\, T_{(k,k+1,\ldots,m)} &= T_{(k,k+1,\ldots,m)} \varepsilon(W_{i-1,j-1}) \quad \text{for } \ 1 \leqslant k < i < j \leqslant m, \end{aligned} \tag{3.2}$$

$$\begin{aligned}\iota(W_{ij}) &= \iota(W_{i,j-1}) \sum_{k=i}^{j} T_{(k,k+1,\ldots,j)} \\ &= \iota(W_{i,j-1}) \left(1 + q \sum_{k=i}^{j-1} q^{k-j} T_{(j,k)}\right), \\ \varepsilon(W_{ij}) &= \varepsilon(W_{i,j-1}) \sum_{k=i}^{j} (-q)^{k-j} T_{(k,k+1,\ldots,j)} \\ &= \varepsilon(W_{i,j-1}) \left(1 - \sum_{k=i}^{j-1} q^{k-j} T_{(j,k)}\right),\end{aligned} \tag{3.3}$$

$$T_{(i,i+1,\ldots,j)} T_{(j,j-1,\ldots,i)} = q^{j-i} + (q-1) \sum_{k=i}^{j-1} q^{k-i} T_{(j,k)}. \tag{3.4}$$

Let h^* denote the image of $h \in \mathscr{H}$ under the antiautomorphism of $\mathscr{H}$ induced by the map $T_w \mapsto T_{w^{-1}}$, $w \in W$. The role of h^* is akin to that of a Hermitian conjugate, and we shall call it the $*$-conjugate of h. Note that if X is a product of subgroups of the form W_{ij} then $\iota(X)$ and $\varepsilon(X)$ are self-conjugate.

Let $\lambda \models n$; for any pair of row-standard λ-tableaux s and t we define $x_{st} = T^*_{d(s)} \iota(W_\lambda) T_{d(t)}$ and $y_{st} = T^*_{d(s)} \varepsilon(W_\lambda) T_{d(t)}$. Note that $x^*_{st} = x_{ts}$ and $y^*_{st} = y_{ts}$. To simplify notation, we adopt the convention that if a subscript is t^λ we may replace it by λ; note that a subscript λ' stands for $t^{\lambda'}$, not $(t^\lambda)'$. Our definitions of $x_{\lambda\lambda}$ and $y_{\lambda\lambda}$ coincide with those of x_λ and y_λ in [2]. Let $v = (i, i+1) \in \mathscr{B}$ and let $u = tv$; then $t \lhd u$ if i is in an earlier row of t than $i+1$ and $t \rhd u$ if it is in a later row. If $i-1$ and i are in the same row of t, and $j-1$, j occupy the same position in t^λ then $(j-1, j)\, d(t) = d(t)(i-1, i)$, and clearly $l((j, j-1)\, d(t)) = l(d(t)) + 1$, since $(j-1, j) \in W_\lambda$, so that $x_{st} T_{(j-1,j)} = q x_{st}$, $y_{st} T_{(j-1,j)} = -y_{st}$. Therefore (cf. [1, Lemma 3.2]) we have

$$x_{st} T_v = \begin{cases} x_{su} & \text{if } i \text{ belongs to an earlier row of } t \text{ than } i+1, \\ q x_{st} & \text{if } i \text{ and } i+1 \text{ are in the same row of } t, \\ q x_{su} + (q-1)\, x_{st} & \text{otherwise,} \end{cases}$$

$$y_{st} T_v = \begin{cases} y_{su} & \text{if } i \text{ belongs to an earlier row of } t \text{ then } i+1, \\ -y_{st} & \text{if } i \text{ and } i+1 \text{ are in the same row of } t, \\ q y_{su} + (q-1)\, y_{st} & \text{otherwise.} \end{cases}$$

The left action of T_v is given by $*$-conjugation, which simply interchanges s with t or u and replaces $x_{st} T_v$ by $T_v x_{ts}$. Thus $\{x_{st}$: s, t are row-standard λ-tableaux$\}$ spans $\mathscr{H} x_{\lambda\lambda} \mathscr{H}$; if $\lambda = (1^n)$ then this is just $\mathscr{H}$, whence, trivially, $\{x_{st}$: s and t are row-standard tableaux$\}$ spans $\mathscr{H}$. $\{x_{st}\}$ and $\{y_{st}\}$ have similar properties; when we prove a property of the former we shall state the corresponding property of the latter, but the proof we shall merely sketch, since it will be similar, replacing x and ι by y and ε.

LEMMA 3.5. *Let s and t be row-standard λ-tableaux, $\lambda \models n$; if either s or t is nonstandard then x_{st} (resp. y_{st}) can be expressed as a linear combination of elements of the form x_{uv} (resp. y_{uv}) such that $(s, t) < (u, v)$.*

Proof. We may consider that t is non-standard, since the corresponding case for s may then be derived simply by $*$-conjugation. If $\lambda \neq \bar{\lambda}$, then we may proceed in the manner of [1, Lemma 4.3]. There is a $d \in \mathscr{D}_{\bar{\lambda}}$ such that $dW_\lambda = W_{\bar{\lambda}} d$, so that $T_d x_{\lambda\lambda} = x_{\bar{\lambda}\bar{\lambda}} T_d$, from which $x_{\lambda\lambda} = T_d^{-1} x_{\bar{\lambda}\bar{\lambda}} T_d$. Consequently x_{st} can be expressed as a linear combination of elements x_{uv}, where u and v are $\bar{\lambda}$-tableaux, and $\lambda < \bar{\lambda}$.

Now suppose that $\lambda \vdash n$; we shall proceed by induction on $\lhd$. Assume that the lemma holds for each pair of tableaux (u, v) such that $(u, v) \rhd (s, t)$. Since t is non-standard, there is at least one m such that m occurs in an earlier row of t than $m-1$, and $t(m-1, m) \rhd t$; if we can choose such an m so that $t(m-1, m)$ is non-standard, then the lemma holds for $t(m-1, m)$ by the inductive hypothesis, and is easily extended to t since if $t(m-1, m) \lhd v$ then $t \lhd v$ and $t \lhd v(m-1, m)$ by Lemma 2.1. Otherwise there can only be one m which occurs in an earlier row of t than $m-1$, and $t(m-1, m)$ is standard; this can only happen if $m = \lambda_{i-2}^{+} + 2j - 1$ for some $i \geqslant 2$, $j \leqslant \lambda_i$ and t is constructed by inserting $1, 2, ..., m$ by rows into the subdiagram $[\lambda_1, \lambda_2, ..., \lambda_{i-2}, j-1, j]$, then inserting $m+1$, $m+2, ..., n$ by rows into the remaining nodes of $[\lambda]$. Let ν be the composition obtained from λ by replacing the subsequence λ_{i-1}, λ_i by $j-1$, $\lambda_{i-1} - j + 1$, j, $\lambda_i - j$, and let μ be the composition obtained from λ by replacing λ_{i-1}, λ_i by $j-1$, $\lambda_{i-1}+1$, $\lambda_i - j$, so that for ν we break two rows into two pieces and for μ we recombine two of the fragments. For example, we might have $\lambda = (6, 5, 4, 2)$, $m = 12$, giving $i = 3$, $j = 3$ and

