Extending Hua’s theorem on the geometry of matrices to Bezout domains

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This paper extends Hua’s theorem on the geometry of rectangular matrices over a division ring to the case of Bezout domains. Let $m, n, m', n'$ be integers $\geq 2$, $R$ and $R'$ be two Bezout domains. Assume that $\varphi : R^{m \times n} \rightarrow R'^{m' \times n'}$ is an adjacency preserving bijective map in both directions. Further, assume that $R'$ is a local ring, or $\varphi$ is an invertibility preserving map, or $\varphi$ is an additive map. This paper obtains the algebraic formulas of $\varphi$. As applications, the ring semi-isomorphisms from $R^{m \times n}$ to $R'^{m' \times n'}$ are characterized, and the group isomorphisms from $GL_n(R)$ to $GL_n'(R')$ are discussed.

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A Bezout domain [1,2] is an integral domain in which every finitely generated left (right) ideal is principal. Bezout domain includes several important types of ring, for example, principal ideal domain (PID), commutative semilocal Prüfer domain [3]. Every Bezout domain is both an Hermite ring and a Ore domain [1]. A local Bezout domain is both a local ring and a Bezout domain. For example, division rings and discrete valuation rings (not necessary commutative) [6] are local Bezout domains.

Let $R$ be a Bezout domain and $0 \neq A \in R^{m \times n}$. Then there is the least positive integer $r$ such that $A = BC$ where $B \in R^{m \times r}$ and $C \in R^{r \times n}$. This number $r$ is called the inner rank of $A$ [1,2] and denoted by $\text{rank}(A)$. Define the inner rank of a zero matrix is 0. The row rank [1,2] of $A \in R^{m \times n}$ is the rank of the submodule of $R^{1 \times n}$ (as left $R$-module) spanned by the rows of $A$. The column rank of $A$ can be defined similarly. For any $A \in R^{m \times n}$, the row rank, column rank and inner rank of $A$ are all equal (cf. p. 285 of [1]). Let $\bar{R}$ be the division ring of fractions of $R$. Then $\text{rank}(A)$ and the usual rank of $A \in \bar{R}^{m \times n}$ (as a matrix over division ring) are equal.

For $A, B \in R^{m \times n}$, $\text{ad}(A, B) := \text{rank}(A - B)$ is called the arithmetic distance of $A$ and $B$. Clearly, $\text{ad}(A, B) \geq 0$, $\text{ad}(A, B) = 0 \iff A = B$; $\text{ad}(A, B) = \text{ad}(B, A)$; $\text{ad}(A, B) \leq \text{ad}(A, C) + \text{ad}(C, B)$. If $\text{ad}(A, B) = 1$, then $A$ and $B$ are said to be adjacent and denoted by $A \sim B$.

Let $R, R'$ be Bezout domains. A map $\varphi : R^{m \times n} \to R'^{m' \times n'}$ is called an adjacency preserving map in both directions ("a.p. map in both directions" for short) if $A \sim B \iff \varphi(A) \sim \varphi(B)$ for all $A, B \in R^{m \times n}$. A map $\varphi : R^{m \times n} \to R'^{m' \times n'}$ is called an invertibility preserving map if $A - B$ has a right or left inverse implies that $\varphi(A) - \varphi(B)$ has a right or left inverse.

The set of all bijective maps

$$X \mapsto P X^\sigma Q + A, \quad \forall X \in R^{m \times n}, \quad (1.1)$$

where $A \in R^{m' \times n'}$, $P$ and $Q$ are invertible matrices over $R'$, and $\sigma$ is an isomorphism from $R$ to $R'$, forms a group, called the group of motions from $R^{m \times n}$ to $R'^{m' \times n'}$. The fundamental problem in the geometry of rectangular matrices is to characterize the group of motions by as few geometrical invariants as possible. When $R = D$ is a division ring, Hua showed that the invariant "adjacency" alone is "almost" sufficient to characterize the group of motions, and proved the fundamental theorem of the geometry of rectangular matrices over a division ring [7,8,20]. Hua’s theorem has many applications to algebra and geometry, and his work was continued by many mathematicians (cf. [9–20]). Recently, the "adjacency preserving bijective maps" on Grassmann spaces and $m \times n$ matrices over Bezout domains are discussed by the author [10,9]. On the other hand, the adjacency (singularity) preserving bijective semi-linear maps on matrices over a local ring were investigated by Wong [22] and Guterman [5].

By using distinct method, this paper extends Hua’s theorem to the case of Bezout domains as follows.

**Theorem 1.1.** Let $m, n, m', n'$ be integers $\geq 2$, and let $R$ and $R'$ be two Bezout domains. Let $\varphi : R^{m \times n} \to R'^{m' \times n'}$ be an a.p. bijective map in both directions. Further, assume that $R'$ is a local ring, or $\varphi$ is an invertibility preserving map, or $\varphi$ is an additive map. Then either $(m, n) = (m', n')$ or $(m, n) = (n', m')$.

If $(m, n) = (m', n')$ with $m \neq n$, then $R$ is isomorphic to $R'$ and $\varphi$ is of the form

$$\varphi(X) = PX^\sigma Q + \varphi(0), \quad \forall X \in R^{m \times n}, \quad (1.2)$$

where $\sigma$ is an isomorphism from $R$ to $R'$, $P \in GL_m(R')$ and $Q \in GL_n(R')$. If $(m, n) = (n', m')$ with $m \neq n$, then $R$ is anti-isomorphic to $R'$ and $\varphi$ is of the form

$$\varphi(X) = P^T X^\tau Q + \varphi(0), \quad \forall X \in R^{m \times n}, \quad (1.3)$$

where $\tau$ is an anti-isomorphism from $R$ to $R'$, $P \in GL_n(R')$ and $Q \in GL_m(R')$. If $m = n = m' = n'$, then $\varphi$ is of the form either (1.2) or (1.3).

Conversely, maps (1.2) and (1.3) are both a.p. bijective maps in both directions and invertibility preserving maps.
Remark 1.2. If \( \varphi : R^{m \times n} \to R^{m' \times n'} \) is an a.p. bijection in both directions (resp. invertibility preserving map, additive map) and \( A \in R^{m \times n} \), then \( \psi(X) := Q_1 [\varphi(P_1XP_2 + A) - \varphi(A)]Q_2 \) is also an a.p. bijection in both directions (resp. invertibility preserving map, additive map) from \( R^{m \times n} \) to \( R^{m' \times n'} \), where \( P_i \) and \( Q_i \) are fixed invertible matrices over \( R \) and \( R' \), respectively, and \( \sigma \) is an isomorphism from \( R \) to \( R' \). Moreover, \( \psi'(X) := Q_1 [\varphi(P_1XP_2 + A) - \varphi(A)]Q_2 \) is an a.p. bijection in both directions (resp. invertibility preserving map, additive map) from \( R^{m \times n} \) to \( R^{m' \times n'} \), where \( P_i \) and \( Q_i \) are fixed invertible matrices over \( R \) and \( R' \), respectively, and \( \rho \) is an anti-isomorphism from \( R \) to \( R' \).

This paper is organized as follows. In Section 2, we apply Theorem 1.1 to algebra, characterize the ring semi-isomorphisms from \( R^{m \times n} \) to \( R^{m' \times n'} \), and discuss the group isomorphism from \( GL_n(R) \) to \( GL_{n'}(R') \). In Section 3, we discuss the affine geometry of a maximal set of rectangular matrices over a Bezout domain. In Section 4, we prove Theorem 1.1 for the case \( m = n = 2 \). Then Section 5 proves Theorem 1.1.

2. Applications to algebra

Now, we discuss the applications of Theorem 1.1 to algebra. Denote by \( I_r \) the \( r \times r \) identity matrix (\( I \) for short). Let \( E_{ij}^{m \times n} \) (\( E_{ij} \) for short) be the \( m \times n \) matrix whose \((i, j)\)-entry is 1 and all other entries are 0s. If \( 0 \neq A \in R^{n \times n} \) and \( A^2 = A \), then \( A \) is called an idempotent matrix. Two \( n \times n \) matrices \( A_1 \) and \( A_2 \) over \( R \) are said to be orthogonal, if \( A_1A_2 = A_2A_1 = 0 \). An idempotent matrix \( A \) is called primitive, if \( A \) cannot be written as a sum of two orthogonal idempotent matrices.

Lemma 2.1 (cf. Lemma 4.4.1 of [9]). Let \( R \) be a Bezout domain, \( A \in R^{n \times n} \). Then \( A \) is an idempotent matrix if and only if there exists \( P \in GL_n(R) \) such that \( A = P \text{ diag}(I_r, 0)P^{-1} \).

By Lemma 2.1, it is easy to prove the following lemmas.

Lemma 2.2 (cf. [9]). Let \( R \) be a Bezout domain. Then \( A \in R^{n \times n} \) is a primitive idempotent matrix if and only if \( A \) is an idempotent matrix of inner rank 1.

Lemma 2.3 (cf. [9]). Let \( R \) be a Bezout domain. Then a non-zero \( n \times n \) matrix \( A \) over \( R \) is of inner rank 1 if and only if there is a primitive idempotent matrix \( E \) over \( R \) such that

\[
(I - E)A(I - E) = 0, \quad \text{and} \quad (A - EAE)^2 = 0. \tag{2.1}
\]

Let \( \Omega \) and \( \Omega' \) be two rings. A bijective map \( f : \Omega \to \Omega' \) is called a ring semi-isomorphism if \( f(a + b) = f(a) + f(b), f(aba) = f(a)f(b)f(a), f(1_\Omega) = 1_{\Omega'} \), for all \( a, b \in \Omega \).

Theorem 2.4. Let \( R \) and \( R' \) be Bezout domains. Then \( \varphi : R^{n \times n} \to R^{n' \times n'} \) \((n, n' \geq 2)\) is a ring semi-isomorphism if and only if \( n = n' \) and \( \varphi \) is a ring isomorphism either of the form

\[
\varphi(X) = P^{-1}X^\sigma P, \quad \forall X \in R^{n \times n}, \tag{2.2}
\]

or a ring anti-isomorphism of the form

\[
\varphi(X) = P^{-1}X^\tau P, \quad \forall X \in R^{n \times n}, \tag{2.3}
\]

where \( \sigma \) is an isomorphism from \( R \) to \( R' \), \( \tau \) is an anti-isomorphism from \( R \) to \( R' \), \( P \in GL_n(R') \).

Proof. Let \( \varphi : R^{n \times n} \to R^{n' \times n'} \) be a ring semi-isomorphism. Then \( \varphi^{-1} \) is also a ring semi-isomorphism, \( \varphi(I_n) = I_{n'} \) and \( \varphi(0) = 0 \). Thus \( \varphi \) and \( \varphi^{-1} \) carry idempotent matrices into idempotent matrices. Let
X, Y be orthogonal idempotent matrices. Then \( \varphi(XY + YX) = \varphi((X + Y)^2 - X^2 - Y^2) = \varphi(X)\varphi(Y) + \varphi(Y)\varphi(X) = 0 \). Hence \( 0 = \varphi(XXY) = \varphi(X)\varphi(Y)\varphi(X) = -(\varphi(X))^2\varphi(Y) = -\varphi(X)\varphi(Y) \), it follows that \( \varphi(X)\varphi(Y) = \varphi(Y)\varphi(X) = 0 \). Therefore, \( \varphi \) and \( \varphi^{-1} \) carry orthogonal idempotent matrices to orthogonal idempotent matrices. Then \( \varphi \) and \( \varphi^{-1} \) carry primitive idempotent matrices to primitive idempotent matrices. By Lemma 2.3, \( \varphi \) is an a.p. bijective map in both directions. By Theorem 1.1, \( n = n' \) and either \( \varphi \) is a ring isomorphism of the form (2.2) or a ring anti-isomorphism of the form (2.3).

Conversely, map (2.2) is a ring isomorphism, and map (2.3) is a ring anti-isomorphism. \( \square \)

Let \( E_n(R) \) be the subgroup of \( GL_n(R) \) generated by elementary matrices of the form \( I + rE_{ij} \) where \( r \in R, 1 \leq i \neq j \leq n \).

**Theorem 2.5.** Let \( R \) and \( R' \) be Bezout domains, and let \( \varphi : GL_n(R) \rightarrow GL_{n'}(R') \) \( (n, n' \geq 4) \) be a group isomorphism. Then \( n = n' \), \( R \) is isomorphic to \( R' \) or \( R \) is anti-isomorphic to \( R' \), and we have either

\[
\varphi(A) = P^{-1}A^\sigma P, \quad \forall A \in E_n(R),
\]

or

\[
\varphi(A) = P^{-1}(tA^T)^{-1}P, \quad \forall A \in E_N(R),
\]

where \( \sigma \) is an isomorphism from \( R \) to \( R' \), \( \tau \) is an anti-isomorphism from \( R \) to \( R' \), \( P \in GL_n(R') \).

**Proof.** By the Golubchik’s theorem [4], there exist central idempotent matrices \( E \in R^{n \times n} \) and \( F \in R^{n' \times n'} \) appropriately together with ring isomorphism \( \theta_1 : ER^{n \times n} \rightarrow FR^{n' \times n'} \) and ring anti-isomorphism \( \theta_2 : (I - E)R^{n \times n} \rightarrow (I - F)R^{n' \times n'} \), such that

\[
\varphi(A) = \theta_1(EA) + \theta_2((I - E)A^{-1}), \quad \forall A \in E_n(R).
\]

By Lemma 2.1, every non-zero central idempotent matrix in \( R^{n \times n} \) is \( I_n \). Therefore, we have either \( E = I_n \) with \( F = I_{n'} \), or \( E = 0 \) with \( F = 0 \). Then \( \varphi(A) = \theta_1(A) \) or \( \varphi(A) = \theta_2(A^{-1}) \) for all \( A \in E_n(R) \). By Theorem 2.4, \( n = n' \), \( R \) is isomorphic to \( R' \) or \( R \) is anti-isomorphic to \( R' \), and \( \varphi|_{E_n(R)} \) is of the form (2.4) or (2.5). \( \square \)

An integral domain \( R \) is called a *Euclidean ring*, if there exists a function \( \delta : R^X \rightarrow \mathbb{N} \) such that \( R \) satisfies the *division algorithm*: for all \( x, y \in R, y \neq 0 \), there exists \( q, r \in R \) such that \( x = qy + r \), and either \( r = 0 \) or \( \delta(r) < \delta(x) \). Every Euclidean ring is a Bezout domain. If \( R \) is a Euclidean ring and \( A \in GL_n(R) \), then \( A = A_1\text{diag}(1, \ldots, 1, a) \), where \( A_1 \in E_n(R) \). By Theorem 2.5 and similar to the proof in [21,8], it is easy to see the following corollary.

