On a Jacobi–Trudi Identity
for Supersymmetric Polynomials

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INTRODUCTION

In the very late eighties a new identity for symmetric polynomials was
discovered. In the form presented here the identity is a generalization of the
Jacobi–Trudi identity. The latter identity expresses the Schur polynomials
in a finite set of variables as a certain symmetrizing operator applied to
monomials in the variables. The new identity involves two sets of variables.
It expresses the super Schur polynomials as a certain symmetrizing
operator applied to very simple polynomials in the two sets of variables.

It is a classical result that the Schur polynomials are the characters of the
polynomial representations of \( SL_m \). Hence the Jacobi–Trudi identity may
be viewed as a character formula for \( SL_m \). This approach was generalized
to other algebraic groups by H. Weyl in his character formula.

The new identity was in fact discovered as a Weyl-type formula for the
characters of polynomial representations of the Lie superalgebra \( sl(m/n) \).
From one side the formula was conjectured by J. van der Jeugt, J. W. B.
side the identity was communicated without proof by A. Serge'ev to the
first author who gave a proof of its validity in [P]. The proof, though
elementary, rested on the characterization of J. Stembridge of super Schur
polynomials via a certain cancellation property, therefore, the proof in [P]
was not self-contained.

The aim of the present note is to give a self-contained and elementary
proof of the new identity. The method used gives a simple insight in the
space of supersymmetric polynomials. Byproducts of the proof are the
above mentioned characterization of super Schur polynomials given by J. Stembridge [S], the basis theorem and the factorization formula for super Schur polynomials given by A. Berele and A. Regev [B-R], and the duality formula.

1. Preliminaries

Setup (1.1). We use throughout for partitions the notation described in Macdonald's book [M]. In particular, a partition

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

is assumed to be decreasing,

$$\lambda_1 \geq \lambda_2 \geq \cdots,$$

and the Ferrers' diagram of $\lambda$ is the set of points $(i, j) \in \mathbb{N}^2$ such that $1 \leq j \leq \lambda_i$. The diagram of $\lambda$ will be denoted $D_\lambda$. The diagram will always be pictured in a system of matrix coordinates where the first index $i$ is a row index and the second $j$ is a column index.

Definition (1.2). Let $m, n$ be non-negative integers. A partition $\lambda$ will be said to be contained in the $(m, n)$-hook, if $\lambda_{m+1} \leq n$, or, equivalently, if the diagram $D_\lambda$ is contained in the "hook,"

$$\{(i, j) \in \mathbb{N}^2 \mid i \leq m \text{ or } j \leq n\}.$$

Denote by $D_{m,n}^\lambda$ the following subset of $D_\lambda$:

$$D_{m,n}^\lambda := \{(i, j) \mid i \leq m, j \leq n, j \leq \lambda_i\}.$$

Assume that $\lambda$ is contained in the $(m, n)$-hook. Then the part of $D_\lambda$ outside $D_{m,n}^\lambda$ consists of two parts which—up to a translation—are diagrams of partitions: The first part consists of the points to the right of the line $j = n$, that is, of points $(i, j) \in D_\lambda$ such that $j > n$, and the second part consists of the points below the line $i = m$, that is, of points $(i, j) \in D_\lambda$ such that $i > m$. Up to a translation of the $i$-axis, the first part is the diagram of a partition $\mu$; more precisely, the non-zero parts of $\mu$ are defined by $\mu_i = \lambda_i - n$ for $\lambda_i > n$. Similarly, the second part is up to a translation the transpose of the diagram of a partition $\nu$, whose non-zero parts are defined by $\nu_j = \lambda'_j - m$ for $\lambda'_j > m$. Here $\lambda'$ denotes the conjugate of $\lambda$, defined by transposing the diagram of $\lambda$, or, equivalently, by $\lambda'_j = \text{Card}\{i \mid \lambda_i \geq j\}$. Note that the partitions $\mu$ and $\nu$ obtained from $\lambda$ depend on $m$ and $n$. 
The three parts of $D_\lambda$ are pictured below, where each point $(i, j)$ is represented by a unit square with $(i, j)$ as its lower right vertex:

\[ D_m^{m,n} \]
\[ D_v \]
\[ D_\mu \]
\[ i = m \]
\[ j = n \]

**Definition (1.3).** Let $X_m = (x_1, \ldots, x_m)$ and $Y_n = (y_1, \ldots, y_n)$ be two independent sets of indeterminates. Define polynomials $S_k(X_m/Y_n)$ for all integers $k$ by the following equation of power series in the variable $T$:

\[
\prod_{i=1}^{m} (1 - x_i T)^{-1} \prod_{j=1}^{n} (1 + y_j T) = \sum_{k} S_k(X_m/Y_n) T^k.
\]

In particular, $S_k = 0$ when $k < 0$. Clearly, the polynomial $S_k$ is homogeneous of degree equal to $k$.

