Pseudodifferential Operators and Hecke Operators

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Abstract—It has been known how to construct pseudodifferential operators from modular forms and Jacobi-like forms. In this paper, we construct Hecke operators on the pseudodifferential operators that are compatible with the usual Hecke operators on modular forms and Jacobi-like forms. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The pseudodifferential operator is the basic object of the modern theory of linear, partial differential equations. It has found many applications throughout pure mathematics including index theory, K-theory, and Hodge theory and provided an extremely powerful technique in applied mathematics for solving variable coefficient PDEs. Recently, Zagier studied the pseudodifferential operator through Rankin-Cohen brackets, which are bilinear operators defined over the spaces of modular forms and Jacobi-like forms. Further properties of pseudodifferential operators constructed from modular forms as well as Jacobi-like forms have been explored and pseudodifferential operators with some kind of automorphic behavior have been studied in detail [1]. On the other hand, it has been well known that Hecke operators play an essential role in the theory of modular forms. Hecke operators are endomorphisms on the vector spaces of modular forms. They are used as a tool to understand arithmetic properties of modular forms in terms of Fourier coefficients and give analytic properties of modular forms through L-series. Moreover, certain sets of Hecke operators possess ring or algebra structure called a Hecke ring or a Hecke algebra. Therefore, various forms of Hecke algebras have been studied extensively over the years in connection with several branches of mathematics such as the theory of finite group representations, the theory of knots, quantum groups, etc.

In this paper, we construct Hecke operators on the space of pseudodifferential operators, which are compatible with the usual Hecke operators on modular forms and Jacobi-like forms. In particular, we study Hecke operators that are compatible with the isomorphism between two exact sequences involving the space of $\Gamma$-invariant pseudodifferential operators and the space of Jacobi-like forms. In [2], a Hecke operator on differential equations has been defined and its

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properties have been studied. We present a theory of Hecke operators on differential equations analogous to those on pseudodifferential operators.

2. DEFINITIONS

First, we recall the definition of Hecke operators acting on the space of $\Gamma$-invariant functions and Jacobi-like forms. We follow definitions and notations given in [3,4].

Let $GL^+(2, \mathbb{R})$ (respectively, $SL(2, \mathbb{R})$) be the multiplicative group of $2 \times 2$ real matrices with positive determinant (respectively, determinant 1). Then, $GL^+(2, \mathbb{R})$ acts on the Poincaré upper half-plane

$$\mathcal{H} = \{ z \in \mathbb{C} | \text{Im } z > 0 \}$$

as linear fractional transformations. If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbb{R})$ and if $f : \mathcal{H} \to \mathbb{C}$ is a function, define a slash operator as

$$(f|kM)(z) = \det(M)^{k/2}(cz+d)^{-k}f(Mz), \quad (2.1)$$

$k \in \mathbb{Z}$.

DEFINITION 2.1. Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group of the first kind, so that the quotient space $\Gamma \backslash \mathcal{H} \cup \{ \text{cusps} \}$ is a compact Riemann surface, and let $k$ be a nonnegative integer. Let $R$ be a $\Gamma$-invariant ring of functions in $\mathcal{H}$.

1. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight $k$ for $\Gamma$ if it satisfies

$$(f|kM)(z) = f(z),$$

for all $M \in \Gamma$ and $\tau \in \mathcal{H}$ and is meromorphic at the cusps. We shall denote by $M_k(\Gamma)$ the space of all modular forms of weight $k$ for $\Gamma$.

2. An element $\Phi$ in the vector space

$$\mathcal{J}(\Gamma) = \left\{ \Phi(z, X) = \sum_{n=1}^{\infty} \phi_n(z) X^n \in R[[X]] \right\}$$

$$\Phi \left( \frac{az+b}{cz+d}, \frac{X}{(cz+d)^2} \right) = e^{cz/(cz+d)} \Phi(z, X), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

is called a Jacobi-like form. Here, $R$ is a $\Gamma$-invariant ring of functions in $\mathcal{H}$.