**Corollary 2.6** [21]. Let \( R \) and \( R' \) be Euclidean rings, and let \( \varphi : GL_n(R) \rightarrow GL_{n'}(R') \) \( (n, n' \geq 4) \) be a group isomorphism. Then \( n = n' \) and we have either

\[
\varphi(A) = \chi(A)P^{-1}A^\sigma P, \quad \forall A \in GL_n(R),
\]

or

\[
\varphi(A) = \chi(A)P^{-1}(tA^T)^{-1}P, \quad \forall A \in GL_n(R),
\]

where \( \sigma \) is an isomorphism from \( R \) to \( R' \), \( \tau \) is an anti-isomorphism from \( R \) to \( R' \), \( P \in GL_n(R') \), and \( \chi : GL_n(R) \rightarrow \text{center}(GL_n(R')) \) is a group homomorphism.
3. Affine geometry and maximal set on $R^{m \times n}$

In this section, let $R, R'$ be Bezout domains, and let $m, n, m', n'$ be integers $\geq 2$. Denote by $0_{m,n}$ the $m \times n$ zero matrix (0 for short), $0_n = 0_{n,n}$. If $S$ is a subset of $R^{m \times n}, P \in GL_m(R), Q \in GL_n(R)$ and $A \in R^{m \times n}$ are fixed, then we write $P SQ + A := \{PX + A : X \in S\}$. For $1 \leq s \leq m$ and $1 \leq t \leq n$, we write that

$$
P \left( \begin{array}{cc} R^{s \times t} & 0 \\ 0 & 0 \end{array} \right) Q = \left\{ \begin{array}{c} P \left( \begin{array}{cc} X & 0 \\ 0 & 0 \end{array} \right) Q \in R^{m \times n} : X \in R^{s \times t} \end{array} \right\}.
$$

For convenience, some zero elements in matrices above may be disappearing.

We say that $(a_1, \ldots, a_n)$ is a right unimodular (unimodular for short) row if $\sum_{i=1}^n a_i R = R$. Similarly, $(b_1, \ldots, b_n)$ is left unimodular (unimodular for short) column if $\sum_{i=1}^n b_i R = R$. For any $\alpha \in R^{1 \times n}$, there exists a unimodular $\alpha' \in R^{1 \times n}$ such that $\alpha = k \alpha'$, and $\langle \alpha \rangle := k\alpha'$ is the submodule of $R^{1 \times n}$ spanned by $\alpha$. Note that $\langle \alpha \rangle := k\alpha$ if $\alpha$ is not unimodular.

A matrix over $R$ is called regular if it is neither 0 nor a left or right zero-divisor. If $A$ be a matrix over $R$, then $A$ is regular if and only if $A$ is a square matrix and $\text{rank}(A)$ is the order of $A$ (cf. Corollary 5.4.5 of [1]).

**Lemma 3.1.** Let $R$ be a Bezout domain. Then every non-zero matrix $A$ over $R$ has a factorization $A = P \text{diag}(A_1, 0)Q$, where $P, Q$ are invertible matrices over $R, A_1$ is an $r \times r$ regular matrix and $\text{rank}(A) = r$.

**Proof.** Since $R$ is a semifir [1], Corollary 2.3.2 of [1] implies that there exist two invertible matrices $P_1, Q_1$ over $R$ such that the non-zero rows and non-zero columns of $P_1AQ_1$ are left and right linearly independent, respectively. Interchanging rows and columns, we can assume that $P_2AQ_2 = \text{diag}(A_1, 0)$, where $P_2, Q_2$ are invertible matrices over $R, A_1$ is an $r \times r$ matrix with $\text{rank}(A_1) = r$. Moreover, $A_1$ is regular. $\square$

**Lemma 3.2** (cf. Lemma 2.3.16 of [9]). Let $R$ be a Bezout domain, and $1 \leq r, s < \min\{m, n\}$. Assume that $\alpha = (i_1, \ldots, i_s), \beta = (j_1, \ldots, j_s)$, where $1 \leq i_1 < \cdots < i_r \leq m$ and $1 \leq j_1 < \cdots < j_s \leq n$. Let $A = (a_{ij}) \in R^{m \times n}, B_i = \sum_{k=0}^{r-1} \sum_{k=1}^{s-1} b_{ijk}^{(i)} c_{ijk}^{(i)}, i = 1, 2$, and $B_1 \neq B_2$. If $A \sim B_i, i = 1, 2$, then either $a_{ij} = 0$ for all $i \notin \alpha$, or $a_{ij} = 0$ for all $j \notin \beta$.

**Proof.** We show that $a_{ij} = 0$ if $i \notin \alpha$ with $j \notin \beta$. Otherwise, there exists submatrices $\begin{pmatrix} d_i & a_1 \\ a_2 & \lambda_1 \end{pmatrix}$ of $A - B_i$, where $d_1 \neq d_2$ and $\lambda_1 \neq 0$, such that $\text{rank} \begin{pmatrix} d_i & a_1 \\ a_2 & \lambda_1 \end{pmatrix} = 1, i = 1, 2$. Let $\overline{R}$ be the division ring of fractions of $R$. Then $\text{rank} \begin{pmatrix} d_i - a_1 \lambda_1^{-1} a_2 \\ a_2 & \lambda_1 \end{pmatrix} = 1$ over $\overline{R}, i = 1, 2$, a contradiction. By $a_{ij} = 0$ if $i \notin \alpha$ with $j \notin \beta$, we have either $a_{ij} = 0$ for all $i \notin \alpha$, or $a_{ij} = 0$ for all $j \notin \beta$. $\square$

Two distinct matrices $A, B \in R^{m \times n}$ are said to be of distance $r$, denoted by $d(A, B) = r$, if $r$ is the least positive integer for which there is a sequence of $r + 1$ points $X_0, X_1, \ldots, X_r \in R^{m \times n}$ with $X_0 = A$ and $X_r = B$ such that $X_{i-1} \sim X_i, i = 1, \ldots, r$. Define $d(A, A) = 0$. We have that $d(A, B) \geq 0, d(A, B) = 0 \iff A = B; d(A, B) = d(B, A); d(A, B) \leq d(A, C) + d(C, B)$.

**Lemma 3.3.** If $R$ is a Bezout domain, then $d(A, B) = d(A, B), \forall A, B \in R^{m \times n}$.
Proof. For any distinct $A, B \in \mathbb{R}^{m \times n}$, let $r = \text{ad}(A, B) = \text{rank}(A - B) > 0$. By Theorem 3.1, there are invertible matrices $P$ and $Q$ such that $A - B = P \text{diag}(A_1, 0)Q$, where $A_1 = \frac{(\alpha_1)}{\alpha_r}$ is regular. Let $B_i = \frac{(\alpha_1)}{\alpha_r}, T_i = P \frac{(B_i \ 0 \ 0 \ 0)}{0 \ 0 \ 0 \ 0} Q$, and let $X_0 = A, X_i = A - T_i, i = 1, \ldots, r$. Then $B = X_r$, and $X_{i-1} \sim X_i$, $i = 1, \ldots, r$. Hence $d(A, B) \leq r$. Suppose that $d(A, B) = s$. Then there exist $s + 1$ points $X_0, X_1, \ldots, X_s$ in $\mathbb{R}^{m \times n}$ with $X_0 = A$ and $X_s = B$ such that $X_{i-1} \sim X_i, i = 1, \ldots, s$. Since $A - B = \sum_{i=1}^s (X_{i-1} - X_i)$, it follows that $\text{rank}(A - B) \leq \sum_{i=1}^s \text{rank}(X_{i-1} - X_i) = s$, i.e. $r \leq d(A, B)$. Thus $r = d(A, B)$.

By Lemma 3.3 and the definition of the distance, we have clearly

**Corollary 3.4.** If $\varphi : \mathbb{R}^{m \times n} \to \mathbb{R}^{m' \times n'}$ is an a.p. bijection in both directions, then $\varphi$ preserves the arithmetic distance between any pair of matrices, that is, $\text{ad}(A, B) = \text{ad}(\varphi(A), \varphi(B))$ for all $A, B \in \mathbb{R}^{m \times n}$.

A nonempty set $\mathcal{M}$ contained in $\mathbb{R}^{m \times n}$ is said to be a **maximal set**, if any two distinct points of $\mathcal{M}$ are adjacent and there is no other point outside $\mathcal{M}$ in $\mathbb{R}^{m \times n}$, which is adjacent to each point of $\mathcal{M}$.

In $\mathbb{R}^{m \times n}$, let

$$\mathcal{M}_i = \left\{ \sum_{j=1}^n x_j E_{ij} : x_j \in \mathbb{R} \right\}, \quad i = 1, \ldots, m,$$

$$\mathcal{N}_j = \left\{ \sum_{i=1}^m y_i E_{ij} : y_i \in \mathbb{R} \right\}, \quad j = 1, \ldots, n.$$ 

By Lemmas 3.1 and 3.2, it is easy to prove the following lemma.

**Lemma 3.5.** In $\mathbb{R}^{m \times n}$, all $\mathcal{M}_i$’s, $\mathcal{N}_j$’s, $i = 1, \ldots, m$, $j = 1, \ldots, n$, are maximal sets. Moreover, any maximal set is of one of the following forms.

Type one. $\mathcal{M} = P \mathcal{M}_1 Q + A = P \mathcal{M}_1 + A$, where $P \in \text{GL}_m(\mathbb{R}), Q \in \text{GL}_n(\mathbb{R})$;

Type two. $\mathcal{M} = P \mathcal{N}_1 Q + A = \mathcal{N}_1 Q + A$, where $P \in \text{GL}_m(\mathbb{R}), Q \in \text{GL}_n(\mathbb{R})$.

The $\mathcal{M}_1$ (resp. $\mathcal{N}_1$) is called the **standard maximal set** of the type one (resp. the type two).

**Corollary 3.6.** Let $A$ and $B$ be two adjacent points in $\mathbb{R}^{m \times n}$ ($m, n \geq 2$). Then there are two and only two maximal sets containing $A$ and $B$.

Let $R^m$ ($^nR_R$) be the left $R$-module (right $R$-module) whose elements are $n$-dimensional row (column) vectors over $R$. By Corollary 0.3.4 of [2], every finitely generated submodule of $R^m$ is free of rank at most $n$. An $r$-dimensional free submodule $M$ of $R^m$ is said to be an $(r - 1)$-dimensional left projective flat if $M$ is a direct summand of $R^m$ (that is, $R^m = M \oplus N$ for some submodule $N$). The right projective flat on $^nR_R$ is defined similarly. An $r$-dimensional free submodule $[\alpha_1, \ldots, \alpha_r]$ of $R^m$ (resp. $^nR_R$) is an $(r - 1)$-dimensional left (resp. right ) projective flat if and only if $\{\alpha_1, \ldots, \alpha_r\}$ is a unimodular basis, i.e. the matrix $\frac{(\alpha_1)}{\alpha_r}$ has a right inverse (resp. $\{\alpha_1, \ldots, \alpha_r\}$ has a left inverse).
Denote by \( (\alpha_1, \ldots, \alpha_r) \) the \((r - 1)\)-dimensional left (or right) projective flat in \( R^n \) (or \( R^R \)), where \( \{\alpha_1, \ldots, \alpha_r\} \) is a unimodular basis.

Let \( V = R^n \) (or \( R^R \)). If \( M \) is an \((r - 1)\)-dimensional left (or right) projective flat in \( V \) and \( a \in V \), then \( M + a \) is called an \( r\)-dimensional left (or right) affine flat (or flat for short) in \( V \). Let \( S \) be an \( m\)-flat in \( V \) (\( m \leq n \)). The set of all flats in \( S \) is called an \( m\)-dimensional left (or right) affine geometry on \( S \) and will be denoted by \( AG(S) \). The **dimension** of \( AG(S) \) is \( m \), written \( \dim(AG(S)) \). In particular, the flats of \( AG(S) \) of dimensions 0, 1, 2 are called **points**, **lines**, **planes** in \( AG(S) \), respectively. \( AG(S) \) is also called a left or right affine geometry over a Bezout domain.

In a left or right affine geometry, the set of points belonging to both flats \( M + a \) and \( N + b \) is called a **left** (or **right** affine geometry) over a Bezout domain.

In a left or right affine geometry, the set of points belonging to both flats \( M + a \) and \( N + b \) is called the **intersection** of \( M + a \) and \( N + b \), which is denoted by \( (M + a) \cap (N + b) \). The minimum dimensional flat containing flats \( M + a \) and \( N + b \) is called a **join** of \( M + a \) and \( N + b \), which is denoted by \( (M + a) \cup (N + b) \). \( (M + a) \cup (N + b) \) is also the intersection of all the flats which contain \( M + a \) and \( N + b \).

In \( AG(S) \), we have (cf. Chapter 3 of [9]):

- Any two distinct points lie on a unique line, or, the intersection of two distinct lines is either a point or empty.
- Any \( r + 1 \) points, not lying on any \((r - 1)\)-flat, lie on a unique \( r\)-flat.

Let \( \mathcal{M} = P.M_1 Q \) be a maximal set of the type \( 1 \) in \( R^{m \times n} \) which contains \( 0 \). Then we have the left affine geometry \( AG(\mathcal{M}) \) such that \( AG(\mathcal{M}) \) and \( AG(R^R) \) are affine isomorphic. In \( AG(\mathcal{M}) \), the parametric equation of a line is

\[
l = P\{xT + A : x \in R\}Q = P\left(\begin{array}{c} R\beta + \alpha \\ 0 \end{array}\right)Q,
\]

where \( T = \begin{pmatrix} \beta \\ 0 \end{pmatrix}, A = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \in M_1 \) are fixed, and \( \beta \in R^{1 \times n} \) is right unimodular.

In \( AG(P.M_1 Q) \), the parametric equation of an \( r\)-flat \( p \) is

\[
p = P\left(\begin{array}{c} R^{1 \times r} Q_r + \alpha \\ 0 \end{array}\right)Q,
\]

where \( Q_r \in R^{1 \times n} \) and \( \alpha \in R^{1 \times n} \) are fixed, and \( Q_r \) has a right inverse, \( 1 \leq r < n \).

Similarly, we have the right affine geometry \( AG(P.N_1 Q) \), and we can write the parametric equations of line and \( r\)-flat in \( AG(P.N_1 Q) \).

**Lemma 3.7.** Let \( \mathcal{M} \) be a maximal set in \( R^{m \times n} \) which contains \( 0 \). Then \( l \) is a line in \( AG(\mathcal{M}) \) if and only if \( l = \mathcal{M} \cap \mathcal{M}' \), where \( \mathcal{M}, \mathcal{M}' \) are two distinct maximal sets and \( |\mathcal{M} \cap \mathcal{M}'| \geq 2 \).

**Proof.** Without loss of generality, we assume \( \mathcal{M} = M_1 \). Let \( l = \{xT + B : x \in R\} \) be a line in \( AG(M_1) \), where \( T = \begin{pmatrix} T_1 \\ 0 \end{pmatrix} \) satisfies that \( T_1 \in R^{1 \times n} \) is unimodular. Then there exists a \( T_2 \in R^{(n - 1) \times n} \) such that

\[
P = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \in GL_n(R).
\]

Let \( M' = N_1 P + B \). Then \( M' \neq M_1 \). By \( M_1 = M_1 P + B \), it is easy to see that

\[
l = \mathcal{M} \cap \mathcal{M}' .
\]

Conversely, let \( \mathcal{M}' \) be a maximal set such that \( \mathcal{M}' \neq M_1 \) and \( |\mathcal{M}_1 \cap \mathcal{M}'| \geq 2 \). By Lemma 3.5, \( \mathcal{M}' = P.M_1 + B \) or \( N_1 Q + B \), where \( P \in GL_m(R), Q \in GL_n(R), B \in R^{m \times n} \). By \( |\mathcal{M}_1 \cap \mathcal{M}'| \geq 2 \), we can assume that \( B \in M_1 \). Suppose \( \mathcal{M}' = P.M_1 + B \). It is easy to check that \( M_1 \cap P.M_1 = \{0\} \), thus
\[ M_1 \cap M' = \{ B \}, \] a contradiction. Thus \( M' = N_1Q + B. \) Then \( M_1 \cap M' = (M_1 \cap N_1)Q + B. \) By Lemma 3.11, \( M_1 \cap N_1 = \text{RE}_{11}, \) \( M_1 \cap M' \) is a line in \( \text{AG}(M_1). \) \( \square \)

By the proof of Lemma 3.7, we have

**Corollary 3.8.** Let \( M \) and \( M' \) be two distinct maximal sets of the same type in \( \mathbb{R}^{m \times n}. \) If \( M \cap M' \neq \emptyset, \) then \( |M \cap M'| = 1. \)

**Corollary 3.9.** Let \( M \) and \( M' \) be two maximal sets of different types in \( \mathbb{R}^{m \times n}. \) Then either \( |M \cap M'| \geq 2 \) or \( M \cap M' = \emptyset. \)

By Corollaries 3.4, 3.8 and 3.9, it is easy to prove the following lemma.