Let $\lambda$ be a partition. Denote by $l$ the length of $\lambda$, that is, the number of non-zero parts of $\lambda$. Define the polynomial $S_\lambda(X_m/Y_n)$ as the following $1 \times 1$ determinant:

\[
S_\lambda := \det[S_{\lambda_i - i + j}(X_m/Y_n)]_{1 \leq i, j \leq l}.
\]

The polynomial $S_\lambda(X_m/Y_n)$ will be called the super Schur polynomial corresponding to the partition $\lambda$. It is homogeneous of degree equal to the weight $|\lambda| = \lambda_1 + \cdots + \lambda_l$ of $\lambda$.

Note that the polynomials $S_\lambda(X_m/Y_n)$ specialize as $m$ and $n$ varies: If $m < \hat{m}$ and $n < \hat{n}$, then the substitution of $x_i = 0$ for $m < i < \hat{m}$ and $y_j = 0$ for $n < j < \hat{n}$ in the polynomial $S_\lambda(X_m/Y_n)$ yields the polynomial $S_\lambda(X_{\hat{m}}/Y_{\hat{n}})$.

Note also that if $n = 0$, then the super Schur polynomial is the usual Schur polynomial $S_\lambda(X_m)$ in the single set of indeterminates $X_m$.

**Definition (1.4).** Let $\lambda$ be a partition. Assume that the length of $\lambda$ is less than or equal to $m$. Define

\[
F_\lambda(X_m) := \sum_w w \left[ \frac{x_1^{\lambda_1 + m - 1} x_2^{\lambda_2 + m - 2} \cdots x_m^{\lambda_m}}{\Delta(X_m)} \right].
\]
The sum is over all \( w \) in the group \( \text{Aut}(X_m) \) of all permutations of the indeterminates \( X_m \). The denominator \( \Delta = \Delta(X_m) \) is the Vandermonde determinant
\[
\Delta = \prod_{1 \leq i < j \leq m} (x_i - x_j).
\]
When the length of \( \lambda \) is greater than \( m \), define \( F_\lambda(X_m) := 0 \).

The sum on the right hand side of (1.4.1) is the result of applying a symmetrizing operator. Note that the sum may be rewritten as
\[
\frac{1}{\Delta(X_m)} \sum_w \text{sign}(w) \, w\prod_{i \geq 1} (x_i - x_i^\lambda). 
\]
Clearly, the sum in the latter expression is an alternating polynomial. Hence the sum is divisible by \( \Delta(X_m) \). Thus \( F_\lambda \) is indeed a symmetric polynomial.

**Lemma (1.5).** (1) For any partition \( \lambda \) the following equation holds:
\[
S_\lambda(X_m) = F_\lambda(X_m). \tag{1.5.1}
\]

(2) The polynomials \( F_\lambda(X_m) \) for all partitions \( \lambda \) of length less than or equal to \( m \) form a basis of the \( \mathbb{Z} \)-module of symmetric polynomials in \( X_m \).

**Proof.** The identity (1.5.1) is the Jacobi–Trudi identity. The reader is referred to Jacobi [J] for the classical proof or to Macdonald [M] for a modern proof.

To prove (2), note that the polynomials of the form
\[
\sum_w \text{sign}(w) \, w\prod_{i \geq 1} (x_i^{\lambda_i} x_i^{\lambda_2} \cdots x_i^{\lambda_m}), 
\]
where the sequence of exponents is strictly decreasing, form a basis of the \( \mathbb{Z} \)-module of alternating polynomials. The latter statement implies the assertion in (2), because multiplication by \( \Delta(X_m) \) is an isomorphism from the \( \mathbb{Z} \)-module of symmetric polynomials onto the \( \mathbb{Z} \)-module of alternating polynomials.