$$t = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 12 & 13 & 14 & \\ 9 & 10 & 11 & 15 & & \\ 16 & 17 & & & & \end{array} \qquad t^\nu = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & & & & \\ 9 & 10 & 11 & & & \\ 12 & 13 & 14 & & & \\ 15 & & & & & \\ 16 & 17 & & & & \end{array} \qquad t^\mu = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & & & & \\ 9 & 10 & 11 & 12 & 13 & 14 \\ 15 & & & & & \\ 16 & 17 & & & & \end{array}$$

Let Δ_k be the subset of $\mathscr{D}_\nu$ which fixes all but rows k and $k+1$ of t^ν, and Δ_k^{-1} the set of inverses of Δ_k; then $W_\lambda = W_\nu \Delta_{i-1} \Delta_{i+1} = \Delta_{i-1}^{-1} \Delta_{i+1}^{-1} W_\nu$, so that $x_{\lambda\lambda} = x_{\nu\nu}\imath(\Delta_{i-1})\,\imath(\Delta_{i+1}) = \imath(\Delta_{i-1})^*\imath(\Delta_{i+1})^* x_{\nu\nu}$, and similarly $x_{\mu\mu} = x_{\nu\nu}\imath(\Delta_i)$. Consequently,

$$x_{s\lambda}\imath(\Delta_i) = T^*_{d(s)}\imath(\Delta_{i-1})^*\imath(\Delta_{i+1})^* x_{\mu\mu}.$$

Note that $t \in t^\lambda \Delta_i$ and that $t^\lambda \Delta_i \trianglerighteq t$, that is, t is the unique least dominant element. Thus the left side of the above equation is the sum of x_{st} and a linear combination of elements x_{su} where u is a λ-tableau such that $t \lhd u$, while the right side can be expressed as a linear combination of $\{x_{uv}: u, v$ are $\bar{\mu}$-tableaux$\}$. Obviously $\lambda < \bar{\mu}$, so that the lemma holds for x_{st}; this completes the proof. ∎

Repeated application of this lemma will express any element x_{st} as a linear combination of standard element, so that $\{x_{st}: s$ and t are standard$\}$ spans $\mathscr{H}$. It is well known that there is a one–one correspondence between pairs of standard tableaux and permutations [12, 14], so that by comparing dimensions we may conclude that we have a basis for $\mathscr{H}$. However, this is not quite enough, and we can prove linear independence directly, using [1, Lemmas 4.1 and 5.1]. Let λ and μ be partitions and $w \in W$; then the first of these states that $x_{\lambda\lambda} w y_{\mu\mu}$ is non-zero only if the double coset $W_\lambda w W_\mu$ has the trivial intersection property, $w^{-1} W_\lambda w \cap W_\mu = \{1\}$, which implies that $\lambda \trianglelefteq \mu'$. In case $\lambda = \mu'$, there is only one double coset having the trivial intersection property, namely $W_\lambda w_\lambda W_{\lambda'}$, of which w_λ is the unique element of minimal length, whence $x_{\lambda\lambda} w y_{\lambda'\lambda'} = 0$ whenever $l(w) < l(w_\lambda)$. Let s and t be row-standard λ-tableaux, t standard, with $t \lhd s$; then by Lemma 2.2, $l(d(s))\, l(d(t')^{-1}) < l(w_\lambda)$, so that $x_{\lambda s} y_{t'\lambda'} = x_{\lambda\lambda} T_{d(s)} T^*_{d(t')} y_{\lambda'\lambda'} = 0$.

On the other hand, the special element

$$x_{\lambda\lambda} T_{w_\lambda} y_{\lambda'\lambda'}$$

is not zero; it consists of the sum of T_{ω_λ} and a linear combination of elements T_w, where w runs over the remaining elements of $W_\lambda w_\lambda W_{\lambda'}$. Let s and t' be row-standard λ and λ' tableaux, respectively; then we define $z_{st} = T^*_{d(s)} x_{\lambda\lambda} T_{\omega_\lambda} y_{\lambda'\lambda'} T_{d(t')}$. Now we have (cf. [1, Lemma 5.1])

LEMMA 3.6. *Let $\lambda \vdash n$; if t is a standard λ-tableau then $z_{\lambda t}$ is linearly independent of*

$$\{z_{\lambda s}: s' \rhd t',\ s' \text{ a row-standard } \lambda'\text{-tableau}\}$$

and $z_{t\lambda}$ is linearly independent of

$$\{z_{\lambda s}: s \rhd t,\ s \text{ a row-standard } \lambda\text{-tableau}\}.$$

Proof. Since t is standard, $w_\lambda d(t') = d(t)$ and $T_{w_\lambda} T_{d(t')} = q^{l(t')} T_{d(t)} +$ a linear combination of elements T_w with $l(w) > l(d(t))$. Since for $w \in W_\lambda$, $l(wd(t)) = l(w) + l(d(t))$, $z_{\lambda t}$ has the same form. Now let $s' \rhd t'$; then $l(s') < l(t')$. Consider the expansion of $T_w T_{w_\lambda} T_{d(s')}$, where $w \in W_\lambda$; this consists of a linear combination of $\{T_{\bar{w}} : \bar{w} \in W,\ l(\bar{w}) \geq l(w) + l(w_\lambda) - l(s')\}$, so that the coefficient of $T_{d(t)}$ in $z_{\lambda s}$ is zero. This proves the lemma for $z_{\lambda s}$; $z_{t\lambda}$ is treated in the same way. ∎

LEMMA 3.7. *Let s, t be standard λ-tableaux; then x_{st} (resp. y_{st}) is linearly independent of $\{x_{uv} : (s, t) < (u, v)\}$ (resp. $\{y_{uv} : (s, t) < (u, v)\}$), so that, letting λ run over all partitions of n, we have $\{x_{st}$: s and t are standard λ-tableaux, $\lambda \vdash n\}$ (resp. $\{y_{st} : s, t$ are standard λ-tableaux, $\lambda \vdash n\}$) is a linearly independent set.*