**Lemma 3.10.** Let \( \varphi : \mathbb{R}^{m \times n} \to \mathbb{R}^{m' \times n'} \) be an a.p. bijective map in both directions. Then \( M \) is a maximal set if and only if \( \varphi(M) \) is a maximal set. Moreover, either \( M \) and \( \varphi(M) \) are of the same type for every maximal set \( M \) containing \( 0, \) or \( M \) and \( \varphi(M) \) are of different types for every maximal set \( M \) containing \( 0. \)

By Corollary 3.4–Lemma 3.10, we have immediately

**Lemma 3.11.** Let \( \varphi : \mathbb{R}^{m \times n} \to \mathbb{R}^{m' \times n'} \) be an a.p. bijective map in both directions. Assume that \( M \) is a maximal set containing \( 0 \) in \( \mathbb{R}^{m \times n}. \) Then \( l \) is a line in \( \text{AG}(M) \) if and only if \( \varphi(l) \) is a line in \( \text{AG}(\varphi(M)). \)

**Theorem 3.12.** Let \( \varphi : \mathbb{R}^{m \times n} \to \mathbb{R}^{m' \times n'} \) be an a.p. bijective map in both directions with \( \varphi(0) = 0. \) Let \( M \) be a maximal set of the type one (resp. type two) containing \( 0 \) in \( \mathbb{R}^{m \times n}. \) Let \( \varphi(M) \) be two maximal sets of different types in \( \mathbb{R}^{m' \times n'}. \) Then \( \varphi \) maps \( r \)-flats containing \( 0 \) in \( \text{AG}(M) \) onto \( r \)-flats containing \( 0 \) in \( \text{AG}(\varphi(M)) \) for all \( 1 \leq r \leq n \) (resp. \( 1 \leq r \leq m \)).

Moreover, for any \( r \)-flat \( \langle \alpha_1, \ldots, \alpha_r \rangle \) containing \( 0 \) in \( \text{AG}(M) \), we have

\[ \varphi(\langle \alpha_1, \ldots, \alpha_r \rangle) = \varphi(\langle \alpha_1, \ldots, \alpha_{r-1} \rangle) \cup \varphi(\langle \alpha_r \rangle). \]  

(3.1)

Thus \( \text{dim}[\text{AG}(M)] = \text{dim}[\text{AG}(\varphi(M))], \) and either \( (m, n) = (m', n') \) or \( (m, n) = (n', m'). \)

**Proof.** Let \( M' = \left\{ \sum_{j=1}^{n'} x_j F_{ij} : x_j \in \mathbb{R}^r \right\}. \) By Lemma 3.10 and Remark 1.2, without loss of generality, we assume that \( M = M_1 \) and \( \varphi(M) = M'_1. \) By the proof of Theorem 4.2.17 of [9], \( \varphi \) carries planes in \( \text{AG}(M_1) \) onto planes in \( \text{AG}(M'_1). \)

We prove that \( \varphi \) maps \( r \)-flats containing \( 0 \) in \( \text{AG}(M_1) \) onto \( r \)-flats containing \( 0 \) for all \( r \) as follows. We prove it by induction on \( r. \) For \( r = 1, \) it is known by Lemma 3.11. Now assume that it is true for \( r = 1 \) \( (r \geq 1), \) \( M = \left\langle \alpha_1, \ldots, \alpha_r \right\rangle \) be any \( r \)-flat containing \( 0 \) in \( \text{AG}(M_1), \) and let \( M_{r-1} = \left\langle \alpha_1, \ldots, \alpha_{r-1} \right\rangle. \)

Then \( M = M_{r-1} + \langle \alpha_r \rangle \) (as the direct sum of left \( R \)-module). By the induction hypothesis, we let \( M_{r-1}' = \left\langle \alpha_r^1, \ldots, \alpha_r^{r-1} \right\rangle \) is an \( (r-1) \)-flat containing \( 0 \) in \( M_1'. \) Let \( \varphi(\langle \alpha_r \rangle) = \langle \alpha_r^* \rangle. \) Since \( \langle \alpha_r^* \rangle \nsubseteq M_{r-1}', \langle M_{r-1}', \langle \alpha_r^* \rangle \) is an \( r \)-flat containing \( 0 \) in \( M_1'. \)

Let \( v \in M. \) Then \( v = u + w, \) where \( u \in M_{r-1} \) and \( w \in \langle \alpha_r \rangle. \) Without loss of generality we let \( u \neq 0 \) and \( w = 0. \) Let \( M_2 \) be the plane containing \( u, w \) and \( 0. \) Let \( v = \varphi(v), u' = \varphi(u), \) and \( w' = \varphi(w). \) Since \( u' \in M_{r-1}', w' \in \langle \alpha_r^* \rangle, u' \) and \( w' \) are not lying on any line containing \( 0. \) By \( u', w' \in \varphi(M_2), \) \( \varphi(M_2) = \langle u' \rangle \cup \langle w' \rangle, \) thus we have \( \varphi(M_2) \subseteq M_{r-1}' \cup \langle \alpha_r^* \rangle. \) Since \( v \in M_2, \) we get \( v' \in \varphi(M_2). \)

Therefore

\[ \varphi(M) \subseteq M_{r-1}' \cup \langle \alpha_r^* \rangle. \]  

(3.2)

Conversely, for any \( u' \in M_{r-1}' \) and \( w' \in \langle \alpha_r^* \rangle, \) let \( v' = u' + w'. \) Without loss of generality we assume \( u' \neq 0 \) and \( w' \neq 0. \) There exists \( 0 \neq u \in M_{r-1} \) and \( 0 \neq w \in \langle \alpha_r \rangle \) such that \( \varphi(u) = u' \) and
Remark 3.13. Assume that (3.1) holds. Let
\[ \phi(M^\prime) \subseteq \phi(M). \] Since \( \phi(M^\prime) \subseteq M \). Consequently, \( \phi(M^2) \subseteq \phi(M) \). Since \( u', w' \in \phi(M^2) \), we have \( v' = u' + w' \in \phi(M^2) \). Then we obtain that
\[ M_{r-1}^\prime + \langle \alpha_r^* \rangle \subseteq \phi(M). \] (3.3)

Since \( \alpha_r^* \notin M_{r-1}^\prime \), there exist \( b_r \in R^\times \) and \( \alpha \in M_{r-1}^\prime \) such that \( \alpha_r^* - \alpha = b_r \alpha_r^* \) and \( M_{r-1}^\prime \cup \langle \alpha_r^* \rangle = M_{r-1}^\prime + \langle \alpha_r^* \rangle \), where \( \alpha_r^* \) is right unimodular. By (3.3), we have \( b_r \alpha_r^* = \alpha_r^* - \alpha \in \phi(M) \), which implies \( \alpha_r^* \in \phi(M) \). It follows that there exists \( \langle \alpha_r^* \rangle \subseteq M \) such that \( \phi(\langle \alpha_r \rangle) = \langle \alpha_r^* \rangle \). Following the proof of (3.3) step by step, but \( \langle \alpha_r^* \rangle \) and \( \langle \alpha_r \rangle \) should be replaced by \( \langle \alpha_r^* \rangle \) and \( \langle \alpha_r^* \rangle \), respectively, we can prove similarly
\[ M_{r-1}^\prime + \langle \alpha_r^* \rangle = M_{r-1}^\prime \cup \langle \alpha_r^* \rangle \subseteq \phi(M). \] It follows from (3.2) that
\[ \phi(M) = M_{r-1}^\prime \cup \langle \alpha_r^* \rangle. \] (3.4)

Thus, \( \phi \) maps every \( r \)-flat containing \( 0 \) in \( AG(M^1) \) onto \( r \)-flat containing \( 0 \) in \( AG(M^1_i) \) for all \( 1 \leq r \leq n \), and we have (3.1). It follows that \( \dim[AG(M^1)] = \dim[AG(\phi(M))] \), hence Lemmas 3.5 and 3.10 imply that either \( (m, n) = (m', n') \) or \( (m, n) = (n', m') \). □

Remark 3.13. Assume that (3.1) holds. Let \( \phi(\langle \alpha_1, \ldots, \alpha_{r-1} \rangle) = \langle \beta_1, \ldots, \beta_{r-1} \rangle \). Then there exist a unimodular \( \beta_r \) such that \( \phi(\langle \alpha_1, \ldots, \alpha_r \rangle) = \langle \beta_1, \ldots, \beta_{r-1}, \beta_r \rangle \).

4. Geometry of \( 2 \times 2 \) matrices over Bezout domains

In this section, we write that \( M_1 \) and \( N_1 \) (resp. \( M'_1 \) and \( N'_1 \)) are the standard maximal sets of the type one and two in \( R^{2 \times 2} \) (resp. \( R^{2 \times 2} \)), respectively.

Lemma 4.1 (cf. Theorem 19.1 of [17]). For any ring \( R \), the following statements are equivalent:

1. \( R \) is a local ring,
2. \( R/\text{rad}(R) \) is a division ring,
3. \( a + b \in R^\times \) implies that \( a \in R^\times \) or \( b \in R^\times \).

Lemma 4.2. Let \( R, R' \) be Bezout domains, and \( \phi : R^{2 \times 2} \to R^{2 \times 2} \) be an a.p. bijective map in both directions with \( \phi(0) = 0 \). Further, assume that \( R' \) is a local ring, or \( \phi \) is an invertibility preserving map, or \( \phi \) is an additive map. Then we have that either
\[ \phi(M_1) = PM_1Q \quad \text{and} \quad \phi(N_1) = PN_1Q, \quad i = 1, 2, \] (4.1)
or
\[ \phi(M_1) = PN_1Q \quad \text{and} \quad \phi(N_1) = PM_1Q, \quad i = 1, 2, \] (4.2)

where \( P, Q \) are fixed and invertible matrices over \( R' \).

Proof. There exist \( P_0, Q_0 \in GL_2(R') \) such that \( \phi(E_{11}) = P_0a_1E_{11}Q_0 \) where \( a_1 \in R^\times \). Replacing \( \phi \) by the transformation \( \phi(X) \mapsto P_0^{-1} \phi(X)Q_0^{-1} \), we have \( \phi(E_{11}) = a_1E_{11} \). By Corollary 3.6, we have that either \( \phi(M_1) = M_1 \) with \( \phi(N_1) = N_1 \), or \( \phi(M_1) = N_1' \) with \( \phi(N_1) = M_1' \). We only prove the case of \( \phi(M_1) = M_1 \) with \( \phi(N_1) = N_1 \); the proof of the other case is similar.

Assume \( \phi(M_1) = M_1 \) with \( \phi(N_1) = N_1 \). We prove (4.1) as follows. (When \( \phi(M_1) = N_1' \) with \( \phi(N_1) = M_1' \), we can prove similarly (4.2).) By \( M_1 \cap N_1 = RE_{11} \), we get \( \phi(RE_{11}) = R'E_{11} \). Let \( \phi(xE_{11}) = x^{\sigma_1}E_{11} \) for all \( x \in R \), where \( \sigma_1 : R \to R' \) is a bijective map with \( 0^{\sigma_1} = 0 \).
Since $l_2 := RE_{12}$ is a line containing 0 in $AG(M_1)$, by Lemma 3.11, $\varphi(l_2)$ is also a line containing 0 in $AG(M'_1)$. Let $\varphi(l_2) = R'(a_1E_{11} + a_2E_{22})$, where $(a_1, a_2)$ is right unimodular. Assume that

$$\varphi(yE_{12}) = y^{\mu_1}_0(a_1E_{11} + a_2E_{12}), \quad \forall y \in R,$$

(4.3)

where $\mu_1 : R \rightarrow R'$ is a bijective map with $0^{\mu_1} = 0$. Since $l'_2 := RE_{21}$ is a line containing 0 in $AG(N_1)$, we have similarly

$$\varphi(zE_{21}) = (b_1E_{11} + b_2E_{21})z^{\mu_2}, \quad \forall z \in R,$$

(4.4)

where $^t(b_1, b_2)$ is left unimodular, and $\mu_2 : R \rightarrow R'$ is a bijective map with $0^{\mu_2} = 0$. We distinguish the following two cases.

**Case 1.** $a_2 \in R^*$. Then there exists an $y_0$ such that $y_0^{\mu_1}_0a_2 = 1$. It follows from (4.3) that $\varphi(y_0E_{12}) = y_0^{\mu_1}_0a_1E_{11} + E_{12} = E_{12}Q_4$, where $Q_4 = \begin{pmatrix} 1 & 0 \\ y_0^{\mu_1}_0a_1 & 1 \end{pmatrix}$. Replacing $\varphi$ by the transformation $\varphi(x) \mapsto \varphi(x)Q_4^{-1}$, we have $\varphi(y_0E_{12}) = E_{12}$. By $\varphi(M_1) = M'_1$ and Corollary 3.6, we obtain

$$\varphi(N_2) = N'_2.$$

(4.5)

Since $N_1 \cap N'_2 = RE_{12}$, $\varphi(RE_{12}) = R'E_{12}$. Let $\varphi(yE_{12}) = y^{\sigma_2}E_{12}$, $\forall y \in R$, where $\sigma_2 : R \rightarrow R'$ is a bijective map with $0^{\sigma_2} = 0$.

We prove $b_2 \in R^*$ as follows. If $\varphi$ is an invertibility preserving map, then $E_{12} - E_{21}$ is invertible implies that $b_2 \in R^*$. If $\varphi$ is an additive map, then

$$\varphi\left( \begin{pmatrix} x \\ z \end{pmatrix} \right) = \varphi(xE_{11}) + \varphi(zE_{21}) = \begin{pmatrix} x^{\mu_1}_0 + b_1z^{\mu_2} \\ b_2z^{\mu_2} \end{pmatrix}, \quad \forall x, z \in R,$$

it follows from $\varphi(N_1) = N'_1$ that $b_2 \in R^*$. Now we assume $R'$ is a local ring by the conditions. For an arbitrary but fixed $z \in R$, $l_{22} := RE_{11} + zE_{21}$ is a line in $AG(N'_1)$. By Lemma 3.11, $\varphi(l_{22})$ is also a line in $AG(N'_1)$. Let $\varphi(l_{22}) = (\gamma zE_{11} + \delta zE_{21})R' + \varphi(zE_{21})$, where $^t(\gamma, \delta)$ is unimodular. Then by (4.4) we can assume that

$$\varphi\left( \begin{pmatrix} x \\ z \end{pmatrix} \right) = \begin{pmatrix} x^{\gamma}_0z^{\tau} + b_1z^{\mu_2} \\ \delta z^{\tau} + b_2z^{\mu_2} \end{pmatrix}, \quad \forall x, z \in R,$$

where $\tau_2 : R \rightarrow R'$ is a bijective map with $0^{\tau_2} = 0$. Here, $^t(\gamma_0, \delta_0) = (1, 0)$. By $\varphi(N'_1) = N'_1$, there are $x_0, z_0 \in R$ such that $z_0 \neq 0$ and $\delta_0x_0^{\tau_2} + b_2z_0^{\mu_2} \in R^*$. Since $b_2z_0^{\mu_2} \notin R^*$ and $R'$ is local, Lemma 4.1 implies $\delta_0x_0^{\tau_2} \in R^*$, which implies $\delta_0 \in R^*$. Let $\varphi\left( \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} \right) = \begin{pmatrix} x_0^{*} \\ z_0^{*} \end{pmatrix}$ for any $y \in R^*$, where $0^*$ is not necessarily zero. We show that $z_0^{*} \neq 0$. In fact, if $\varphi\left( \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} \right) = \begin{pmatrix} x_0^{*} \\ z_0^{*} \end{pmatrix}$, then by

$$\varphi(RE_{11}) = R'E_{11}, \quad \begin{pmatrix} x_0^{*} \\ 0 \end{pmatrix} \sim x_0^{*}E_{11} := \varphi(x'_0E_{11}),$$

which implies $\begin{pmatrix} x_0 \\ 0 \end{pmatrix} \sim x'_0E_{11}$, a contradiction.