We now recall definitions of Hecke operators on the above vector spaces. If $\Gamma_1$ and $\Gamma_2$ are subgroups of $GL^+(2, \mathbb{R})$, then we say that $\Gamma_1$ and $\Gamma_2$ are commensurable and write $\Gamma_1 \sim \Gamma_2$ if $\Gamma_1 \cap \Gamma_2$ has finite index in both $\Gamma_1$ and $\Gamma_2$.

DEFINITION 2.2. Let $\Gamma$ be the Fuchsian group of the first kind as above, and set

$$\tilde{\Gamma} = \{ M \in GL^+(2, \mathbb{R}) | M\Gamma M^{-1} \sim \Gamma \}.$$ 

If $M \in \tilde{\Gamma}$, then the double coset $\Gamma M \Gamma$ has disjoint coset decomposition of the form $\Gamma M \Gamma = \bigcup_{\nu=1}^{s} \Gamma M_{\nu}$ for some $M_{\nu} \in GL^+(2, \mathbb{R})$. We will denote it by $\Gamma M \Gamma = \prod_{\nu=1}^{s} \Gamma M_{\nu}$. Then, we have the following.

1. The Hecke operator on $M_k(\Gamma)$ associated to $M \in \tilde{\Gamma}$ is a map $T(M) : M_k(\Gamma) \to M_k(\Gamma)$ defined by

$$T(M)f = \det(M)^{k/2-1} \sum_{\nu=0}^{s} (f|kM_{\nu}), \quad (2.2)$$

for all $f \in M_k(\Gamma)$.
2. The Hecke operator on $J(\Gamma)$ associated to $M \in \Gamma$ is a map $V(M) : J(\Gamma) \to J(\Gamma)$ defined by

$$V(M)\Phi = \sum_{\nu=0}^3 e^{-cz/(cz+dz)}\Phi \left( M_{\nu}z, \frac{(\det M)X}{(cz+dz)^2} \right), \tag{2.3}$$

for all $\Phi \in J(\Gamma)$.

REMARK 2.3. The Hecke operator $V(M)$ on $J(\Gamma)$ is a Hecke operator on the space of Jacobi forms (see [5]). For $\Phi(z, X) \in J(\Gamma)$, we note the function $\Phi(z, X^2)$, $\Phi(z, X) \in J(\Gamma)$ satisfies one of the functional equation for Jacobi forms of weight 0 and index 1.

3. MODULAR FORMS AND PSEUDODIFFERENTIAL OPERATORS

We start from a formal definition of pseudodifferential operators given in [1,4].

Let $\partial = \frac{d}{dt}$ be the associated differential operator which transforms under a coordinate change $\tau \to \tau' \equiv \partial \equiv (\frac{dt}{d\tau})^{-1} \partial$. Let $R$ be a ring of functions on $\mathbb{C}$ on which $\partial$ acts, so that the pair $(R, \partial)$ is a ring with derivation. By a pseudodifferential operator ($\Psi DO(R)$) over $R$, we mean a formal Laurent series in the formal inverse $\partial^{-1}$ of $\partial$ with coefficients in $R$, i.e., an element of the vector space

$$\psi DO(R) = \left\{ \sum_{n \in \mathbb{Z}} h_n \partial^n : h_n \in R, \ h_n = 0, \text{ if } n \gg 0 \right\}.$$

The multiplication of pseudodifferential operators is implied by Leibniz rule

$$\left( \sum_m f_m(\tau) \partial^m \right) \left( \sum_n g_n(\tau) \partial^n \right) = \sum_{m,n} \sum_{r \geq 0} \binom{m}{r} f_m(\tau) g_n^r(\tau) \partial^{n+m-r},$$

and $\psi DO(R)$ in this way forms an associative ring. Let

$$\psi DO(R)_{w} = \left\{ \sum_{n=0}^{\infty} f_n \partial^{w-n} : f_n \in R \right\}$$

be a subspace of $\psi DO(R)$, for each $w \in \mathbb{Z}$. Then the following is a short exact sequence:

$$0 \to \psi DO_{w-1}(R) \to \psi DO_w(R) \to R \to 0,$$

for every $w \in \mathbb{Z}$, where the final map sends $\sum_{m \geq 0} f_m \partial^{w-m}$ to $f_0$.