Proof. Obviously the second statement follows from the first. Suppose that the first is false; then we have elements $a_{uv} \in K$ such that

$$x_{st} = \sum_{(u,v) > (s,t)} a_{uv} x_{uv},$$

the sum running over all pairs of row-standard (not necessarily standard) tableaux of the same shape. We may annihilate all the terms with $v \neq t$ by multiplying on the right by $y_{t'\lambda'}$, and obtain

$$z_{s\lambda} = \sum_{\{u : u \lhd s,\ u \text{ is a } \lambda\text{-tableau}\}} a_{u\lambda} z_{u\lambda},$$

which contradicts the last lemma. ∎

LEMMA 3.8. *If t' is a row-standard λ'-tableau, $\lambda \vdash n$, but not standard, then $z_{\lambda t}$ can be expressed as a linear combination of $\{z_{\lambda s} : s' \rhd t',\ s'$ a row-standard λ'-tableau$\}$.*

Proof. Apply Lemma 3.5 to $y_{\lambda' t'}$ and multiply on the left by $x_{\lambda\lambda} T_{w_\lambda}$; the result is immediate. ∎

THEOREM 3.9. *Each of $\{x_{st}$: s and t are standard λ-tableaux, $\lambda \vdash n\}$ and $\{y_{st}$: s and t are standard λ-tableaux, $\lambda \vdash n\}$ is a basis for $\mathscr{H}$, and $\{z_{\lambda t}$: t is a standard λ-tableau$\}$ is a basis for the right ideal $S^\lambda = z_{\lambda t}\mathscr{H}$ (the Specht module of [1]).*

Proof. The first part follows immediately from Lemma 3.5 and 3.7, the second from Lemmas 3.6 and 3.8. ∎

4. THE ACTION OF $\{L_m\}$ ON $\mathscr{H}$

Define elements $L_m \in \mathscr{H}$ by

$$L_m = q^{-1}T_{(m-1,m)} + q^{-2}T_{(m-2,m)} + \cdots + q^{-m}T_{(1,m)} \qquad \text{for} \quad 1 \leqslant m \leqslant n.$$

These are the same as in [2], except that $L_1 = 0$; the following commutation relations are easily checked. Let $2 \leqslant i \leqslant n$, $2 \leqslant m \leqslant n$; then

$$\begin{aligned} T_{(i-1,i)}L_m &= L_m T_{(i-1,i)} \qquad \text{if} \quad i-1 \neq m \neq i, \\ T_{(m-1,m)}L_m &= 1 + L_{m-1}T_{(m-1,m)} + (q-1)\,L_m, \\ L_m T_{(m-1,m)} &= 1 + T_{(m-1,m)}L_{m-1} + (q-1)\,L_m. \end{aligned} \tag{4.1}$$

If we set $(m-1, m) = v$, the last pair of equations can be rewritten as

$$\begin{aligned} qT_v^{-1}L_m &= 1 + L_{m-1}T_v, \\ T_v L_{m-1} &= qL_m T_v^{-1} - 1. \end{aligned} \tag{4.2}$$

It follows that $T_{(m-1,m)}$ commutes with $L_{m-1} + L_m$ and with $m_{m-1}L_m$, and so with any polynomial in $\{L_i\}$ which is symmetric in L_{m-1} and L_m. From (3.3) we see that

$$\iota(W) = \prod_{m=1}^{n} (1 + qL_m), \qquad \varepsilon(W) = \prod_{m=1}^{n} (1 - L_m). \tag{4.3}$$

For any integer k, define $[k]_q = (1-q^k)/(1-q)$; for positive k we have $[k]_q = 1 + q + \cdots + q^{k-1}$, $[-k]_q = -q^{-1} - q^{-2} - \cdots - q^{-k}$, and $[0]_q = 0$. For $q = 1$ we take the natural limit, so that $[k]_1 = k$. The residue or content of the node (i, j) of a Young diagram is $j - i$; in [2] this is generalised to $[j-i]_q$. We denote the generalised residue of the node occupied by m in a tableau t (not necessarily row-standard) by $r_t(m)$. A diagonal of $[\lambda]$ is a set of nodes of constant residue; a λ-tableau whose elements increase strictly from left to right along the diagonals we shall call a regular tableau. Obviously a regular tableau is uniquely determined by its residues. Let $\alpha(\lambda) = \sum_{i=1}^{\infty} i\lambda_i$; for t a λ-tableau, let $\alpha(t) = \alpha(\lambda)$.

LEMMA 4.4. *Let $k > 0$ and $\mu \models n-1$ such that μ has $\bar{k} < k$ parts, and let s be a row-standard μ-tableau. Let σ, τ be the tableaux obtained by adjoining n to respectively the first and the kth rows of s; then*

$$\sum_{\{t : \tau \trianglelefteq t \trianglelefteq \sigma\}} q^{-\alpha(t) - l(d(t))} x_{tt} = (L_n - [-k]_q)\, q^{-\alpha(s) - l(d(s))} x_{ss},$$

$$\sum_{\{t : \tau \trianglelefteq t \trianglelefteq \sigma\}} q^{\alpha(t) - l(d(t))} y_{tt} = q([k]_q - L_n)\, q^{\alpha(s) - l(d(s))} y_{ss}.$$

Proof. Let t be obtained by attaching n to the ith row of s, and let ν be the corresponding composition of n. Suppose that $n = md(t)$, so that m occupies the same position in t^{ν} as n in t; then $d(t) = (n, n-1, ..., m)\, d(s)$. Therefore, the left side of each identity has a common left-factor $T^*_{d(s)}$ and a common right-factor $T_{d(s)}$; x_{ss} and y_{ss} have, of course, the same factors. Since L_n commutes with $\mathscr{H}_{K,q}[W_{1,n-1}]$, these factors may be cancelled throughout, so that it is enough to prove the lemma in the case that $s = t^{\mu}$, which we now assume. Let the ith row of t^{ν} be $j, j+1, ..., m$; we now have

$$x_{tt} = T^*_d x_{\nu\nu} T_d,$$

where $d = d(t) = (n, n-1, ..., m)$. Let the entries in some row of t^{μ} be $a, a+1, ..., b$; then this row supplies a factor $\iota(W_{ab})$ to $x_{\mu\mu}$. The corresponding row supplies the same factor to $x_{\nu\nu}$ if it is not later than the ith, in which case the factor commutes with T_d; otherwise it supplies a factor $\iota(W_{a+1,b+1})$ and $\iota(W_{a+1,b+1})\, T_d = T_d \iota(W_{ab})$ by (3.2). The ith row supplies an extra factor to $x_{\nu\nu}$, which by (3.3) is