Similarly, $y^* \neq 0$. Since

$$\begin{pmatrix} x_0 \\ z_0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_0^{*} \\ z_0^{*} \end{pmatrix} \sim \begin{pmatrix} b_1z_0^{\mu_2} \\ b_2z_0^{\mu_2} \end{pmatrix}, \quad \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} \sim \begin{pmatrix} b_1z_0^{\mu_2} \\ b_2z_0^{\mu_2} \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} x_0^{*} \\ z_0^{*} \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} b_1z_0^{\mu_2} \\ b_2z_0^{\mu_2} \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} z_0^{*} \\ b_2z_0^{\mu_2} \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
By \((x_0^* - b_1 z_0^{(2)} y^*) = 0\), there exist \(k_1, k_2 \in \mathbb{R}^\times\) such that
\[
\begin{pmatrix}
  x_0^* - b_1 z_0^{(2)} \\
  z_0^* - b_2 z_0^{(2)}
\end{pmatrix}
= \begin{pmatrix}
  y^* \\
  0^*
\end{pmatrix}
k_1 = \begin{pmatrix}
  y^* \\
  0^*
\end{pmatrix}k_2,
\]
which implies that \(\text{rank} \begin{pmatrix}
  x_0^* - b_1 z_0^{(2)} - y_0^* x_0^{(2)}^* y^* \\
  z_0^* - b_2 z_0^{(2)} - \delta_{z_0} x_0^{(2)}^* 0^*
\end{pmatrix} = \text{rank} \begin{pmatrix}
  y_0^* x_0^{(2)}^* y^* \\
  \delta_{z_0} x_0^{(2)}^* 0^*
\end{pmatrix} = 1\). Therefore we have \((y_2 \delta_{z_0} x_0^{(2)}^* 0^*) \sim \delta_{z_0} x_0^{(2)}^* 0^*\).

0. Recall that \(\delta_{z_0} x_0^{(2)}^* \in \mathbb{R}^*\). We obtain \((y_2 \delta_{z_0} x_0^{(2)}^* 0^*) \sim \delta_{z_0} x_0^{(2)}^* 0^*\), and hence \(y^* = y_2 \delta_{z_0} x_0^{(2)}^* 0^*\). Since the arithmetic distance between \(\begin{pmatrix}
  x_0 \\
  y
\end{pmatrix}
\) and \(x E_{11}\) is 2, for all \(x \in \mathbb{R}\), the arithmetic distance between \(\begin{pmatrix}
  x_0^* \\
  y^*
\end{pmatrix}
\) and \(x^{(2)} E_{11}\) is also 2, for all \(x^{(2)} \in \mathbb{R}'\). Hence
\[
\text{rank} \begin{pmatrix}
  x_0^* - x^{(2)} y^* \\
  z_0^* 0^*
\end{pmatrix} = \text{rank} \begin{pmatrix}
  x_0^* - y_2 \delta_{z_0} x_0^{(2)}^* - x^{(2)} 0 \\
  z_0^* 0^*
\end{pmatrix} = 2, \ \forall x^{(2)} \in \mathbb{R}',
\]
this is a contradiction. Then we have proved \(b_2 \in \mathbb{R}^*\).

Similar to the proof of (4.5), replacing \(\varphi\) by the bijection \(\varphi(X) \mapsto P_2^{-1} \varphi(X)\), we have \(\varphi(M_2) = M_2'\). Then we have proved that \(\varphi(M_i) = M_i'\) and \(\varphi(N_i) = N_i', i = 1, 2\). Recall that we have made the transformations. We have (4.1) and this lemma holds.

**Case 2.** \(a_2 \notin \mathbb{R}^*\). We show that this is a contradiction as follows. If \(\varphi\) is an invertibility preserving map, then \(E_{12} - E_{21}\) is invertible implies \(a_2 \in \mathbb{R}^*\), a contradiction. If \(\varphi\) is an additive map, then
\[
\varphi \begin{pmatrix}
  x \\
  0
\end{pmatrix} = \varphi(x E_{11}) + \varphi(y E_{12}) = \begin{pmatrix}
  x^{(2)} + y^{(2)} a_1 \\
  y^{(2)} a_2
\end{pmatrix}, \ \forall x, y \in \mathbb{R},
\]
thus \(\varphi(M_1) = M_1'\) implies \(a_2 \in \mathbb{R}^*\), a contradiction. Now we assume \(\mathbb{R}'\) is a local ring by the conditions. For an arbitrary but fixed \(y \in \mathbb{R}, l_{1y} := R E_{11} + y E_{12}\) is a line in \(AG(M_1)\), it follows from Lemma 3.11 that \(\varphi(l_{1y})\) is also a line in \(AG(M_1)\). Let \(\varphi(l_{1y}) = R' (a y E_{11} + b y E_{22}) + \varphi(y E_{12})\), where \((a y, b y)\) is unimodular. By (4.3), we can assume that
\[
\varphi \begin{pmatrix}
  x \\
  0
\end{pmatrix} = \begin{pmatrix}
  x^{(2)} \alpha y + y^{(2)} a_1 x^{(2)} \beta y + y^{(2)} a_2
\end{pmatrix}, \ \forall x, y \in \mathbb{R},
\]
where \(\alpha y : R \rightarrow R'\) is a bijective map with \(0^{(2)} \alpha y = 0\). Here, \((\alpha y, \beta y) = (1, 0)\).

By \(\varphi(M_1) = M_1'\), there are \(x_0, y_0 \in \mathbb{R}\) such that \(y_0 \neq 0\) and \(x_0^{(2)} \beta y_0 + y_0^{(2)} a_2 \in \mathbb{R}^*\). Since \(y_0^{(2)} a_2 \notin \mathbb{R}^*\) and \(\mathbb{R}'\) is local, Lemma 4.1 implies \(x_0^{(2)} \beta y_0 \in \mathbb{R}^*\). Let \(\varphi \begin{pmatrix}
  x_0 \\
  z
\end{pmatrix} = \begin{pmatrix}
  x_0^* \\
  z^*
\end{pmatrix} y_0^*\) for any \(z \in \mathbb{R}^	imes\). Similar to the Case 1, we can prove that \(y_0^* \neq 0\) and \(z^* \neq 0\). Since
\[
\begin{pmatrix}
  x_0^* - y_0^{(2)} a_1 - x_0^{(2)} \alpha y_0 \\
  z^*
\end{pmatrix}
\neq \begin{pmatrix}
  y_0^{(2)} a_2 - x_0^{(2)} \beta y_0 \\
  0^*
\end{pmatrix},
\]
\[
\begin{pmatrix}
  x_0 - y_0^{(2)} a_1 - x_0^{(2)} \alpha y_0 \\
  z
\end{pmatrix}
\neq \begin{pmatrix}
  y_0^{(2)} a_2 - x_0^{(2)} \beta y_0 \\
  0^*
\end{pmatrix} \sim 0.
\]
Theorem 4.3. Let $R, R'$ be Bezout domains, and $\varphi : R^{2 \times 2} \to R'^{2 \times 2}$ be an a.p. bijective map in both directions such that $\varphi(M_1)$ is a maximal set of the type one. Further, assume that $R'$ is a local ring, or $\varphi$ is an invertibility preserving map, or $\varphi$ is an additive map. Then $R$ is isomorphic to $R'$, and $\varphi$ is of the form

$$\varphi(X) = PX^\sigma Q + \varphi(0), \quad \forall X \in R^{2 \times 2},$$

(4.6)

where $P, Q \in GL_2(R')$ are fixed, and $\sigma$ is an isomorphism from $R$ to $R'$.

**Proof.** When $|R^\times| = 1$, $R = \mathbb{F}_2$ is the finite field of 2 elements. By Lemma 4.2, it is easy to see that $R' = \mathbb{F}_2$, thus Theorem 4.3 holds by Hua’s theorem. From now on we assume $|R^\times| \geqslant 2$. Then Lemma 4.2 implies $|R'^\times| \geqslant 2$. Replacing $\varphi$ by the transformation $\varphi(X) \longmapsto \varphi(X) - \varphi(0)$, we can assume that $\varphi(0) = 0$.

**Step 1.** We need to prove some formulas of $\varphi$ for the proof of Theorem 4.3. First, we prove five formulas of $\varphi$ as follows.

By Lemma 4.2, there are fixed $P_1, Q_1 \in GL_2(R')$ such that $\varphi(M_i) = P_1M'_iQ_1$ and $\varphi(N_i) = P_1N'_iQ_1$, $i = 1, 2$. Replacing $\varphi$ by the a.p. bijection $\varphi(X) \longmapsto P_1^{-1}\varphi(X)Q_1^{-1}$, we have

$$\varphi(M_i) = M'_i \quad \text{and} \quad \varphi(N_i) = N'_i, \quad i = 1, 2.$$  
(4.7)

Since $M_i \cap N_j = ME_{ij}, \varphi(RE_{ij}) = R'E_{ij}, i, j = 1, 2$. Let $\varphi(xE_{ij}) = x^{\sigma_{ij}}E_{ij}$ for all $x \in R$, $i, j = 1, 2$, where $\sigma_{ij} : R \to R'$ is a bijective map with $0^{\sigma_{ij}} = 0, i, j = 1, 2$.

For an arbitrary but fixed $y \in R^\times$, let $\psi_y(X) = \varphi(X + yE_{22}) - \varphi(yE_{22}) = \varphi(X + yE_{22}) - y^{\sigma_{22}}E_{22}$. Then $\psi_y : R^{2 \times 2} \to R'^{2 \times 2}$ is an a.p. bijective map in both directions with $\psi_y(0) = 0$. If $\varphi$ is an invertibility preserving map (or additive), then $\psi_y$ is also an invertibility preserving map (or additive). By the conditions, we have that $\psi_y(M_2) = M'_2$ and $\psi_y(N_2) = N'_2$. By Lemma 4.2, we have that

$$\psi_y(M_i) = P_yM'_iQ_y \quad \text{and} \quad \psi_y(N_i) = P_yN'_iQ_y, \quad i = 1, 2,$$

where $P_y, Q_y \in GL_2(R')$. Since $\psi_y(M_2) = M'_2 = P_yM'_2$ and $\psi_y(N_2) = N'_2 = N'_2Q_y$, we can assume that

$$P_y = \begin{pmatrix} 1 & 0 \\ p_1 & 1 \end{pmatrix}, \quad Q_y = \begin{pmatrix} 1 & q_2 \\ p_1 & 0 \end{pmatrix}.$$  

Since $\psi_y(RE_{ij}) = P_y(R'E_{ij})Q_y$, we let $\psi_y(xE_{ij}) = P_yx^{\tau_{ij}}E_{ij}Q_y, \forall x \in R, i, j = 1, 2$, where $\tau_{ij} : R \to R'$ is a bijective map with $0^{\tau_{ij}} = 0$, and $\tau_{ij}$ does not depend on the choice of $y, i, j = 1, 2$. Then

$$\varphi(xE_{ij} + yE_{22}) = P_yx^{\tau_{ij}}E_{ij}Q_y + y^{\sigma_{22}}E_{22}, \quad \forall x \in R, \quad y \in R^\times, \quad i, j = 1, 2.$$  
(4.8)
By (4.8) and computing, we have
\[
\varphi \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x^T 1 & x^T 2 \\ p_1 x^T 1 & p_1 x^T 2 + y^T 22 \end{pmatrix}, \quad \forall x \in R, \ y \in R^x, \tag{4.9}
\]
\[
\varphi \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x^T 1 & x^T 2 + y^T 22 \end{pmatrix}, \quad \forall x \in R, \ y \in R^x, \tag{4.10}
\]
\[
\varphi \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & x^T 1 \\ 0 & p_1 x^T 1 + y^T 22 \end{pmatrix}, \quad \forall x \in R, \ y \in R^x, \tag{4.11}
\]
\[
\varphi \begin{pmatrix} 0 & 0 \\ 0 & x + y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x^T 2 + y^T 22 \end{pmatrix}, \quad \forall x \in R, \ y \in R^x, \tag{4.12}
\]
where \(p_1\) and \(q_2\) are functions of \(y\).

For an arbitrary but fixed \(y \in R^x\), let \(\psi'_y(X) = \varphi(X + y E_{12}) - \varphi(y E_{12}) = \varphi(X + y E_{12}) - y^T 12 E_{12}\). Then \(\psi'_y : R^{2 \times 2} \rightarrow R^{2 \times 2}\) is an a.p. bijective map in both directions with \(\psi'_y(0) = 0\). If \(\varphi\) is an invertibility preserving map (or additive), then \(\psi'_y\) is also an invertibility preserving map (or additive). By (4.7), we have that \(\psi'_y(M_1) = M'_1\) and \(\psi'_y(N_2) = N'_2\). By Lemma 4.2, we have that
\[
\psi'_y(M_1) = C_y M'_1 D_y \quad \text{and} \quad \psi'_y(N_1) = C_y N'_1 D_y, \quad i = 1, 2,
\]
where \(C_y, D_y \in GL_2(R')\). Since \(\psi'_y(M_1) = M'_1 = C_y M'_1\) and \(N_2 = N'_2 D_y\), we can let \(C_y = \begin{pmatrix} 1 & c_3 \\ 0 & 1 \end{pmatrix}\) and
\[
D_y = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}.
\]

By \(\psi'_y(RE_{ij}) = C_y (R'E_{ij}) D_y\), we assume that \(\psi'_y(x E_{ij}) = C_y x^T 12 E_{ij} D_y, \forall x \in R, i, j = 1, 2\), where \(\tau'_j : R \rightarrow R'\) is a bijective map with \(0_{ij} = 0\), and \(\tau'_j\) does not depend on the choice of \(y, i, j = 1, 2\). Then
\[
\varphi(x E_{ij} + y E_{12}) = C_y x^T 12 E_{ij} D_y + y^T 12 E_{12}, \quad \forall x \in R, \ i, j = 1, 2.
\]
By computing, we have that
\[
\varphi \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x^T 11 & x^T 11 d_2 + y^T 12 \\ 0 & 0 \end{pmatrix}, \quad \forall x \in R, \ y \in R^x, \tag{4.13}
\]
where \(d_2\) is a function of \(y\).

**Step 2.** Secondly, we prove five new formulas of \(\varphi\). Computing these formulas, we obtain seven simple formulas of \(\varphi\).