If the coordinate change is a fractional linear transformation $\tilde{\tau} = M(z) = (az + b/cz + d)$ with $M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{R})$, then

$$\tilde{\partial}^w = [(cz + d)^2 \partial]^w = \sum_{n=0}^{\infty} n! \left( \begin{array}{c} w \\ n \end{array} \right) \left( w - 1 \right) \left( c^2 + d^2 \right)^{2w-n} \partial^{w-n}.$$
PROPOSITION 3.1. (See [1].) For \( k \in \mathbb{N} \), define an operator \( \mathcal{L}_k : R \to \Psi DO(R)_{-k} \) by
\[
\mathcal{L}_k(f) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+k)!((n+k-1)!)f(n)\partial^{-k-n}}{n!(n+2k-1)!}.
\]
Then, \( \mathcal{L}_k(f)|_{2kM} = \mathcal{L}_k(f)|_0 M, \) for all \( M \in SL(2, \mathbb{R}) \). In particular, if \( f \in \text{M}_2k(\Gamma) \), then \( \mathcal{L}_k(f) \in \Psi DO(R)_{k} \).

According to Proposition 3.2 given in [1], the sequence
\[
0 \to \mathcal{J}(\Gamma)_{k+1} \to \mathcal{J}(\Gamma)_k \to \text{M}_{2k}(\Gamma) \to 0
\]
(3.2)
splits and is canonically isomorphic to the split short exact sequence in (3.1), for \( k \in \mathbb{N} \). Here \( \mathcal{J}(\Gamma)_k = \mathcal{J} \cap X^k R[[X]] \).

The following proposition states explicit relation between space elliptic modular forms and that of Jacobi-like forms.

PROPOSITION 3.2. (See [1].) Let \( \phi_k \) be an element of \( R \) for each \( k \in \mathbb{N} \). Then the following are equivalent:
1. \( \Phi(z, X) := \sum_{k=1}^{\infty} \phi_k(z)X^k \in \mathcal{J}(\Gamma) \),
2. \( \psi(z) := \sum_{k=1}^{\infty} (-1)^k k!(k-1)!\phi_k(z)\partial^{-k} \in \Psi DO(R)_{\Gamma} \),
3. \( \phi_k|_{2k} \gamma(z) = \sum_{n=1}^{k} 1/(c/x + d)^n \phi_{k-n}(z), \) for all \( k \geq 1 \) and \( \gamma = (c^* \ d^*) \in \Gamma \),
4. \( \sum_{r=0}^{k-1} (-1)^r ((2k - 2 - r)!/r!\phi_k^{(r)}(z) \in \text{M}_{2k}(\Gamma), \) for all \( k \geq 1 \),
5. \( \phi_n(z) = \sum_{r=0}^{n-1} (1/r!(2n - r - 1)!\psi_{k-n}^{(r)}(z), \) where \( f_k \in \text{M}_{2k}(\Gamma) (k \geq 1) \).

4. HECKE OPERATORS, PSEUDODIFFERENTIAL OPERATORS, AND JACOBI-LIKE FORMS

We define a Hecke operator on the pseudodifferential operators as the following way.

DEFINITION 4.1. Let \( M \in \tilde{\Gamma} \) and let \( T(M) \) be the Hecke operator on the space \( \text{M}_{2k}(\Gamma) \) of modular forms of weight \( 2k \) on \( \Gamma \) described in (2.2). A Hecke operator \( T_\psi(M) \) on the space \( \Psi DO(R)_{\Gamma}^{\Lambda} \), \( k \in \mathbb{N} \), is a map
\[
T_\psi(M) : \Psi DO(R)_{\Gamma}^{\Lambda} \to \Psi DO(R)_{\Gamma}^{\Lambda}
\]
such that for each \( \psi \in \Psi DO(R)_{\Gamma}^{\Lambda} \) the modular form \( f_{T_\psi(M)\psi} \) corresponding to \( T_\psi(M)\psi \) is equal to the image \( T(M)f_\psi \) of modular form corresponding to \( \psi \); namely, \( T(M)f_\psi = f_{T_\psi(M)\psi} \) where \( f_\psi \) is a modular form corresponding to \( \psi \in \Psi DO(R)_{\Gamma}^{\Lambda} \).