$$1 + \sum_{c=j}^{m-1} q^{c-m+1} T_{(c,m)},$$

whence, since $l(d) = n - m$ and $i = \alpha(\nu) - \alpha(\mu)$, we have, for $i < \bar{k}$,

$$\begin{aligned} q^{-\alpha(t) - l(d(t))} x_{tt} &= q^{m-n-i} T^*_d \left(1 + \sum_{c=j}^{m-1} q^{c-m+1} T_{(c,m)}\right) T_d q^{-\alpha(\mu)} x_{\mu\mu} \\ &= q^{-i} \left(1 + \sum_{c=j}^{n-1} q^{c-n+1} T_{(c,n)}\right) q^{-\alpha(\mu)} x_{\mu\mu} \\ &\quad - q^{-i-1} \sum_{c=m}^{n-1} q^{c-n+1} T_{(c,n)} q^{-\alpha(\mu)} x_{\mu\mu}. \end{aligned}$$

If $i \geqslant \bar{k}$ then d is the identity, and we have

$$q^{-\alpha(\nu)} x_{\nu\nu} = q^{-i} \left(1 + \sum_{c=j}^{n-1} q^{c-n-1} T_{(c,n)}\right) q^{-\alpha(\mu)} x_{\mu\mu}.$$

Observe that when i is decreased by 1, the previous j becomes the new m, so that summing over i from 1 to k leads to partial cancellation between alternate terms and gives.

$$\begin{aligned} \sum_{\{t:\, \tau \trianglelefteq t \trianglelefteq \sigma\}} q^{-\alpha(t)} x_{tt} &= (q^{-k} + q^{-k+1} + \cdots + q^{-1} + L_n)\, q^{-\alpha(\mu)} x_{\mu\mu} \\ &= (L_n - [-k]_q)\, q^{-\alpha(\mu)} x_{\mu\mu}. \end{aligned}$$

For y we have

$$q^{\alpha(t)-l(d(t))}y_{tt}=q^{m-n+i}T_d^*\left(1-\sum_{c=j}^{m-1}q^{c-m}T_{(c,m)}\right)T_d q^{\alpha(\mu)}y_{\mu\mu}$$

$$=q^i\left(1-\sum_{c=j}^{n-1}q^{c-n}T_{(c,n)}\right)q^{\alpha(\mu)}y_{\mu\mu}$$

$$-q^{i-1}\sum_{c=m}^{n-1}q^{c-n}T_{(c,n)}q^{\alpha(\mu)}y_{\mu\mu},$$

for $i<\bar{k}$, and

$$q^{\alpha(v)}y_{vv}=q^i\left(1-\sum_{c=j}^{n-1}q^{c-n}T_{(c,n)}\right)q^{\alpha(\mu)}y_{\mu\mu}$$

otherwise. Collecting terms now gives

$$\sum_{\{t:\tau\trianglelefteq t\trianglelefteq\sigma\}}q^{\alpha(t)-l(d(t))}y_{tt}=(q^k+q^{k-1}+\cdots+q-qL_n)\,q^{\alpha(\mu)}x_{\mu\mu}$$

$$=q([k]_q-L_n)\,q^{\alpha(\mu)}y_{\mu\mu}. \quad \blacksquare$$

For $\lambda\models n$, let $\rho_\lambda(m)$ be the number of the row occupied by m in t^λ, $1\leqslant m\leqslant n$. We define

$$\xi_{\lambda\lambda}=q^{\alpha(\lambda)}\prod_{m=1}^{n}(L_m-[-\rho_\lambda(m)]_q),$$

$$\eta_{\lambda\lambda}=q^{-\alpha(\lambda)}\prod_{m=1}^{n}q([\rho_\lambda(m)]_q-L_m),$$

and $\xi_{st}=T^*_{d(s)}\,\xi_{\lambda\lambda}T_{d(t)}$, $\eta_{st}=T^*_{d(s)}\,\eta_{\lambda\lambda}T_{d(t)}$ for any row-standard λ-tableaux s and t. A simple induction on the last lemma gives

THEOREM 4.5. *For any $\lambda\models n$,*

$$\xi_{\lambda\lambda}=\sum_{\{t:t^\lambda\trianglelefteq t\}}q^{\alpha(\lambda)-\alpha(t)-l(d(t))}x_{tt},$$

$$\eta_{\lambda\lambda}=\sum_{\{t:t^\lambda\trianglelefteq t\}}q^{-\alpha(\lambda)+\alpha(t)-l(d(t))}y_{tt},$$

where the sum runs over row-standard tableaux.

If s and t are standard, then by Lemma 3.7, ξ_{st} is the sum of x_{st} and a linear combination of $\{x_{uv}:(u,v)\rhd(s,t)\}$, and similarly for η_{st}; consequently, $\{\xi_{st}: s, t \text{ standard}\}$ and $\{\eta_{st}: s, t \text{ standard}\}$ are both bases for $\mathscr{H}$, being derived by linear transformations of unit moduli from known bases.

THEOREM 4.6. *Let s, t be row-standard λ-tableaux; then, for $1 \leqslant m \leqslant n$, each of $\xi_{st}(L_m - r_t(m))$, $(L_m - r_s(m))\, \xi_{st}$ (resp. $\eta_{st}(L_m - r_{t'}(m))$, $(L_m - r_{s'}(m))\, \eta_{st}$) is a linear combination of elements $\xi_{\sigma\tau}$ (resp. $\eta_{\sigma\tau}$) such that σ and τ are row-standard tableaux with $(s, t) \lhd (\sigma, \tau)$, or, equivalently, σ and τ are standard tableaux with $(s, t) < (\sigma, \tau)$.*

Proof. Note that $\xi_{st}(L_m - r_t(m))$ is the $*$-conjugate of $(L_m - r_t(m))\, \xi_{ts}$, so that we need only consider the former case; similarly for η_{st}. Moreover, we need only consider the case where $s = t_\lambda$, since the general result is then obtained by multiplying on the left by $T^*_{d(s)}$. Observe also that we might have written x instead of ξ; indeed, these are interchangeable throughout, since x_{st} and ξ_{st} differ only in terms ξ_{uv}(or x_{uv}) with $(u, v) \rhd (s, t)$, and similarly for y, η. Since

$$\xi_{(n)(n)} = \iota(W), \qquad \eta_{(n)(n)} = \varepsilon(W),$$

and

$$r_{(n)}(m) = [m-1]_q, \qquad r_{(1^n)}(m) = [1-m]_q,$$

we have

$$\xi_{(n)(n)}(L_m - r_{(n)}(m)) = 0 = \eta_{(n)(n)}(L_m - r_{(1^n)}(m))$$

by (3.1), so the theorem holds trivially in this case. Let us suppose that it holds for every $\xi_{\sigma\tau}$ with $(\sigma, \tau) \rhd (s, t)$ for all m, and for $\xi_{\lambda t}$ for all $m < k$ for some $k \leqslant n$. Now, either $t = t^\lambda$ or there is some $j > 1$ in an earlier row of t than $j-1$, in which case if $v = (j-1, j)$ then $u = tv \rhd t$ is row-standard, and so $\xi_{\lambda\mu}$ satisfies the inductive hypothesis. Set $w = (k-1, k)$; there are five cases to consider.