For an arbitrary but fixed \(x \in R^x\), let \(\Phi_x(X) = \varphi(X + x E_{11}) - \varphi(x E_{11})\). Then \(\Phi_x : R^{2 \times 2} \rightarrow R^{2 \times 2}\) is an a.p. bijective map in both directions with \(\Phi_x(0) = 0\). If \(\varphi\) is an invertibility preserving map (or additive), then \(\Phi_x\) is also an invertibility preserving map (or additive). By (4.7), we have that \(\Phi_x(M_1) = M'_1\) and \(\Phi_x(N_1) = N'_1\). By Lemma 4.2, we obtain that
\[
\Phi_x(M_i) = S_x M'_i T_x \quad \text{and} \quad \Phi_x(N_i) = S_x N'_i T_x, \quad i = 1, 2.
\]
where $S_x, T_x \in GL_2(R')$. Since $\Phi_x(M_1) = M'_1 = S_xM'_1$ and $\Phi_x(N_1) = N'_1 = N'_1T_x$, we can assume that $S_x = \begin{pmatrix} 1 & -s_3 \\ 0 & 1 \end{pmatrix}$ and $T_x = \begin{pmatrix} 1 & 0 \\ t_3 & 1 \end{pmatrix}$.

By $\Phi_x(RE_{ij}) = S_x(RE_{ij})T_x$, we let $\Phi_x(yE_{ij}) = S_xy^{\mu_{ij}}E_{ij}T_x, \forall y \in R, i, j = 1, 2$, where $\mu_{ij} : R \to R'$ is a bijective map with $0^{\mu_{ij}} = 0$, and $\mu_{ij}$ does not depend on the choice of $x, i, j = 1, 2$. Then

$$\varphi(yE_{ij} + xE_{11}) = S_xy^{\mu_{ij}}E_{ij}T_x + x^{\sigma_{ij}}E_{11}, \forall y \in R, i, j = 1, 2.$$  

By computing, we have

$$\varphi\left(\begin{array}{c} x \\ 0 \\ 0 \\ y \end{array}\right) = \begin{pmatrix} s_3 y^{\mu_{22}}t_2 + x^{\sigma_{11}} & -s_3 y^{\mu_{22}} \\ -y^{\mu_{22}}t_2 & y^{\mu_{22}} \end{pmatrix}, \forall x \in R^\times, y \in R, \quad (4.14)$$

$$\varphi\left(\begin{array}{c} x \\ y \\ 0 \\ 0 \end{array}\right) = \begin{pmatrix} -y^{\mu_{12}}t_3 + x^{\sigma_{11}} & y^{\mu_{12}} \\ 0 & 0 \end{pmatrix}, \forall x \in R^\times, y \in R, \quad (4.15)$$

$$\varphi\left(\begin{array}{c} x \\ 0 \\ y \\ 0 \end{array}\right) = \begin{pmatrix} -s_3 y^{\mu_{21}} + x^{\sigma_{11}} & 0 \\ y^{\mu_{21}} & 0 \end{pmatrix}, \forall x \in R^\times, y \in R, \quad (4.16)$$

$$\varphi\left(\begin{array}{c} x + y \\ 0 \end{array}\right) = \begin{pmatrix} x^{\sigma_{11}} + y^{\mu_{11}} & 0 \\ 0 & 0 \end{pmatrix}, \forall x \in R^\times, y \in R, \quad (4.17)$$

where $s_3$ and $t_3$ are functions of $x$.

For an arbitrary but fixed $x \in R^\times$, let $\Phi_x'(X) = \varphi(X + xE_{21}) - \varphi(xE_{21})$. Then $\Phi_x' : R^{2 \times 2} \to R^{2 \times 2}$ is an a.p. bijective map in both directions with $\Phi_x'(0) = 0$. If $\varphi$ is an invertibility preserving map (or additive), then $\Phi_x'$ is also an invertibility preserving map (or additive). By (4.7), we have that $\Phi_x'(M_2) = M'_2$ and $\Phi_x'(N_1) = N'_1$. By Lemma 4.2, we get that

$$\Phi_x'(M_1) = G_xM_xH_x \quad \text{and} \quad \Phi_x'(N_j) = G_xN_jH_x, \quad i, j = 1, 2,$$

where $G_x, H_x \in GL_2(R')$. Since $\Phi_x'(M_2) = M'_2 = G_xM_2$ and $N_1 = N'_1H_x$, we assume $G_x = \begin{pmatrix} 1 & 0 \\ g_1 & 1 \end{pmatrix}$ and $H_x = \begin{pmatrix} 1 & 0 \\ -h_3 & 1 \end{pmatrix}$.

By $\Phi_x'(RE_{ij}) = G_x(RE_{ij})H_x$, we assume that $\Phi_x'(yE_{ij}) = G_xy^{\mu_{ij}}E_{ij}H_x, \forall y \in R, i, j = 1, 2$, where $\mu_{ij} : R \to R'$ is a bijective map with $0^{\mu_{ij}} = 0$, and $\mu_{ij}$ does not depend on the choice of $x, i, j = 1, 2$. Then

$$\varphi(yE_{ij} + xE_{21}) = G_xy^{\mu_{ij}}E_{ij}H_x + x^{\sigma_{21}}E_{21}, \forall y \in R, i, j = 1, 2.$$  

By computing, it is easy to see that

$$\varphi\left(\begin{array}{c} x \\ 0 \\ 0 \\ y \end{array}\right) = \begin{pmatrix} 0 \\ 0 \\ -y^{\mu_{22}}h_3 + x^{\sigma_{22}}y^{\mu_{22}} \end{pmatrix}, \forall x \in R^\times, y \in R, \quad (4.18)$$

where $h_3$ is a function of $x$. 

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By (4.18) and (4.10), we get
\[ y^{q_{22}} = x^{\tau_{21}} q_2 + y^{\sigma_{22}}, \quad \forall x, y \in R^\times. \tag{4.19} \]

By (4.13) and (4.15), we have
\[ x^{\tau_{11}} = -y^{\mu_{12}} t_3 + x^{\sigma_{11}}, \quad \forall x, y \in R^\times. \tag{4.20} \]

By the comparison between (4.9) and (4.14), we have
\[ x^{\tau_{11}} q_2 = -s_3 y^{\mu_{22}}, \quad \forall x, y \in R^\times, \tag{4.21} \]
\[ p_1 x^{\tau_{11}} q_2 = -y^{\mu_{22}} t_3, \quad \forall x, y \in R^\times, \tag{4.22} \]
\[ p_1 x^{\tau_{11}} q_2 + y^{\sigma_{22}} = y^{\mu_{22}}, \quad \forall x, y \in R^\times, \tag{4.23} \]
where \( p_1, q_2 \) are functions of \( y \), and \( s_3, t_3 \) are functions of \( x \).

Since \( |R^\times| \geq 2 \), by (4.19), there are two distinct \( x, x' \in R^\times \) such that \( (x^{\tau_{21}} - x'^{\tau_{21}}) q_2 = 0 \) for all \( y \in R^\times \), which implies \( q_2 = 0 \) for all \( y \in R^\times \). Then (4.22) implies \( s_3 = 0 \) for all \( x \in R^\times \). By (4.20), there are two distinct \( y, y' \in R^\times \) such that \( (y^{\mu_{12}} - y'^{\mu_{12}}) t_3 = 0 \) for all \( x \in R^\times \), which implies \( t_3 = 0 \) for all \( x \in R^\times \). Then (4.23) implies that \( p_1 = 0 \) for all \( y \in R^\times \). Thus (4.21) and (4.24) become
\[ x^{\tau_{11}} = x^{\sigma_{11}}, \quad \forall x \in R^\times, \quad \text{and} \quad y^{\sigma_{22}} = y^{\mu_{22}}, \quad \forall y \in R^\times. \]

By \( p_1 = q_2 = 0, t_3 = s_3 = 0, (4.9)-(4.12) \) and (4.14)-(4.17), we have
\[ \varphi \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x^{\sigma_{11}} & 0 \\ 0 & y^{\sigma_{22}} \end{pmatrix}, \quad \forall x, y \in R, \tag{4.25} \]
\[ \varphi \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x^{\tau_{21}} & y^{\sigma_{22}} \end{pmatrix}, \quad \forall x, y \in R, \tag{4.26} \]
\[ \varphi \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & x^{\tau_{12}} \\ 0 & y^{\sigma_{22}} \end{pmatrix}, \quad \forall x, y \in R, \tag{4.27} \]
\[ \varphi \begin{pmatrix} 0 & 0 \\ 0 & x+y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x^{\tau_{22}} + y^{\sigma_{22}} \end{pmatrix}, \quad \forall x, y \in R, \tag{4.28} \]
\[ \varphi \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x^{\sigma_{11}} & y^{\mu_{12}} \\ 0 & 0 \end{pmatrix}, \quad \forall x, y \in R, \tag{4.29} \]
\[ \varphi \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} x^{\sigma_{11}} & 0 \\ y^{\mu_{12}} & 0 \end{pmatrix}, \quad \forall x, y \in R, \tag{4.30} \]
\[ \varphi \begin{pmatrix} x+y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x^{\sigma_{11}} + y^{\mu_{11}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall x, y \in R. \tag{4.31} \]

**Step 3.** Finally, we prove Theorem 4.3 by formulas (4.25)-(4.31).
For any $k, y \in R^x$, let $\varphi \begin{pmatrix} 0 & y \\ k & 0 \end{pmatrix} = \begin{pmatrix} 0^* & y^* \\ k^* & 0^{**} \end{pmatrix}$. Since $\begin{pmatrix} 0 & y \\ k & 0 \end{pmatrix} \sim \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ for all $x \in R$, (4.29) implies $\begin{pmatrix} 0^* & y^* \\ k^* & 0^{**} \end{pmatrix} \sim \begin{pmatrix} x^{\sigma_{11}} & y^{\sigma_{12}} \\ 0 & 0 \end{pmatrix}$ for all $x \in R$. Thus $\begin{pmatrix} x & y^* - y^{\sigma_{12}} \\ k^* & 0^{**} \end{pmatrix} \sim 0$ for all $x \in R$. By $|R^x| \geq 2$ and Lemma 3.2, we get $0^{**} = 0$. Then $\varphi \begin{pmatrix} 0 & y \\ k & 0 \end{pmatrix} = \begin{pmatrix} 0^* & y^* \\ k^* & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & y \\ k & 0 \end{pmatrix} \sim yE_{12}$, $\begin{pmatrix} 0^* & y^* \\ k^* & 0 \end{pmatrix} \sim y^{\sigma_{12}}E_{12}$. Similarly, $k^* = k^{\sigma_{21}}E_{21}$. Hence $\varphi \begin{pmatrix} 0 & y \\ k & 0 \end{pmatrix} = \begin{pmatrix} 0^* & y^{\sigma_{12}} \\ k^{\sigma_{21}} & 0 \end{pmatrix}$.

By $\begin{pmatrix} 0 & y \\ k & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ k & z \end{pmatrix}$ $\forall z \in R$ and (4.26), $\begin{pmatrix} 0^* & y^{\sigma_{12}} \\ k^{\sigma_{21}} & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ k^{\sigma_{21}} & z^{\sigma_{22}} \end{pmatrix}$ for all $z \in R$. It follows that $\begin{pmatrix} 0^* & y^{\sigma_{12}} \\ k^{\sigma_{21}} & 0 \\ 0 & z^{\sigma_{22}} \end{pmatrix} \sim 0$ for all $z \in R$. By $|R^x| \geq 2$ and Lemma 3.2, we have similarly $0^* = 0$ and $k^{\sigma_{21}} = k^{\sigma_{21}}$. Then we have proved that

$$\varphi \begin{pmatrix} 0 & y \\ k & 0 \end{pmatrix} = \begin{pmatrix} 0 & y^{\sigma_{12}} \\ k^{\sigma_{21}} & 0 \end{pmatrix}, \quad \forall k, y \in R, \quad (4.32)$$

$$\varphi \begin{pmatrix} 0 & 0 \\ k & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ k^{\sigma_{21}} & z^{\sigma_{22}} \end{pmatrix}, \quad \forall k, z \in R. \quad (4.33)$$

Since $\begin{pmatrix} 0 & y \\ k & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix}$, it follows from (4.32) and (4.27) that

$$\varphi \begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & y^{\sigma_{12}} \\ 0 & z^{\sigma_{22}} \end{pmatrix}, \quad \forall y, z \in R. \quad (4.34)$$

Since $\begin{pmatrix} 0 & y \\ k & 0 \end{pmatrix} \sim \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$, by (4.32) and (4.29) we get

$$\varphi \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x^{\sigma_{11}} & y^{\sigma_{12}} \\ 0 & 0 \end{pmatrix}, \quad \forall x, y \in R. \quad (4.35)$$

Since $l := R(E_{11} + E_{12})$ is a line containing $0$ in $AG(M_1)$, by Lemma 3.11, $\varphi(l)$ is also a line containing $0$ in $AG(M_1^l)$. Let $\varphi(l) = R'(a_1E_{11} + a_2E_{12})$, where $(a_1, a_2)$ is unimodular. Thus we can assume that $\varphi \begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x^T a_1 & x^T a_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x^{\sigma_{11}} & x^{\sigma_{12}} \\ 0 & 0 \end{pmatrix}, \quad \forall x \in R,$

where $\tau : R \rightarrow R'$ is a bijective map with $0^\tau = 0$. Then $x^T a_1 = x^{\sigma_{11}}$ and $x^T a_2 = x^{\sigma_{12}}$ for all $x \in R$. Thus $a_1, a_2 \in R^x$ and $x^{\sigma_{12}} = x^{\sigma_{11}} a_1^{-1} a_2$ for all $x \in R$. Let $\sigma = \sigma_{11}$. Replacing $\varphi$ by the bijective map $\varphi(X) \mapsto \varphi(X) \text{diag}(1, a_2^{-1} a_1)$, we have

$$x^{\sigma_{11}} = x^{\sigma_{12}} = x^\tau, \quad \forall x \in R, \quad (4.36)$$
By (4.31), \((x + y)^\sigma = (y + x)^\sigma = x^{\mu_{11}} + y^\sigma\) for all \(x, y \in R\). Taking \(y = -x\) we have \(x^{\mu_{11}} = -(-x)^\sigma\) for all \(x \in R\). Thus \((x + y)^\sigma = -(-x)^\sigma + y^\sigma\) for all \(x, y \in R\), which implies that \((y - x)^\sigma = y^\sigma - x^\sigma\) for all \(x, y \in R\). Taking \(y = 0\), we obtain that

\[-x^\sigma = -x^\sigma, \quad (x + y)^\sigma = x^\sigma + y^\sigma, \quad \forall x, y \in R.\]  

By (4.28), we can prove similarly that

\[-x^{\sigma_{22}} = -x^{\sigma_{22}}, \quad (x + y)^{\sigma_{22}} = x^{\sigma_{22}} + y^{\sigma_{22}}, \quad \forall x, y \in R.\]  

Let \(x, z, k \in R^\times\) and \(\varphi\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right) = \left(\begin{array}{c} x^* \\ 0 \\ k^* \\ z^* \end{array}\right)\). Since \(\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right)\) is adjacent with both \(\left(\begin{array}{c} x \\ 0 \\ 0 \\ z \end{array}\right)\) and \(\left(\begin{array}{c} 0 \\ 0 \\ k \\ z \end{array}\right)\), \(\left(\begin{array}{c} x^* \\ 0 \\ k^* \\ z^* \end{array}\right)\) is adjacent with both \(\left(\begin{array}{c} x^\sigma \\ 0 \\ 0 \\ z^{\sigma_{22}} \end{array}\right)\) and \(\left(\begin{array}{c} 0 \\ 0 \\ 0 \\ z^{\sigma_{22}} \end{array}\right)\). It follows from Lemma 3.2 that \(z^* = z^{\sigma_{22}}\) and \(0^*k^* = 0\). Since \(\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right)\) is adjacent with both \(\left(\begin{array}{c} x \\ 0 \\ 0 \\ z \end{array}\right)\) and \(\left(\begin{array}{c} x \\ 0 \\ 0 \\ 0 \end{array}\right)\), by Lemma 3.2, we have similarly \(x^* = x^\sigma\). Then \(\varphi\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right) = \left(\begin{array}{c} x^\sigma \\ 0^* \\ k^* \\ z^{\sigma_{22}} \end{array}\right)\), where \(0^*k^* = 0\).