Now, we have the following result.

THEOREM 4.2.
1. A Hecke operator \( T_\psi(M) \) given in the definition on the space \( \Psi DO(R)_{\Gamma}^{\Lambda} \) is an endomorphism.
2. Let \( M \in \tilde{\Gamma} \) with the decomposition of double coset
\[
\tilde{\Gamma} M \tilde{\Gamma} = \prod_{\nu=1}^{s} \tilde{\Gamma} M_\nu
\]
for some \( M_\nu \in GL^+(2, \mathbb{R}) \). For each \( \psi(\tau) = \sum_{n=0}^{\infty} g_n(\tau)\partial^{-n-k} \in \Psi DO(R)_{\Gamma}^{\Lambda}, \) \( g_k \neq 0, \) \( k \in \mathbb{N} \), the explicit formula of \( T_\psi(M) \) is given by
\[
T_\psi(M)(\psi) = (\det M)^{k-1} \sum_{n=0}^{\infty} (n+k)!(n+k-1)! \sum_{\nu=0}^{s} \sum_{\ell=0}^{n} \frac{(c_\nu)^{n-\ell}g_{2k+2\ell M_\nu}}{(n-\ell)!(\ell+k)!(\ell+k-1)!}(c_\nu \tau + d_\nu)^{-n-\ell}\partial^{-n-k}.
\]

Here, \( M_\nu = \left( \begin{smallmatrix} c_\nu & * \\ * & d_\nu \end{smallmatrix} \right) \).

To prove this, we need the following lemma.
Lemma 4.3. Let \( f(\tau) \in M_{2k}(\Gamma) \) and \( M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \bar{\Gamma} \) with \( \Gamma M \Gamma = \prod_{\nu=1}^{s} \Gamma M_{\nu} \).

1. \( D^{n}(f|_{2k} M) = n!(n + 2k - 1)! \sum_{\ell=0}^{n} \frac{(-c)^{n-\ell} D^{\ell}(f)|_{2k+2\ell} M}{\ell!(n-\ell)!(\ell + 2k - 1)!(cr + d)^{n-\ell}}, \)

where \( D(f) = \frac{df}{d\tau} \).

2. Let \( T(M) \) be the Hecke operator on \( M_{2k}(\Gamma) \) given in (2.2). Then, for \( f \in M_{2k}(\Gamma) \),

\[
L_{k}(T(M)f) = (\det M)^{k-1} \sum_{n=0}^{\infty} \sum_{\nu=0}^{s} (-1)^{n} (n+k)!(n+k-1)! \sum_{\ell=0}^{n} \frac{(-c_{\nu})^{n-\ell} f|_{2k+2\ell} M_{\nu}}{\ell!(n-\ell)!(\ell + 2k - 1)!(c_{\nu} \tau + d_{\nu})^{n-\ell}} \theta^{-n-k}.
\]

Proof. These can be proved by induction on \( n \).

We now prove the theorem.

Proof of Theorem.

1. Since \( \Gamma \)-invariant pseudodifferential operator can be expanded as a sum of lifts from modular form \( f \) from (3.1) and (3.2), the operator \( T_{\psi}(M) \) on \( \Psi DO(R)^{\Gamma_{k}} \) can be determined by \( T_{\psi}(M)(L_{k}(f)) \). So, if we let \( T_{\psi}(M)(L_{k}(f)) = L_{k}(T(M)f) \), then the operator \( T_{\psi}(M) \) is a well-defined endomorphism on \( \Psi DO(R)^{\Gamma_{k}} \) such that, for each \( \psi \in \Psi DO(R)^{\Gamma_{k}} \), the modular form \( f_{T_{\psi}(M)\psi} \) corresponding to \( T_{\psi}(M)\psi \) is equal to the image \( T(M)f_{\psi} \) of modular form corresponding to \( \psi \) under \( T_{\psi}(M) \).