(i) $t = t^\lambda$ and k is the first element of a row (the ith say) of t^λ; then $r_\lambda(k) = [1-i]_q$, and if $\mu = (\lambda_1, \lambda_2, ..., \lambda_{i-2}, \lambda_{i-1}+1, \lambda_i - 1, \lambda_{i+1}, ...)$ so that $\mu \rhd \lambda$, then

$$\xi_{\lambda\lambda}(L_k - r_\lambda(k)) = \lambda_{\mu\mu}(L_k - [-i]_q).$$

(ii) $t = t^\lambda$ and k is in the same row of t^λ as $k-1$; then $x_{\lambda\lambda} T_w^{-1} = q^{-1} x_{\lambda\lambda}$ and

$$r_\lambda(k) = 1 + q r_\lambda(k-1).$$

Therefore,

$$x_{\lambda\lambda}(L_k - r_\lambda(k)) = q x_{\lambda\lambda} T_w^{-1}(L_k - r_\lambda(k)) = x_{\lambda\lambda}(L_{k-1} - r_\lambda(k-1))\, T_w.$$

(iii) $t \neq t^\lambda$, $j-1 \neq k \neq j$; here $r_t(k) = r_k(k)$ and

$$\xi_{\lambda t}(L_k - r_t(k)) = \xi_{\lambda u} T_v (L_k - r_t(k)) = \xi_{\lambda u}(L_k - r_u(k))\, T_v.$$

(iv) $t \neq t^\lambda$, $j = k$; here $r_t(k) = r_u(k-1)$ and

$$\begin{aligned}\xi_{\lambda t}(L_k - r_t(k)) &= \xi_{\lambda u} T_v (L_k - r_u(k-1)) \\ &= \xi_{\lambda u}(L_{k-1} - r_u(k-1))\, T_v + \xi_{\lambda u} + (q-1)\, \xi_{\lambda u} L_k.\end{aligned}$$

(v) $t \neq t^\lambda$, $j = k+1$; here $r_t(k) = r_u(k+1)$ and

$$\begin{aligned}\xi_{\lambda t}(L_k - r_t(k)) &= \xi_{\lambda u} T_v (L_k - r_u(k+1)) \\ &= \xi_{\lambda u}(L_{k+1} - r_u(k+1))\, T_v - \xi_{\lambda u} - (q-1)\, \xi_{\lambda u} L_k.\end{aligned}$$

In each case, it follows from the inductive hypothesis, together with Lemma 2.1 in cases (iii)–(v), that the right-hand side is a sum of elements $\xi_{\sigma\tau}$ with $(s, t) \lhd (\sigma, \tau)$; these may be replaced by standard tableaux by application of Lemma 3.5. ∎

5. The Seminormal Basis

We now assume that $[2]_q$, $[3]_q, \ldots, [n]_q$ are invertible elements of K. Note that this ensures that any regular tableau is precisely determined by its generalised residues. Let $\mathscr{R}(m)$ be the set of possible generalised residues $r_t(m)$ for standard tableaux t, i.e.,

$$\begin{aligned}\mathscr{R}(m) &= \{[k]_q : -m < k < m\} \qquad &\text{if } m \geqslant 4,\\ \mathscr{R}(m) &= \{[k]_q : -m < k < m\} \backslash \{0\} \qquad &\text{if } m = 2, 3.\end{aligned}$$

For any tableau t, we define

$$E_t = \prod_{m=1}^{n} \prod_{c \in \mathscr{R}(n) \backslash \{r_t(m)\}} \frac{L_m - c}{r_t(m) - c}.$$

Let s and t be standard tableaux, and u an arbitrary regular tableau; note that either $t = u$ or there is a k such that $r_t(k) \neq r_u(k)$. Therefore, from Theorem 4.6 we have

$$\xi_{st} E_u = \delta_{tu} \xi_{st} + \sum_{\{\sigma, \tau :\, \sigma, \tau \text{ standard},\, (\sigma, \tau) \rhd (s, t)\}} a_{\sigma\tau} \xi_{\sigma\tau}, \; a_{\sigma\tau} \in K, \tag{5.1}$$

where δ_{su} is the Kronecker delta, one if its arguments are identical, zero otherwise. Thus $E_t \neq 0$ by Lemma 3.7, since t is standard. Since there are finitely many standard tableaux, it follows that there is some power of E_t, the jth say, which annihilates each ξ_{su} with $t<u$; thus by Theorem 4.6,

$$\xi_{st} E_t^j (L_m - r_t(m)) = \xi_{st}(L_m - r_t(m)) E_t^j = 0, \qquad m = 1, 2, ..., n.$$

Clearly $\{\xi_{st} E_t^j : s, t \text{ standard}\}$ is a basis for $\mathscr{H}$, since it is obtained from $\{\xi_{st}: s, t \text{ standard}\}$ by a unimodular linear transformation; consequently, any polynomial over K in the elements $\{L_m : 1 < m \leqslant n\}$ is zero if for each standard t it contains a factor $L_m - r_t(m)$ for some m (not necessarily the same m for each t, of course), since it then annihilates $\mathscr{H}$. In particular, $E_u = 0$ if u is regular but not standard, and

$$\prod_{c \in \mathscr{R}(m)} (L_m - c) = 0, \qquad m = 2, 3, ..., n. \tag{5.2}$$

Note that we cannot remove any if the factors and still retain zero, since the result is a factor of some E_t, where t is standard, so that (5.2) is the minimum polynomial for $L_{\tilde{m}}$. Define

$$E(m, k) = \prod_{c \in \mathscr{R}(m) \setminus \{k\}} \frac{L_m - c}{k - c}, \qquad k \in \mathscr{R}(m).$$

Note that $E(m, k) \neq 0$, since it is a factor of E_t; (5.2) now shows that $\{E(m, k): k \in \mathscr{R}(m)\}$ is a set of orthogonal idempotents of $\mathscr{H}$. Let u be indeterminate; a variant of a well-known polynomial identity gives

$$\sum_{k \in \mathscr{R}(m)} \prod_{c \in \mathscr{R}(m) \setminus \{k\}} \frac{u - c}{k - c} \equiv 1;$$

this is most easily proved by noting that the left side is a polynomial of maximum degree $|\mathscr{R}(m)| - 1$, which takes the value 1 at each $u \in \mathscr{R}(m)$. Substituting L_m for u shows that the set of idempotents is complete.