For \(k, z \in R^\times\), \(\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right) \sim \left(\begin{array}{c} x \\ 0 \\ y \\ 0 \end{array}\right)\), thus (4.38) implies

\[\varphi\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right) = \left(\begin{array}{c} x^\sigma \\ 0 \\ k^* \\ z^{\sigma_{22}} \end{array}\right) \sim \left(\begin{array}{c} x^\sigma \\ 0 \\ y^{\sigma_{21}} \\ 0 \end{array}\right)\]  

for all \(y \in R\). Thus Lemma 3.2 implies \(0^* = 0\) and \(\varphi\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right) = \left(\begin{array}{c} x^\sigma \\ 0 \\ k^* \\ z^{\sigma_{22}} \end{array}\right)\). Since \(\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right) \sim \left(\begin{array}{c} 0 \\ -x \\ 0 \\ z - k \end{array}\right)\), by (4.34), (4.36), (4.39) and (4.40), \(\left(\begin{array}{c} x^\sigma \\ 0 \\ k^* \\ z^{\sigma_{22}} \end{array}\right) \sim \left(\begin{array}{c} 0 \\ -x^\sigma \\ 0 \\ z^{\sigma_{22}} - k^{\sigma_{22}} \end{array}\right)\). Thus \(\left(\begin{array}{c} x^\sigma \\ x^\sigma \end{array}\right) \sim 0\), and hence \(k^* = k^{\sigma_{22}}\). Thus we have proved that

\[\varphi\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right) = \left(\begin{array}{c} x^\sigma \\ 0 \\ k^{\sigma_{22}} \\ z^{\sigma_{22}} \end{array}\right), \quad \forall x, k, z \in R.\]
Let $\varphi \left(\begin{array}{c} x \\ y \\ k \\ z \end{array}\right) = \left(\begin{array}{c} x^* \\ y^* \\ k^* \\ z^* \end{array}\right)$, where $x, k, y, z \in R^*$. By (4.41), we see $y^* \neq 0$. Since $\varphi \left(\begin{array}{c} x \\ y \\ k \\ z \end{array}\right)$ is adjacent with both $\left(\begin{array}{c} x \\ 0 \\ k \\ z \end{array}\right)$ and $\left(\begin{array}{c} 0 \\ 0 \\ k \\ z \end{array}\right)$, $\varphi \left(\begin{array}{c} x^* \\ y^* \\ k^* \\ z^* \end{array}\right)$ is adjacent with both $\left(\begin{array}{c} x^\sigma \\ 0 \\ k^\sigma 22 \\ z^\sigma 22 \end{array}\right)$ and $\left(\begin{array}{c} 0 \\ 0 \\ k^\sigma 22 \\ z^\sigma 22 \end{array}\right)$.

By Lemma 3.2, we have that $z^* - z^\sigma 22 = 0$ and $k^* - k^\sigma 22 = 0$. Then $\varphi \left(\begin{array}{c} x \\ y \\ k \\ z \end{array}\right) = \left(\begin{array}{c} k^\sigma 22 \\ z^\sigma 22 \end{array}\right)$. Since $\varphi \left(\begin{array}{c} x \\ y \\ k \\ z \end{array}\right)$ is adjacent with both $\left(\begin{array}{c} x \\ 0 \\ k \\ 0 \end{array}\right)$ and $\left(\begin{array}{c} 0 \\ 0 \\ k \\ 0 \end{array}\right)$, $\varphi \left(\begin{array}{c} x^* \\ y^* \\ k^* \\ z^* \end{array}\right)$ is adjacent with both $\left(\begin{array}{c} x^\sigma \\ 0 \\ k^\sigma 22 \\ 0 \end{array}\right)$ and $\left(\begin{array}{c} 0 \\ y^\sigma \\ 0 \\ z^\sigma 22 \end{array}\right)$. It follows that $x^* = x^\sigma$ and $y^* = y^\sigma$. Then

$$\varphi \left(\begin{array}{c} x \\ y \\ k \\ z \end{array}\right) = \left(\begin{array}{c} x^\sigma \\ y^\sigma \\ k^\sigma 22 \\ z^\sigma 22 \end{array}\right), \quad \forall x, y, k, z \in R. \tag{4.42}$$

There are $y_0, k_0 \in R$ such that $y_0^\sigma = k_0^\sigma 22 = 1$. Since $\left(\begin{array}{c} 1 \\ y_0 \\ k_0 \end{array}\right) \sim 0$, $\left(\begin{array}{c} 1 \\ 1 \\ (k_0 y_0)^\sigma 22 \end{array}\right) \sim 0$. Thus $1^\sigma (k_0 y_0)^\sigma 22 = 1$ and $1^\sigma \in R^*$. Similarly, $1^\sigma 22 \in R^*$. Replacing $\varphi$ by the transformation

$$\varphi(X) \longrightarrow \text{diag} \left((1^\sigma)^{-1}, (1^{\sigma 22})^{-1}\right) \varphi(X),$$

without loss of generality, we assume that (4.42) holds and $1^\sigma = 1^{\sigma 22} = 1$.

Since $\left(\begin{array}{c} x \\ 1 \\ 1 \\ x \end{array}\right) \sim 0$, $\left(\begin{array}{c} x^\sigma \\ 1 \\ x^\sigma 22 \\ 1 \end{array}\right) \sim 0$. Thus $x^\sigma 22 = x^\sigma$ for all $x \in R$. Then we have proved that

$$\varphi \left(\begin{array}{c} x \\ y \\ k \\ z \end{array}\right) = \left(\begin{array}{c} x^\sigma \\ y^\sigma \\ k^\sigma \\ z^\sigma \end{array}\right), \quad \forall x, y, k, z \in R.$$

Since $\left(\begin{array}{c} 1 \\ y \\ x y \end{array}\right) \sim 0$, $\left(\begin{array}{c} 1 \\ y^\sigma \\ x^\sigma (y^\sigma) \end{array}\right) \sim 0$, which implies $(xy)^\sigma = x^\sigma y^\sigma$ for all $x, y \in R$. Thus $\sigma$ is an isomorphism from $R$ to $R'$.

Recalling the transformations we have made, the original $\varphi$ must be of the form (4.6). The proof of Theorem 4.3 ends. $\Box$

Similar to the proof of Theorem 4.3, we can prove the following theorem.

**Theorem 4.4.** Let $R, R'$ be Bezout domains, and $\varphi : R^{2 \times 2} \to R'^{2 \times 2}$ be an a.p. bijective map in both directions such that $\varphi(M_1)$ is a maximal set of the type two. Further, assume that $R'$ is a local ring, or $\varphi$ is an invertibility preserving map, or $\varphi$ is an additive map. Then $R$ is anti-isomorphic to $R'$, and $\varphi$ is of the form

$$\varphi(X) = P^T X^T Q + \varphi(0), \quad \forall X \in R^{2 \times 2}, \tag{4.43}$$

where $P, Q \in GL_2(R')$ are fixed, and $\tau$ is an anti-isomorphism from $R$ to $R'$. 
Combination Theorems 4.3 and 4.4, we have immediately the following fundamental theorem of the geometry of $2 \times 2$ matrices over Bezout domains.

**Theorem 4.5.** Let $R$, $R'$ be Bezout domains, and $\varphi : R^{2 \times 2} \rightarrow R'^{2 \times 2}$ be an a.p. bijective map in both directions. Further, assume that $R'$ is a local ring, or $\varphi$ is an invertibility preserving map, or $\varphi$ is an additive map. Then $\varphi$ is of the form either

$$\varphi(X) = PX^\sigma Q + \varphi(0) \quad \forall X \in R^{2 \times 2},$$

or

$$\varphi(X) = P^tX^tQ + \varphi(0) \quad \forall X \in R^{2 \times 2},$$

where $P, Q \in GL_2(R')$ are fixed, $\sigma$ is an isomorphism from $R$ to $R'$, and $\tau$ is an anti-isomorphism from $R$ to $R'$.

5. **Geometry of $m \times n$ matrices over Bezout domains**

In this section, we assume that $\mathcal{M}_1$ and $\mathcal{N}_1$ (resp. $\mathcal{M}'_1$ and $\mathcal{N}'_1$) are the standard maximal sets of the type one and two in $R^{m \times n}$ (resp. $R'^{m' \times n'}$), respectively.

**Lemma 5.1.** Let $R$, $R'$ be Bezout domains and let $m$, $m'$, $n'$ be integers $\geq 2$. Assume that $\varphi : R^{m \times n} \rightarrow R'^{m' \times n'}$ is an a.p. bijective map in both directions such that $\varphi(0) = 0$. If $\varphi(\mathcal{M}_1)$ is a maximal set of the type one, then $(m, n) = (m', n')$, and there exist fixed $P_1 \in GL_m(R')$, $Q_1 \in GL_n(R')$, such that

$$\varphi \left( \begin{array}{c} R^{s \times t} 0 \\ 0 0 \end{array} \right) = P_1 \left( \begin{array}{c} R^{s \times t} 0 \\ 0 0 \end{array} \right) Q_1, \quad s = 1, \ldots, m, \ t = 1, \ldots, n. \quad (5.1)$$

If $\varphi(\mathcal{M}_1)$ is a maximal set of the type two, then $(m, n) = (n', m')$, and there exist fixed $P_2 \in GL_n(R')$, $Q_2 \in GL_m(R')$, such that

$$\varphi \left( \begin{array}{c} R^{s \times t} 0 \\ 0 0 \end{array} \right) = P_2 \left( \begin{array}{c} R^{t \times s} 0 \\ 0 0 \end{array} \right) Q_2, \quad s = 1, \ldots, m, \ t = 1, \ldots, n. \quad (5.2)$$

**Proof.** There exist two fixed invertible matrices $P_0 \in GL_{m'}(R')$, $Q_0 \in GL_{n'}(R')$, such that $\varphi(E_{11}) = P_0a_1E_{11}Q_0$ where $a_1 \in R^{x \times}$.

**Case 1.** $\varphi(\mathcal{M}_1)$ is a maximal set of the type one. Since $\mathcal{M}_1$ and $\mathcal{N}_1$ are two maximal sets containing $0$ and $E_{11}$ (or $a_1E_{11}$), Corollary 3.6 implies that $\varphi(\mathcal{M}_1) = P_0\mathcal{M}'_1Q_0$ with $\varphi(\mathcal{N}_1) = P_0\mathcal{N}'_1Q_0$. By Theorem 3.12, we have $(m, n) = (m', n')$. Replacing $\varphi$ by the transformation $\varphi(X) \mapsto P_0^{-1}\varphi(X)Q_0^{-1}$, we have

$$\varphi(\mathcal{M}_1) = \mathcal{M}', \quad \varphi(\mathcal{N}_1) = \mathcal{N}', \quad \varphi(\mathcal{E}_{11}) = \mathcal{E}_{11}. \quad (5.3)$$

Let $2 \leq t < n$. Then $\text{diag}(R^{1 \times t}, 0)$ is a $t$-flat containing $0$ in $AG(\mathcal{M}_1)$. By Theorem 3.12, $\varphi(\text{diag}(R^{1 \times t}, 0))$ is also $t$-flat containing $0$ in $AG(\mathcal{M}_1')$. Thus $\varphi(\text{diag}(R^{1 \times t}, 0)) = \text{diag}(R^{1 \times t}, 0) \left( \begin{array}{c} Q_t \\ 0 \end{array} \right)$, where $Q_t \in R^{t \times n}$ has a right inverse. By $\varphi(\mathcal{E}_{11}) = R'E_{11}$, we can choose $Q_1 := (1, 0, \ldots, 0)$. By Theorem 3.12 and Remark 3.13, we can choose $q_{t+1} \in R^{1 \times n}$ such that $Q_t = \left( \begin{array}{c} Q_t \\ q_{t+1} \end{array} \right)$ for all $t = 1, \ldots, n - 1$.

Then $Q_n = \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{22} \end{array} \right) \in GL_n(R')$, and

$$\varphi \left( \begin{array}{c} R^{1 \times t} 0 \\ 0 0 \end{array} \right) = \left( \begin{array}{c} R^{1 \times t} 0 \\ 0 0 \end{array} \right) Q_n, \quad t = 1, \ldots, n. \quad (5.4)$$
By \( \varphi(\mathcal{N}_1) = \mathcal{N}_1' \), similarly, there exists a \( S_m = \begin{pmatrix} 1 & * \\ 0 & S_{22} \end{pmatrix} \in GL_m(R') \) such that

\[
\varphi \begin{pmatrix} R^s \times 1 \\ 0 & 0 \end{pmatrix} = S_m \begin{pmatrix} R^s \times 1 \\ 0 & 0 \end{pmatrix}, \quad s = 1, \ldots, m. \tag{5.5}
\]

Clearly, we have

\[
\varphi \begin{pmatrix} R^1 \times t \\ 0 & 0 \end{pmatrix} = S_m \begin{pmatrix} R^1 \times t \\ 0 & 0 \end{pmatrix} Q_n, \quad t = 1, \ldots, n, \tag{5.6}
\]

\[
\varphi \begin{pmatrix} R^s \times 1 \\ 0 & 0 \end{pmatrix} = S_m \begin{pmatrix} R^s \times 1 \\ 0 & 0 \end{pmatrix} Q_n, \quad s = 1, \ldots, m. \tag{5.7}
\]

Replacing \( \varphi \) by the transformation \( \varphi(X) \mapsto S_m^{-1} \varphi(X) Q_n^{-1} \), we have

\[
\varphi \begin{pmatrix} R^1 \times t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R^1 \times t \\ 0 & 0 \end{pmatrix}, \quad t = 1, \ldots, n, \tag{5.6}
\]

\[
\varphi \begin{pmatrix} R^s \times 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R^s \times 1 \\ 0 & 0 \end{pmatrix}, \quad s = 1, \ldots, m. \tag{5.7}
\]

Let \( R_k^s \times t = \{ X \in R^s \times t : \text{rank}(X) = k \} \), where \( k \leq \min(s, t) \). Similarly, define \( R_k^s \times t \). We are going to prove that

\[
\varphi \begin{pmatrix} R_1^s \times t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R_1^s \times t \\ 0 & 0 \end{pmatrix}, \quad s = 1, \ldots, m, \quad t = 1, \ldots, n. \tag{5.8}
\]