2. For each \( M \in \bar{\Gamma} \) and \( f \in M_{2k}(\Gamma) \), we let

\[
g_{n}(\tau) = (-1)^{n} \frac{(n+k)!(n+k-1)!}{n!(n+2k-1)!} f^{(n)}(\tau).
\]

Then, we have \( \psi(\tau) = \sum_{n=0}^{\infty} g_{n}(\tau) \theta^{-n-k} = L_{k}(f) \). To check the explicit formula in the theorem, we need to compute \( L_{k}(T(M)f) \). This follows from Lemma 4.3:

\[
(T_{\psi}(M)\psi)(z) = L_{k}(T(M)f)
\]

\[
= (\det M)^{k-1} \sum_{n=0}^{\infty} \sum_{\nu=0}^{s} (-1)^{n}(n+k)!(n+k-1)! \sum_{\ell=0}^{n} \frac{(-c_{\nu})^{n-\ell} f|_{2k+2\ell} M_{\nu}}{\ell!(n-\ell)!(\ell + 2k - 1)!(c_{\nu} \tau + d_{\nu})^{n-\ell}} \theta^{-n-k}
\]

Since \( J(\Gamma)_{k} \) is isomorphic to \( \Psi DO(R)^{\Gamma_{k}} \) by (3.1) and (3.2), we state the compatibility of Hecke operators \( T_{\psi}(M) \), \( V(M) \), and \( T(M) \).

Proposition 4.4. Let \( V(M) \), \( T(M) \), \( T_{\psi}(M) \), be Hecke operators on the spaces \( J(\Gamma) \), \( M_{k}(\Gamma) \), and \( \Psi DO(R)^{\Gamma_{k}} \), respectively. These Hecke operators are compatible with respect to the isomorphism between two exact sequences (3.1),(3.2). In other words,

1. let \( \Phi(z, X) = \sum_{n=1}^{\infty} \phi_{n}(z) X^{k} \in J(\Gamma) \) and let \( (V(M)\Phi)(z, X) = \sum_{n=1}^{\infty} \tilde{\phi}_{n}(z) X^{k} \). Then,

\[
\sum_{n=1}^{\infty} (-1)^{n} n!(n-1)! \phi_{n}(z) \theta^{-n} = (T_{\psi}(M)\psi)(z) \in \Psi DO(R)^{\Gamma},
\]

where \( \psi(z) = \sum_{n=1}^{\infty} (-1)^{n} n!(n-1)! \phi_{n}(z) \theta^{-n} ; \)
\[ \tilde{\phi}_n(z) = \sum_{r=0}^{n-1} \frac{1}{r!(2n-r-1)!} D^r(T(M)f_{n-r}), \]

where \( f_k \in M_{2k}(\Gamma) \).

**Proof of Proposition.** Proposition 3.2 implies that \( \psi(z) = \sum_{n=1}^{\infty} (-1)^n n!(n-1)! \phi_n(z) \partial^{-n} \in \Psi DO(R)^\Gamma \).

So \( \sum_{n=1}^{\infty} (-1)^n n!(n-1)! \tilde{\phi}_n(z) \partial^{-n} \in \Psi DO(R)^\Gamma \). Since

\[ \tilde{\phi}_n(z) = \sum_{\nu=1}^{g} \sum_{j=0}^{n} \frac{(-c_\nu)^{n-j} \partial^{j-1} M_\nu}{(n-j-1)! (c_\nu z + d_\nu)^{n-j+1}} X^n, \]

one can check the equality from the explicit formula \( T_\psi(M) \) given in Theorem 4.2. Therefore, the results follows.

**References**