It follows that $\{E_t : t \text{ is standard}\}$ is also a set of mutually orthogonal idempotents. In fact, some factors cancel, giving

$$E_t = \prod_{m=2}^{n} E(m, r_t(m)).$$

Summing over regular tableaux gives

$$\sum_{\{t \text{ standard}\}} E_t = \sum_{\{t \text{ regular}\}} E_t = \prod_{m=2}^{n} \sum_{k \in \mathscr{R}(m)} E(m, k) = 1,$$

so that this set also is complete. From (5.1) we see that $\mathscr{H}$ has K-basis

$$\{E_s \xi_{st} E_t : s, t \text{ are standard}\},$$

(bear in mind that E_t is self $*$-conjugate) whence $E_t \mathscr{H} E_t$ is a linear K-module and so E_t is primitive. We set

$$\zeta_{st} = E_s \xi_{st} E_t;$$

$\{\zeta_{st}\}$ we shall call the seminormal basis for $\mathscr{H}$. From (5.1) we have

$$\zeta_{st} = \xi_{st} + \sum_{\{\sigma, \tau:\, \sigma,\, \tau \text{ standard}, (\sigma, \tau) > (s, t)\}} b_{\sigma\tau} \xi_{\sigma\tau}, \qquad b_{\sigma\tau} \in K, \tag{5.3}$$

which may be solved to give

$$\xi_{st} = \zeta_{st} + \sum_{\{\sigma, \tau:\, \sigma, \tau \text{ standard}, (\sigma, \tau) > (s, t)\}} c_{\sigma\tau} \zeta_{\sigma\tau}, \qquad c_{\sigma\tau} \in K. \tag{5.4}$$

The orthogonality relations for the idempotents now give

$$\xi_{st} E_u = 0 \qquad \text{unless} \quad t \trianglelefteq u \tag{5.5}$$

for any standard tableaux s, t, u.

Let

$$E^\lambda = \sum_{\{t:\, t \text{ a standard } \lambda\text{-tableau}\}} E_t = \sum_{\{t:\, t \text{ a regular } \lambda\text{-tableau}\}} E_t.$$

E^λ is explicitly symmetric in $\{L_m\colon m = 2, 3, ..., n\}$, and so commutes with each T_v, $v \in \mathscr{B}$, and therefore each element of $\mathscr{H}$; it is thus a central idempotent. If s and t are standard λ-tableaux, then $E_s \xi_{st} E_t \neq 0$, so that E_s and E_t belong to the same block; E^λ is therefore a central primitive, or block, idempotent. It follows that if s and t are respectively standard λ- and μ-tableaux, $\lambda \vdash n$, $\mu \vdash n$, $\lambda \neq \mu$, then for any $h \in \mathscr{H}$

$$E_s h E_t = E_s E^\lambda h E_t = E_s h E^\lambda E_t = 0.$$

If $h \in \mathscr{H}$ commutes with E_t, t standard, then $hE_t = E_t h E_t = h_t E_t$ for some $h_t \in K$ since E_t is primitive. Therefore if h commutes with each of the primitive idempotents then it can be expressed as a linear combination of them, so that the primitive idempotents span a maximal commutative subalgebra of $\mathscr{H}$. This is, of course, the subalgebra generated over K by $\{L_m\colon m = 2, 3, ..., n\}$; indeed

$$L_m = \sum_{\{\text{standard } t\}} r_t(m)\, E_t.$$

6. The Dimensions of the Irreducible Representations

Let us remind ourselves of hooks and various products and quotients derived from them [9, 1]. Let $\lambda \vdash n$; the (i, j)-hook of $[\lambda]$ is the set of nodes $\{(i, k)\colon k = j, j+1, ..., \lambda_i\} \cup \{(k, j)\colon k = i, i+1, ..., \lambda'_j\}$; its length h_{ij} is the number of these nodes. The hook-product $h_\lambda = h_\lambda(q)$ is defined by

$$h_\lambda = \prod_{(i,j) \in [\lambda]} [h_{ij}]_q.$$

The hook-length is related to the residues of the extreme nodes; note that

$$h_{ij} - 1 = (\lambda_i - i) - (j - \lambda'_j).$$

Let t be a λ-tableau; if n is in the ith row of t then we define the hook-quotient γ_{tn} to be

$$\gamma_{tn} = \prod_{j=1}^{\lambda_i} \frac{[h_{ij}]_q}{[h_{ij} - 1]_q};$$

hooks of length one are excluded. For $m < n$ we define γ_{tm} similarly, except that the hooks are computed in the tableau obtained by removing the nodes $m+1, m+2, ..., n$ from t. Let

$$\gamma_t = \prod_{m=2}^{n} \gamma_{tm}.$$

Let $[\mu]$ be the diagram obtained from $[\lambda]$ by removing the node n from t. Consider the product $\gamma_{tn}\gamma_{t'n}$; the numerator corresponds to those hooks of $[\lambda]$ which are not hooks of $[\mu]$, the denominator to those hooks of $[\mu]$ which are not also hooks of $[\lambda]$. A simple induction on n now gives

$$\gamma_t \gamma_{t'} = h_\lambda.$$

In the case of t^λ, the hooks which contribute to γ_λ lie entirely within the rows, and

$$\gamma_\lambda = \prod_{i>0} \prod_{j=1}^{\lambda_i} [j]_q,$$

empty products taken as unity.