When \( s = 1 \) or \( t = 1 \), (5.8) is (5.6) or (5.7). When \( (s, t) = (m, n) \), (5.8) is clear. Without loss of generality we assume that \( 2 \leq s < m \leq n \) and \( 2 \leq t < n \). Let \( A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \in R^1 \times t \) such that \( A \notin \mathcal{M}_1 \) and \( A \notin \mathcal{N}_1 \). Let \( \varphi(A) = \begin{pmatrix} A_{11}^* & 0_1^* \\ A_{21}^* & 0_2^* \\ 0_3^* & 0_4^* \end{pmatrix} = \begin{pmatrix} A_{11}' & A_{12}' & 0_{1t}' \\ 0_{2t}' & 0_{3t}' & 0_{4t}' \end{pmatrix} \), where \( A_{11}^* \in R^1 \times t \), \( A_{21}^* \in R^{(s-1) \times t} \), \( A_{11}' \in R^{s \times 1} \), and \( A_{12}' \in R^{s \times (t-1)} \). There exists \( B_i = \begin{pmatrix} \beta_i & 0 \\ 0 & 0 \end{pmatrix} \) where \( \beta_i \in R^1 \times t \), \( i = 1, 2 \), such that \( B_1 \neq B_2 \) and \( A \sim B_i, i = 1, 2 \). By (5.6), \( \varphi(B_i) = \begin{pmatrix} \beta_i^* & 0 \\ 0 & 0 \end{pmatrix} \) where \( \beta_i^* \in R^1 \times t \), \( i = 1, 2 \). Since \( \varphi(A) \sim \varphi(B_i), \begin{pmatrix} A_{11}^* & -\beta_i^* \\ A_{21}^* & 0_2^* \\ 0_3^* & 0_4^* \end{pmatrix} \sim 0, i = 1, 2 \). Then Lemma 3.2 implies that either
\[ \varphi(A) = \begin{pmatrix} A^*_1 & 0^*_1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_4 \] or \[ \varphi(A) = \begin{pmatrix} 0 & A^*_1 \\ 0 & 0 \end{pmatrix} \). Since \( A \notin \mathcal{M}_1 \), (5.6) implies that \( \varphi(A) \notin \mathcal{M}_1 \). Thus we must have \( \varphi(A) = \begin{pmatrix} A^*_1 & 0^*_1 \\ 0 & 0 \end{pmatrix} \) or \( \varphi(A) = \begin{pmatrix} 0 & A^*_1 \\ 0 & 0 \end{pmatrix} \). Let \( \varphi(C_i) = \begin{pmatrix} \alpha_i & 0 \\ 0 & 0 \end{pmatrix} \) where \( \alpha_i \in \mathbb{R}^{s \times 1}, i = 1, 2 \), such that \( C_1 \neq C_2 \) and \( A \sim C_i, i = 1, 2 \). By (5.7), \( \varphi(C_i) = \begin{pmatrix} \alpha_i^* & 0 \\ 0 & 0 \end{pmatrix} \) where \( \alpha_i^* \in \mathbb{R}^{s \times 1}, i = 1, 2 \). Since \( \varphi(A) \sim \varphi(C_i) \), \( \begin{pmatrix} A^*_1 - \alpha^*_i A^*_1 \alpha^*_i & 0^*_1 \\ 0^*_2 & 0^*_3 \end{pmatrix} \sim 0 \).

\[ \begin{align*}
\text{Then we have proved that } \varphi &\left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \subseteq \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \text{. Considering } \varphi^{-1}, \text{ we have } \varphi \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \\
&\begin{pmatrix} R^{s \times t} & 0 \\ 0 & 0 \end{pmatrix}. \text{ Thus (5.8) holds.} \\
\text{Let } 1 \leq s < m \text{ and } 1 \leq t < n. \text{ Now we prove that} \\
\varphi \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad k = 1, 2, \ldots, \min(s, t). \quad (5.9)
\end{align*} \]

When \( s = 1 \) or \( t = 1 \), (5.9) holds by (5.6) and (5.7). Now we assume \( s, t \geq 2 \). We prove (5.9) by induction on \( k \). When \( k = 1 \), (5.9) holds by (5.8). Suppose that (5.9) holds for \( k - 1 \) (\( k \geq 2 \)). Let \( A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \in \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \) and \( \varphi(A) = \begin{pmatrix} 0^*_1 \\ 0^*_2 \end{pmatrix} \), where \( A_1 \in R^{s \times k} \) and \( A^*_1 \in R^{s \times t} \). There exists \( D_i = \begin{pmatrix} D_{ii} & 0 \\ 0 & 0 \end{pmatrix} \) where \( D_{ii} \in R^{s \times t}, i = 1, 2 \), such that \( ad(D_1, D_2) \geq 2 \) and \( A \sim D_i, i = 1, 2 \).

By the induction hypothesis, \( \varphi(D_i) = \begin{pmatrix} D_{ii}^* & 0 \\ 0 & 0 \end{pmatrix} \) where \( D_{ii}^* \in R^{s \times t}, i = 1, 2 \). Since \( \varphi(A) \sim \varphi(D_i) \), \( \begin{pmatrix} A^*_i - D_{ii}^* & 0^*_i \\ 0^*_2 & 0^*_3 \end{pmatrix} \sim 0, i = 1, 2 \). By Lemma 3.2, we have that either \( \varphi(A) = \begin{pmatrix} 0^*_1 \\ 0^*_2 \end{pmatrix} \) or \( \varphi(A) = \begin{pmatrix} 0^*_1 \\ 0^*_2 \end{pmatrix} \).

Assume \( \varphi(A) = \begin{pmatrix} 0^*_1 \\ 0^*_2 \end{pmatrix} \). Then \( \text{rank}(A^*_1 - D_{ii}^*, 0^*_1) = 1, i = 1, 2 \). Since \( ad(D_1, D_2) \geq 2 \), \( ad(D_{11}^*, D_{22}^*) \geq 2 \). Thus \( ad(A^*_1 - D_{11}^*, A^*_1 - D_{22}^*) \geq 2 \). It follows that there are two columns \( Y_1 \) and \( Y_2 \) of \( A^*_1 - D_{11}^* \) and \( A^*_1 - D_{22}^* \), respectively, such that \( Y_1 \) and \( Y_2 \) are right linearly independent. Then every
column of $0_1^*$ and $Y_1, Y_2$ are right linearly dependent, thus every column of $0_1^*$ must be 0, and hence $0_1^* = 0$. Then $\varphi(A) = \left( \begin{array}{c} A_1^* \ 0 \\ 0 \ 0 \end{array} \right) \in \left( \begin{array}{c} R_{k_1}^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right)$.

Assume $\varphi(A) = \left( \begin{array}{c} A_1^* \ 0 \\ 0 \ 0 \end{array} \right)$ Similarly, we can prove $\varphi(A) = \left( \begin{array}{c} A_1^* \ 0 \\ 0 \ 0 \end{array} \right) \in \left( \begin{array}{c} R_{k_1}^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right)$. Then we always have $\varphi(A) \in \left( \begin{array}{c} R_{k_1}^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right)$ . Thus $\varphi \left( \begin{array}{c} R_{k_1}^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right) \subseteq \left( \begin{array}{c} R_{k_1}^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right)$. Considering $\varphi^{-1}$, we get

$$\varphi \left( \begin{array}{c} R_{k_1}^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right) = \left( \begin{array}{c} R_{k_1}^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right),$$

Therefore, we have proved (5.9). It follows that

$$\varphi \left( \begin{array}{c} R_{k_1}^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right) = \left( \begin{array}{c} R_{k_1}^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right), \quad s = 1, \ldots, m - 1, \quad t = 1, \ldots, n - 1. \quad (5.10)$$

By (5.10), we can prove similarly that $\varphi(R^{m \times t}, 0) = \left( R^{m \times t}, 0 \right), \varphi \left( \begin{array}{c} R^{s \times n} \ 0 \\ 0 \ 0 \end{array} \right) = \left( \begin{array}{c} R^{s \times n} \ 0 \\ 0 \ 0 \end{array} \right)$, $t = 2, \ldots, n - 1, s = 2, \ldots, m - 1$. Thus

$$\varphi \left( \begin{array}{c} R^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right) = \left( \begin{array}{c} R^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right), \quad s = 1, \ldots, m, \quad t = 1, \ldots, n. \quad (5.11)$$

**Case 2.** $\varphi(\mathcal{M}_1)$ is a maximal set of the type two. Similar to the proof of the Case 1, we have that $(m, n) = (n', m')$ and (5.2). This proof ends. □

Now, we prove Theorem 1.1 as follows.

**Proof of Theorem 1.1.** Let $\varphi : R^{m \times n} \rightarrow R^{m' \times n'}$ be an a.p. bijective map in both directions. Further, assume that $R$ is a local ring, or $\varphi$ is an invertibility preserving map, or $\varphi$ is an additive map. Without loss of generality, we assume $2 \leq m \leq n$. Replacing $\varphi$ by the transformation $\varphi(X) \mapsto \varphi(X) - \varphi(0)$, we have $\varphi(0) = 0$.

When $m = n = m' = n' = 2$, the Theorem holds by Theorem 4.5. From now on we assume $(m, n) \neq (2, 2)$. By Corollary 3.4, $\varphi$ preserves the arithmetic distance. By Theorem 3.12, we have $(m, n) = (m', n')$ or $(m, n) = (n', m')$.

If $\varphi(\mathcal{M}_1)$ is a maximal set of the type one (resp. type two), then by Theorem 3.12, $(m, n) = (m', n')$ (resp. $(m, n) = (n', m')$). When $(m, n) = (m', n')$ with $m \neq n$, Theorem 3.12 implies that $\varphi(\mathcal{M}_1)$ must be a maximal set of the type one. Similarly, when $(m, n) = (n', m')$ with $m \neq n$, $\varphi(\mathcal{M}_1)$ must be a maximal set of the type two. When $m = n = m' = n'$, we have two cases: $\varphi(\mathcal{M}_1)$ is a maximal set of the type either one or two.

**Case 1.** $\varphi(\mathcal{M}_1)$ is a maximal set of the type one. Then $(m, n) = (n', m') \neq (2, 2)$ and $2 \leq m \leq n$. By Lemma 5.1, there exist two fixed invertible matrices $P_1 \in GL_m(R), Q_1 \in GL_n(R')$, such that

$$\varphi \left( \begin{array}{c} R^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right) = P_1 \left( \begin{array}{c} R^{s \times t} \ 0 \\ 0 \ 0 \end{array} \right) Q_1, \quad s = 1, \ldots, m, \quad t = 1, \ldots, n. \quad (5.12)$$
Replacing \( \varphi \) by the bijection \( \varphi(X) \mapsto P_1^{-1} \varphi(X)Q_1^{-1} \), we have

\[
\varphi \left( \begin{pmatrix} R^{s \times t} & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} R^{s \times t} & 0 \\ 0 & 0 \end{pmatrix}, \quad s = 1, \ldots, m, \ t = 1, \ldots, n.
\] (5.13)

**Subcase 1.1.** \( m = 2 < n \). By (5.13), we have

\[
\varphi(R^{2 \times 2}, 0) = (R^{2 \times 2}, 0).
\] (5.14)

By (5.14), we can assume that

\[
\varphi(X, 0) = (X^*, 0), \quad \forall X \in R^{2 \times 2}, \quad \text{where } X^* \in R^{2 \times 2}.
\] (5.15)

Then \( \varphi \) induces an a.p. bijective map in both directions \( \varphi': R^{2 \times 2} \to R^{2 \times 2} \) by \( \varphi'(X) = X^* \), where \( X^* \) is defined by (5.15). Clearly, \( \varphi'(0) = 0 \). If \( \varphi \) is an invertibility preserving map (or additive), then \( \varphi' \) is also an invertibility preserving map (or additive). By Theorem 4.5, we have either \( \varphi'(X) = P_2X^\sigma Q_2 \) \( \forall X \in R^{2 \times 2} \), or \( \varphi'(X) = P_2X^\tau Q_2 \forall X \in R^{2 \times 2} \), where \( P_2, Q_2 \in GL_2(R) \) are fixed, \( \sigma \) is an isomorphism from \( R \) to \( R' \), and \( \tau \) is an anti-isomorphism from \( R \) to \( R' \). Suppose that \( \varphi'(X) = P_2X^\tau Q_2 \) for all \( X \in R^{2 \times 2} \).

By (5.13), we have

\[
\varphi' \left( \begin{pmatrix} R^{1 \times 2} \\ 0 \end{pmatrix} \right) = \begin{pmatrix} R^{1 \times 2} \\ 0 \end{pmatrix} = P_2(R^{2 \times 1}, 0)Q_2 = (R^{2 \times 1}, 0)Q_2, \quad \text{a contradiction. Thus we must have } \varphi'(X) = P_2X^\sigma Q_2 \forall X \in R^{2 \times 2}.
\] (5.13)

and

\[
\varphi' \left( \begin{pmatrix} R^{2 \times 1} \\ 0 \end{pmatrix} \right) = \begin{pmatrix} R^{2 \times 1} \\ 0 \end{pmatrix}, \quad \text{thus } P_2 = \begin{pmatrix} p_{11} & p_{12} \\ 0 & p_{22} \end{pmatrix} \quad \text{and } Q_2 = \begin{pmatrix} q_{11} & 0 \\ q_{21} & q_{22} \end{pmatrix}. \quad \text{Then } \varphi(X, 0) = P_2(X, 0)^\sigma \text{ diag}(Q_2, I_{n-2}). \quad \text{Replacing } \varphi \text{ by the a.p. bijective map in both directions}
\]

\[
\varphi(X) \mapsto \left[ P_2^{-1} \varphi(X) \text{ diag}(Q_2^{-1}, I_{n-2}) \right]^{\sigma^{-1}},
\]

\( \varphi \) becomes an a.p. bijection in both directions from \( R^{2 \times n} \) to itself, and we have

\[
\varphi(X, 0) = (X, 0), \quad \forall X \in R^{2 \times 2}.
\] (5.16)

Moreover, (5.13) holds for \( R = R' \).

For \( 2 \leq k \leq n \), we are going to prove that

\[
\varphi(X_k, 0) = (X_k, 0)Q_k, \quad \forall X_k \in R^{2 \times k}.
\] (5.17)

where \( Q_k \in GL_n(R) \) is of the form \( Q_k = \sum_{t=1, t \neq k}^{n} E_{it} + d_k E_{kk} + \sum_{j=1}^{k-1} d_j E_{kj} \).

We prove (5.17) by induction on \( k \). When \( k = 2 \), (5.17) holds by (5.16). Suppose that (5.17) holds for \( k-1 \) (\( k \geq 3 \)). For an arbitrary but fixed \( A_1 \in R^{2 \times 1} \), let \( A = (A_1, 0_{2,n-1}) \) and let

\[
\psi(X) = \varphi(XT + A) - A, \quad \forall X \in R^{2 \times n},
\]

where \( T \) is an \( n \times n \) permutation matrix such that \( (X_{k-1}, 0_{2,n-k+1})^T = (0_{2,1}, X_{k-1}, 0_{2,n-k}) \) for all \( X_{k-1} \in R^{2 \times (k-1)} \). Recall \( \varphi(A) = A \). Then \( \psi : R^{2 \times n} \to R^{2 \times n} \) is an a.p. bijective map in both directions with \( \psi(0) = 0 \). If \( \varphi \) is an invertibility preserving map (or additive), then \( \psi \) is also an invertibility preserving map (or additive). Similar to the proof of the induction hypothesis, there are
\( P_A \in GL_2(R) \) and \( Q_A \in GL_n(R) \) such that \( \psi(X_{k-1}, 0) = P_A(X_{k-1}, 0)Q_A , \forall X_{k-1} \in R^{2 \times (k-1)} \), where \( \mu \) is an automorphism of \( R \). In other words, we have

\[
\varphi(A_1, X_{k-1}, 0, 0, n-\mu) = P_A(X_{k-1}, 0, 0, n-\mu+1)Q_A \quad \forall X_{k-1} \in R^{2 \times (k-1)}.
\] (5.18)

By the induction hypothesis, we have \( \varphi(A_1, X_{k-2}, 0, 2, n-k+1) = (A_1, X_{k-2}, 0, 2, n-k+1) \) for all \( X_{k-2} \in R^{2 \times (k-2)} \), thus (5.18) implies that

\[
P_A(X_{k-2}, 0, 2, n-k+2)Q_A = (0, 1, X_{k-2}, 0, 2, n-k+1), \quad \forall X_{k-2} \in R^{2 \times (k-2)}.
\] (5.19)

Let \( X \in R^{2 \times (k-2)} \), and let \( \beta_i \) be the \( i \)-th row of \( Q_A \), \( i = 1, \ldots, n \). By (5.19), we get that \( P_A \beta_i = (0, 1, X_{k-2}, 0, 2, n-i+1) \) for all \( i = 1, \ldots, k-2 \). By calculating, we have that

\[
P_A = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \quad \forall p \in R^*, \quad i = 1, \ldots, k-2, \text{ and } x^{(\mu)} = pxp^{-1} \text{ for all } x \in R. \]

Thus \( X_{k-1} = (x_1, \ldots, x_{k-1}) \) and \( p^{-1} \beta_k = (d_1, \ldots, d_n) \), where \( d_j \in R \) is a function of \( A_1, j = 1, \ldots, n \). Then (5.18) can be written as

\[
\varphi(A_1, X_{k-1}, 0) = \sum_{i=1}^{k-1} x_i p^{-1} \beta_i + A
\]

for all \( X_j \in R^{2 \times 1}, j = 1, \ldots, k - 1 \). Since \( (A_1, X_{k-1}, 0, 2, n-k) \in (R^{2 \times k}, 0) \), (5.13) implies that \( d_{k+1} = \cdots = d_n = 0. \) Thus

\[
\varphi(A_1, X_{k-1}, 0) = (A_1 + x_k d_1, x_1 + x_k d_2, \ldots, x_{k-2} + x_k d_{k-1}, x_{k-1} d_k, 0), \quad (5.20)
\]

for all \( X_j \in R^{2 \times 1}, j = 1, \ldots, k - 1 \).

Let \( A_1 = 0 \). Then we have similarly

\[
\varphi(0_{2,1}, X_{k-1}, 0) = (x_{k-1} d'_1, x_1 + x_k d'_2, \ldots, x_{k-2} + x_k d'_{k-1}, x_{k-1} d'_k, 0), \quad (5.21)
\]

where \( d'_j \in R, j = 1, \ldots, k \).

For any \( A_1 \neq 0 \), since \( (A_1, X_{k-1}, 0) \sim (0_{2,1}, X_{k-1}, 0, 0) \), (5.20) and (5.21) imply that

\[
(A_1 + x_{k-1}(d_1 - d'_1), x_{k-1}(d_2 - d'_2), \ldots, x_{k-2}(d_{k-1} - d'_{k-1}), x_{k-1}(d_k - d'_k), 0) \sim 0,
\]

for all \( x_{k-1} \in R^{2 \times 1} \). Thus \( d_i = d'_i, i = 2, \ldots, k \). In other words, all \( d_i = 2, \ldots, k \), do not depend on the choice of \( A_1 \in R^{2 \times 1} \). By (5.20) and (5.13), it is easy to see that \( d_k \in R^* \). Let \( Y = (0_{2,k-2}, A_1) \in R^{2 \times (k-1)} \). Then by (5.21) we have

\[
\varphi(0_{2,1}, X_{k-1} + Y, 0) = ((x_{k-1} + A_1) d'_1, x_1 + (x_{k-1} + A_1) d_2, \ldots, x_{k-2} + (x_{k-1} + A_1) d_{k-1}, (x_{k-1} + A_1) d_k, 0),
\] (5.22)

for all \( X_{k-1} \in R^{2 \times 1} \). Since \( (0_{2,1}, X_{k-1} + Y, 0) \sim (A_1, X_{k-1} + Y, 0) \), (5.20) and (5.22) imply that

\[
((x_{k-1} - d'_1) + A_1 (d'_1 - 1), A_1 d_2, \ldots, A_1 d_k, 0) \sim 0,
\]

for all \( x_{k-1} \in R^{2 \times 1} \). Then we must have \( d_1 = d'_1 \), i.e. \( d_1 \) does not depend on the choice of \( A_1 \in R^{2 \times 1} \). Thus (5.20) can be written as \( \varphi(A_1, X_{k-1}, 0) = (A_1, X_{k-1}, 0)Q_k, \forall (A_1, X_{k-1}) \in R^{2 \times k} \), where \( Q_k = \sum_{i=1, i \neq k}^{n} E_{ii} + d_k E_{kk} + \sum_{j=1}^{k-1} d_{j} E_{kj} \in GL_n(R) \) is fixed. In other words, we have

\[
\varphi(X, 0) = (X, 0)Q_k, \quad \forall X_k \in R^{2 \times k}.
\]

By the mathematical induction, we have proved (5.17). Taking \( k = n \) in (5.17), there exists a fixed \( Q' \in GL_n(R) \) such that

\[
\varphi(X) = XQ', \quad \forall X \in R^{2 \times n}.
\]
Recalling the transformations we have made, the original $\varphi$ must be of the form (1.2).

**Subcase 1.2.** $3 \leq m \leq n$. By (5.13), we have $\varphi \begin{pmatrix} R^{2 \times n} \\ 0 \end{pmatrix} = \begin{pmatrix} R^{2 \times n} \\ 0 \end{pmatrix}$. Let $\varphi \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X^* \\ 0 \end{pmatrix}$ for all $X \in R^{2 \times n}$, where $X^* \in R^{2 \times n}$. Then $\varphi$ induces an a.p. bijective map in both directions $\varphi_1: R^{2 \times n} \to R^{2 \times n}$ by $\varphi_1(X) = X^*$. If $\varphi$ is an invertibility preserving map (or additive), then it is easy to prove that $\varphi_1$ is also an invertibility preserving map (or additive). We have $\varphi_1(0) = 0$. By Case 1, $\varphi_1(X) = P_2X^T Q_2$ for all $X \in R^{2 \times n}$, where $P_2 \in GL_2(R')$, $Q_2 \in GL_n(R')$ are fixed and $\sigma$ is an isomorphism from $R$ to $R'$.

Then $\varphi \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} P_2 \\ I_{n-2} \end{pmatrix} \begin{pmatrix} X^\sigma \\ 0 \end{pmatrix} Q_2$, $\forall X \in R^{2 \times n}$. Let $P_3 = \text{diag}(P_2, I_{n-2})$. Replacing $\varphi$ by the a.p. bijective map $\varphi(X) \mapsto [P_3^{-1} \varphi(X) Q_2^{-1}]^{\sigma^{-1}}$, $\varphi$ becomes an a.p. bijection in both directions from $R^{m \times n}$ to itself, and we obtain

$$\varphi \begin{pmatrix} X_2 \\ 0 \end{pmatrix} = \begin{pmatrix} X_2 \\ 0 \end{pmatrix}, \forall X_2 \in R^{2 \times n}. \quad (5.23)$$

For $2 \leq k \leq n$, we are going to prove that

$$\varphi \begin{pmatrix} X_k \\ 0 \end{pmatrix} = P_k \begin{pmatrix} X_k \\ 0 \end{pmatrix}, \forall X_k \in R^{k \times n}, \quad (5.24)$$

where $P_k = \sum_{i=1}^n E_{ii} + d_k E_{kk} + \sum_{j=1}^{k-1} d_{j} E_{jk} \in GL_m(R)$ is fixed.

We prove (5.24) by induction on $k$. When $k = 2$, (5.24) is (5.23). Suppose that (5.24) holds for $k - 1$ ($k \geq 3$). For an arbitrary but fixed $A_1 \in R^{1 \times n}$, let $A = \begin{pmatrix} A_1 \\ 0 \end{pmatrix}$ and let $\psi(X) = \varphi(TX + A) - A, \forall X \in R^{m \times n}$, where $T$ is a permutation matrix such that $T \begin{pmatrix} X_k-1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0_{1,n} \\ X_{k-1} \\ 0 \end{pmatrix}$ for all $X_{k-1} \in R^{(k-1) \times n}$.

Recall $\varphi(A) = A$. Then $\psi: R^{m \times n} \to R^{m \times n}$ is an a.p. bijective map in both directions with $\psi(0) = 0$. If $\varphi$ is an invertibility preserving map (or additive), then $\psi$ is also an invertibility preserving map (or additive). Similar to the proof of the induction hypothesis, there are $P_A \in GL_m(R)$ and $Q_A \in GL_n(R)$ such that $\psi \begin{pmatrix} X_{k-1} \\ 0 \end{pmatrix} = P_A \begin{pmatrix} X_{k-1}^\mu \\ 0 \end{pmatrix} Q_A, \forall X_{k-1} \in R^{(k-1) \times n}$, where $\mu$ is an automorphism of $R$. In other words, we have

$$\varphi \begin{pmatrix} A_1 \\ X_{k-1} \end{pmatrix} = P_A \begin{pmatrix} X_{k-1}^\mu \\ 0_{1,n} \end{pmatrix} Q_A + A, \forall X_{k-1} \in R^{(k-1) \times n}. \quad (5.25)$$

By the induction hypothesis on $\varphi$, we have $\varphi \begin{pmatrix} A_1 \\ X_{k-2} \end{pmatrix} = \begin{pmatrix} A_1 \\ X_{k-2} \end{pmatrix}$, $\forall X_{k-2} \in R^{(k-2) \times n}$. Thus by (5.25) we get that
Let $x_i$ be the $i$th row of $X_{k-2}$, and let $\alpha_j$ be the $j$th column of $P_A$. By (5.26), we have $\sum_{i=1}^{k-2} \alpha_i x_i^\mu = QA = \begin{pmatrix} 0_1, n \\ X_{k-2} \\ 0 \end{pmatrix}$, and hence $\alpha_i x_i^\mu = QA = \begin{pmatrix} 0_1, n \\ X_{k-2} \\ 0 \end{pmatrix}$, $\forall x_i \in R^1 \times n$, $i = 1, \ldots, k$. By computing, we have $QA = p^{-1} I_n$, $\alpha_i = p E_{i+1,1}^m$, where $p \in R$, $i = 1, \ldots, k-1$, and $x^\mu = p^{-1} x p$ for all $x \in R$. Let $X_{k-1} = \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \end{pmatrix}$, where $x_{k-1} \in R^1 \times n$, and let $\alpha_{k-1} p^{-1} = t(d_1, \ldots, d_m)$, where $d_i \in R$ is a function of $A_1$, $i = 1, \ldots, m$. Then (5.25) can be written as

$$
\varphi \begin{pmatrix} A_1 \\ X_{k-1} \\ 0 \end{pmatrix} = \sum_{i=1}^{k-1} \alpha_i p^{-1} x_i + A = \begin{pmatrix} A_1 + d_1 x_{k-1} \\ x_1 + d_2 x_{k-1} \\ \vdots \\ x_{k-2} + d_{k-1} x_{k-1} \\ d_k x_{k-1} \\ \vdots \\ d_m x_{k-1} \end{pmatrix}, \quad \forall x_i \in R^1 \times n.
$$

(5.27)

By (5.13) we have $d_{k+1} = \cdots = d_m = 0$. Thus

$$
\varphi \begin{pmatrix} A_1 \\ X_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 + d_1 x_{k-1} \\ x_1 + d_2 x_{k-1} \\ \vdots \\ x_{k-2} + d_{k-1} x_{k-1} \\ d_k x_{k-1} \\ 0 \end{pmatrix}, \quad \forall x_i \in R^1 \times n, \ i = 1, \ldots, k-1.
$$

(5.27)

Let $A_1 = 0$. Similarly, there exist $d^\prime_i \in R, i = 1, \ldots, k$, such that

$$
\varphi \begin{pmatrix} 0_1, n \\ X_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} d^\prime_1 x_{k-1} \\ x_1 + d^\prime_2 x_{k-1} \\ \vdots \\ x_{k-2} + d^\prime_{k-1} x_{k-1} \\ d^\prime_k x_{k-1} \\ 0 \end{pmatrix}, \quad \forall x_i \in R^1 \times n, \ i = 1, \ldots, k-1.
$$

(5.28)
For any $A_1 \neq 0$, \[
\begin{pmatrix}
A_1 \\
X_{k-1} \\
0
\end{pmatrix} \sim \begin{pmatrix}
0_{1,n} \\
X_{k-1} \\
0
\end{pmatrix},
\] thus by (5.27) and (5.28), we have
\[
\begin{pmatrix}
A_1 + (d_1 - d'_1)x_{k-1} \\
(d_2 - d'_2)x_{k-1} \\
\vdots \\
(d_{k-1} - d'_{k-1})x_{k-1} \\
(d_k - d'_k)x_{k-1} \\
0_m - 0_m
\end{pmatrix} \sim 0, \quad \forall x_{k-1} \in R^{1 \times n}.
\]
Thus we must have $d_i = d'_i$, $i = 2, \ldots, k$. In other words, all $d_i$, $i = 2, \ldots, k$, do not depend on the choice of $A_1 \in R^{1 \times n}$. By (5.27) and (5.13), it is easy to see that $d_k \in R^*$. Let $Y = \begin{pmatrix}
0_{k-2,n} \\
A_1
\end{pmatrix} \in R^{(k-1) \times n}$. Then by (5.28) we get
\[
\varphi \left( X_{k-1} + Y \right) = \begin{pmatrix}
d'_1(x_{k-1} + A_1) \\
x_1 + d_2(x_{k-1} + A_1) \\
\vdots \\
x_{k-2} + d_{k-1}(x_{k-1} + A_1) \\
d_k(x_{k-1} + A_1) \\
0_m - 0_m
\end{pmatrix}, \quad \forall x_i \in R^{1 \times n}, \ 1 \leq i \leq k - 1. \tag{5.29}
\]
\[
\begin{pmatrix}
0 \\
x_{k-1} + Y \\
0
\end{pmatrix} \sim \begin{pmatrix}
A_1 \\
x_{k-1} \\
0
\end{pmatrix}, \text{ by (5.27) and (5.29)}, \quad \begin{pmatrix}
(d'_1 - d_1)x_{k-1} + (d'_1 - 1)A_1 \\
d_2A_1 \\
\vdots \\
d_kA_1 \\
0_m - 0_m
\end{pmatrix} \sim 0 \text{ for all } x_{k-1} \in R^{1 \times n},
\]
which implies $d_1 = d'_1$, i.e. $d_1$ does not depend on the choice of $A_1 \in R^{1 \times n}$. Thus (5.27) can be written as
\[
\varphi \left( X_{k-1} \right) = P_k \begin{pmatrix}
A_1 \\
x_{k-1} \\
0
\end{pmatrix}, \quad \forall x_{k-1} \in R^{(k-1) \times n}, \forall A_1 \in R^{1 \times n}, \tag{5.30}
\]
where $P_k = \sum_{i=1, i \neq k}^n E_{ii} + d_kE_{kk} + \sum_{j=1}^{k-1} d_jE_{jk} \in GL_m(R)$ is fixed. Therefore (5.24) holds.
Replacing $\varphi$ by the transformation $\varphi(X) \mapsto P_k^{-1}\varphi(X)$, we have
\[
\varphi \left( X_k \right) = \begin{pmatrix}
X_k \\
0
\end{pmatrix}, \quad \forall X_k \in R^{k \times n}.
\]
Moreover, (5.13) still holds for $1 \leq s \leq m$ and $t = n$. By the mathematical induction and taking $k = n$ in (5.24), there exists a fixed $P' \in GL_m(R)$ such that

$$\varphi(X) = P'X, \quad \forall X \in R^{m \times n}.$$ 

Recalling the transformations we have made, the original $\varphi$ is of the form (1.2).

**Case 2.** $\varphi(M_1)$ is a maximal set of the type two. Then $(m, n) = (n', m') \neq (2, 2)$ and $2 \leq m \leq n$. Similarly, we can prove that $\varphi$ is of the form (1.3).

The converse part of Theorem 1.1 is trivial. The proof of Theorem 1.1 ends. □

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**References**