LEMMA 6.1. *Let $\lambda \vdash n$, and let $t = t^{\lambda} w_{\lambda}$; then*

$$\xi_{\lambda\lambda} = \gamma_\lambda E_\lambda + \sum_{\{s:\, s \rhd t^\lambda\}} a_s E_s, \qquad a_s \in K,$$

$$\eta_{\lambda'\lambda'} = q^{n-\alpha(\lambda)} \gamma_{\lambda'} E_t + \sum_{\{s:\, s \lhd t\}} b_s E_s, \qquad b_s \in K.$$

Proof. Let s be a standard tableau; then either $s \trianglerighteq t^\lambda$ or there is some m in a later row of s than of t^λ. The first such m will be the first node in a row, and $r_t(m) = [-\rho_\lambda(m)]_q$, whence $\xi_{\lambda\lambda} E_s = 0$. On the other hand, if m occupies node (i, j) in t^λ, so that $i = \rho_\lambda(m)$, then

$$(L_m - [-\rho_\lambda(m)]_q)\, E_\lambda = (r_\lambda(m) - [-\rho_\lambda(m)]_q)\, E_\lambda = q^{-i}[j]_q E_\lambda,$$

so that

$$\xi_{\lambda\lambda} E_\lambda = \gamma_\lambda E_\lambda.$$

Therefore, postmultiplying $\xi_{\lambda\lambda}$ by 1 in the form of the sum of primitive idempotents gives the required identity. In the second case we have factors of the form $q([\rho_{\lambda'(m)}]_q - L_m) t$, which will annihilate E_s if there is some m in a later column of s than of t, which means unless $t \trianglerighteq s$. Acting on E_t this gives $q[\rho_{\lambda'}(m)]_q - r_t(m))$; if m occupies the node (i, j) in t then $\rho_{\lambda'}(m) = [j]_q$, so that we have

$$q([p_{\lambda'}(m)]_q - r_t(m)) = q([j]_q - [j-i]_q) = q^{j-i+1}[i]_q.$$

Observe that $[i]_q$ is the length of a partial column in $[\lambda]$, and so a partial row in $[\lambda']$. Taking product over all $(i, j) \in \lambda$ gives $q^{\alpha(\lambda') - \alpha(\lambda) + n} \gamma_{\lambda'}$, as required. ∎

LEMMA 6.2. *Let t be a standard λ-tableau, $v = (m-1, m)$ for some $m \leqslant n$. Let $s = tv$, and let a and b be the residues of the nodes occupied by m and $m-1$ in t; let $h = b - a$. Then*

$$\left(T_v + \frac{1}{[-h]_q}\right) E_t = E_s \left(T_v + \frac{1}{[h]_q}\right),$$

and if s is standard, $s \rhd t$, then

$$\gamma_t E_t = q\gamma_s \left(T_v + \frac{1}{[h]_q}\right) E_s \left(T_v + \frac{1}{[h]_q}\right).$$

Proof. Let $E = E_s + E_t$; since E is symmetric in L_{m-1} and L_m, it commutes with T_v. Since $r_s(m) = r_t(m-1)$ and $r_t(m) = r_s(m-1)$, we have

$$E_t = \frac{L_m - r_s(m)}{r_t(m) - r_s(m)} E, \qquad E_s = \frac{L_{m-1} - r_s(m)}{r_t(m) - r_s(m)} E.$$

Therefore, using (4.2) we have

$$qT_v^{-1} E_t = \frac{L_{m-1} - r_s(m)}{r_t(m) - r_s(m)} ET_v + \frac{1 + (q-1)\, r_s(m)}{r_t(m) - r_s(m)} E.$$

But

$$\frac{1 + (q-1)\, r_s(m)}{r_t(m) - r_s(m)} = \frac{q^b(q-1)}{q^a - q^b} = \frac{q-1}{q^h - 1} = \frac{1}{[h]_q},$$

so that

$$\left(qT_v^{-1} - \frac{1}{[h]_q}\right) E_t = E_s \left(T_v + \frac{1}{[h]_q}\right).$$

Now

$$qT_v^{-1} - \frac{1}{[h]_q} = T_v - (q-1) - \frac{q-1}{q^h - 1} = T_v - \frac{q-1}{1 - q^{-h}} = T_v + \frac{1}{[-h]_q},$$

whence

$$\left(T_v + \frac{1}{[-h]_q}\right) E_t = E_s \left(T_v + \frac{1}{[h]_q}\right).$$

Now

$$\left(qT_v^{-1} - \frac{1}{[h]_q}\right)\left(T_v + \frac{1}{[h]_q}\right) = \frac{q(q^{h+1} - 1)(q^{h-1} - 1)}{(q^h - 1)^2} = \frac{q[h+1]_q [h-1]_q}{[h]_q^2},$$

which is non-zero if $|h| > 1$, i.e., if s is standard; if $s \rhd t$ then $h > 0$. In this case, the factors of γ_s and γ_t are identical, except that the former has a factor $[h]_q / [h-1]_q$ where the latter has $[h+1]_q / [h]_q$, so that

$$\gamma_t = \gamma_s \frac{[h+1]_q [h-1]_q}{[h]_q^2}$$

and

$$\gamma_t E_t = q\gamma_s \left(T_v + \frac{1}{[h]_q}\right) E_s \left(T_v + \frac{1}{[h]_q}\right). \quad \blacksquare$$

We can iterate Lemma 6.2 to obtain elements Ψ_t and Φ_t of $\mathscr{H}$ such that

$$\Psi_t E_t = E_\lambda \Phi_t, \qquad \text{where} \quad \Phi_t^* \Psi_t = \frac{q^{l(d(t))}\gamma_t}{\gamma_\lambda},$$

and so

$$q^{l(d(t))}\gamma_t E_t = \Phi_t^* \gamma_\lambda E_\lambda \Phi_t.$$

The elements Φ_t and Ψ_t are, in general, not uniquely determined, since they are constructed essentially from a reduced form for $d(t)$, each transposition $v \in \mathscr{B}$ being replaced by an appropriate factor $T_v + [\pm h]_q^{-1}$, and generally different reduced forms give different expressions. However, we shall assume some uniform construction. Note that Φ_t and Ψ_t^* both differ from $T_{d(t)}$ by a linear combination of terms T_w for which $w \rhd d(t)$, so that, using (5.5) and Lemma 2.1 we have

$$\zeta_{st} = E_s \zeta_{st} E_t = E_s \Psi_s^* \xi_{\lambda\lambda} \Psi_t E_t = \Phi_s^* E_\lambda \xi_{\lambda\lambda} E_\lambda \Phi_t = \Phi_s^* \gamma_\lambda E_\lambda \Phi_t, \tag{6.3}$$

and in particular

$$\zeta_{tt} = q^{l(d(t))}\gamma_t E_t.$$

Also

$$\zeta_{st} = \gamma_\lambda \Phi_s^* \Psi_t E_t,$$

from which we can derive the multiplication rule

$$\zeta_{su}\zeta_{ut} = \gamma_u q^{l(d(u))}\zeta_{st}$$

for any standard tableaux s, t, u of the same shape. It is now easy to construct the matrices representing transpositions with respect to the seminormal basis.

THEOREM 6.4 (Young's Seminormal Form). *Let s, u be standard tableaux of the same shape, $v = (i-1, i) \in \mathscr{B}$, $t = sv$. Let (a, b) and (a', b') be* *the nodes occupied by $i-1$ and i respectively in s, and let $h=b-b'-a+a'$; then*

$$\zeta_{us}T_v=\begin{cases}\dfrac{1}{[h]_q}\zeta_{us} & \text{if} \quad |h|=1,\\[2ex] \dfrac{1}{[h]_q}\zeta_{us}+\zeta_{ut} & \text{if} \quad h>1,\\[2ex] \dfrac{1}{[h]_q}\zeta_{us}+\dfrac{q[h+1]_q\,[h-1]_q}{[h]_q^2}\zeta_{ut} & \text{if} \quad h<-1.\end{cases}$$

Proof. If $|h|=1$ then $i-1$ and i are in the same row or column, so that t is regular but not standard, whence $E_t=0$; the first case now follows from Lemma 6.1. The other two cases follow directly from (6.3). ▌

It follows from Theorem 4.5 that

$$z_{st}=T^*_{d(s)}\xi_{\lambda\lambda}T_{w_\lambda}\eta_{\lambda'\lambda'}T_{d(t')}.$$

As in Lemma 6.1, let $t=t^\lambda w_\lambda$; then

$$z_{\lambda t}=\lambda_{\lambda\lambda}T_{w_\lambda}\eta_{\lambda'\lambda'}.$$

Lemma 6.1 shows that E_λ and E_t are the only idempotents in the respective expansions of $\xi_{\lambda\lambda}$ and $\eta_{\lambda'\lambda'}$ which belong to the same block. Therefore

$$\begin{aligned}z_{\lambda t}&=q^{n-\alpha(\lambda)}\gamma_\lambda\gamma_{\lambda'}E_\lambda T_{w_\lambda}E_t\\&=q^{n-\alpha(\lambda)}\gamma_{\lambda'}E_\lambda\xi_{\lambda\lambda}T_{w_\lambda}E_t\\&=q^{n-\alpha(\lambda)}\gamma_{\lambda'}E_\lambda\xi_{\lambda t}E_t\\&=q^{n-\alpha(\lambda)}\gamma_{\lambda'}\zeta_{\lambda t},\end{aligned}$$

whence

$$\begin{aligned}z_{\lambda t}\Psi_t^*&=q^{n-\alpha(\lambda)}\gamma_{\lambda'}\gamma_\lambda E_\lambda\Phi_t\Psi_t^*\\&=q^{n-\alpha(\lambda)}\gamma_{\lambda'}\gamma_t q^{l(d(t))}E_\lambda\\&=q^{n-\alpha(\lambda)-l(d(t))}h_\lambda E_\lambda.\end{aligned}$$

It is useful to determine the coefficient of T_1 in the primitive idempotents. As we know, the coefficient of T_{w_λ} in $z_{\lambda t}$ is unity, as it is in Ψ_t; in the former, w_λ is the unique element of minimal length corresponding to a non-zero coefficient, in the latter, the unique element of maximal length.

Therefore, the coefficient of T_1 in $z_{\lambda t}\Psi_t^*$ is $q^{l(w_\lambda)} = q^{l(d(t))}$, so that the coefficient of T_1 in E_λ is

$$q^{\alpha(\lambda)-n}h_\lambda^{-1}.$$

THEOREM 6.5. *Let M be a left $\mathscr{H}$-module with K-basis $m_1, m_2, ..., m_k$. Suppose that, for each $w \in W\backslash\{1\}$ and each $i \leqslant k$, the coefficient of m_i in the expansion of $T_w m_i$ is* 0; *then*

$$\dim_K(E_\lambda M) = kq^{\alpha(\lambda)-n}h_\lambda^{-1}.$$

Proof. Consider the matrix representing the action of T_w on M. With respect to the basis $\{m_i\}$, each diagonal element is $1/h_\lambda$. However, $M = E_\lambda M \oplus_K (1 - E_\lambda)M$; with respect to a basis reflecting this decomposition, we have 1 occurring $\dim_K(E_\lambda M)$ times, otherwise 0. Equating characters gives

$$\dim_K(E_\lambda M) = kq^{\alpha(\lambda)-n}h_\lambda^{-1}. \quad \blacksquare$$

THEOREM 6.6 (Hook Theorem). *Over any domain the Specht module S^λ has dimension $n!/h_\lambda(1)$.*

Proof. Clearly, the dimension depends neither on the ground-ring nor on q. We therefore take K to be a field of characteristic zero and $q = 1$. If we now set $M = \mathscr{H}$ and take $\{m_i\}$ to be $\{T_w : w \in W\}$ then the conditions of Theorem 6.5 are clearly satisfied, so that, since $\dim_K(\mathscr{H}) = n!$,

$$\dim_K(S^\lambda) = \dim_K(E_\lambda \mathscr{H}) = n!/h_\lambda(1). \quad \blacksquare$$

THEOREM 6.7. *Let $G = GL_n(q)$, the general linear group over a field of characteristic q; the dimension of the ordinary irreducible unipotent representation of G corresponding to the partition λ of n is $[n]_q[n-1]_q \cdots [1]_q q^{\alpha(\lambda)-n}/h_\lambda(q)$.*

Proof. We take K to be a field of characteristic zero. Let B be the set of lower triangular matrices in G, a Borel subgroup; we take $m = B$, and M the right KG-module generated by B. Thus the distinct right cosets of B in G furnish a K-basis for the permutation module M, which affords the unipotent representations of G. The orders of G and B are $|G| = (q^n - 1)(q^n - q^2) \cdots (q^n - q^{n-1})$ and $|B| = q^{n(n-1)/2}(q-1)^n$ respectively, so that $\dim_K(M) = |G|/|B| = [n]_q[n-1]_q \cdots [1]_q$.

It is well known [4] that $\mathscr{H}$ is isomorphic to the KG-endomorphism algebra of M, and may be embedded in KG as follows. Let us identify W with the subgroup of permutation matrices in G; then we may set

$$T_w = |B|^{-1} \sum_{u \in BwB} u, \qquad w \in W,$$

and consider the left action on M. Clearly $BwB \cap B = \emptyset$ unless w is the identity. Since M is a cyclic module generated by B, the conditions of Theorem 6.5 are satisfied, and the required result follows immediately. ∎

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