**Definition (1.6).** Let \( \lambda \) be a partition. Assume that \( \lambda \) is contained in the \((m, n)\)-hook. Define
\[
F_\lambda(X_m/Y_n) := \sum_w w \prod_{(i,j) \in D_{\lambda,n}} (x_i + y_j) 
\]
where
\[
\prod_{(i,j) \in D_{\lambda,n}} (x_i + y_j) = \frac{\Delta(X_m) \Delta(Y_n)}{\Delta(X_m) \Delta(Y_n)}. 
\]
where the partitions $\mu$ and $\nu$ are obtained from $\lambda$ as in (1.2). The sum is over all permutations $w = uv$ in the product group $\text{Aut}(X_m) \times \text{Aut}(Y_n)$.

When $\lambda$ is not contained in the $(m, n)$-hook, define $F_{\lambda}(X_m/Y_n) := 0$.

**Lemma (1.7).** (1) The polynomial $F_{\lambda}(X_m/Y_n)$ is homogeneous of degree equal to $|\lambda|$.

(2) If $\lambda$ is fixed, then the polynomials $F_{\lambda}(X_m/Y_n)$ for varying $m$ and $n$ specialize, that is, if $\hat{m} \geq m$ and $\hat{n} \geq n$, then the substitution of $x_{\hat{m}} = \cdots = x_{m+1} = 0$ and $y_{\hat{n}} = \cdots = y_{n+1} = 0$ in the polynomial $F_{\lambda}(X_m/Y_n)$ yields the polynomial $F_{\lambda}(X_{\hat{m}}/Y_{\hat{n}})$.

**Proof.** The assertion in (1) is trivial.

Clearly, it suffices to prove the assertion in (2) for $\hat{m} = m + 1$ and $\hat{n} = n$. Moreover, it may be assumed that $\lambda$ is contained in the $(m + 1, n)$-hook. Then, by definition,

$$F_{\lambda}(X_{m+1}/Y_n) = \sum_{\hat{\psi}} \hat{\psi} \left[ \frac{x_1^{\mu_1} \cdots x_{m+1}^{\mu_{m+1}} y_1^{\nu_1} \cdots y_n^{\nu_n} \prod_{(i,j) \in D_{n+1}^1} (x_i + y_j)}{A(X_{m+1}) A(Y_n)} \right],$$

(1.7.1)

where the sum is over all permutations $\hat{\psi} = \hat{u} \times \nu$, where $\hat{u} \in \text{Aut}(X_{m+1})$ and $\nu \in \text{Aut}(Y_n)$. The partitions $\mu$ and $\nu$ are those obtained from $\lambda$ using the $(m + 1, n)$-hook.

Note first that the numerator of the fraction in (1.7.1) contains as a factor a power of the indeterminate $x_i$ with an exponent that is positive for $i < m + 1$. Therefore, to evaluate the sum (1.7.1) for $x_{m+1} = 0$ it suffices to evaluate the terms corresponding to permutations $\hat{u} \times \nu$ such that $\hat{u}(m+1) = m + 1$. In other words, evaluation is performed by evaluating the fraction in (1.7.1) and then forming the sum over all permutations $u \times v$ in $\text{Aut}(X_m) \times \text{Aut}(Y_n)$.

To prove the assertion of the lemma, assume first that $\lambda$ is not contained in the $(m, n)$-hook. Then, by definition, $F_{\lambda}(X_m/Y_n) = 0$. On the other hand, the assumption implies that $\lambda_{m+1} > n$. Hence $x_{m+1}$ appears with the non-zero exponent $\mu_{m+1}$ in the numerator of the fraction in (1.7.1). Therefore, evaluation of the sum (1.7.1) yields zero. Hence the assertion holds under the first assumption.

Assume next that $\lambda$ is contained in the $(m, n)$-hook. Then $F_{\lambda}(X_m/Y_n)$ is given by the formula in (1.6). As noted above, it suffices to prove that evaluation at $x_{m+1} = 0$ of the fraction in (1.7.1) yields the fraction in (1.6). Evaluate each factor in the fraction of (1.7.1) and divide the result by the
corresponding factor of the fraction of (1.6). Clearly, the result is the fraction
\[
\frac{x_1 \cdots x_m y_1^{y_1} \cdots y_n^{y_n} \prod_{1 \leq j \leq \lambda_{m+1}} y_j}{x_1 \cdots x_m \cdot 1 \cdot y_1^{y_1} \cdots y_n^{y_n}}.
\]
It follows from the definition of \( \tilde{\nu} \) and \( \nu \) that the latter fraction is equal to 1. Hence the assertion of the lemma holds under the second assumption. Thus the lemma has been proved.

2. Super Symmetric Polynomials

Setup (2.1). Fix as in Section 1 two independent sets of indeterminates \( X_m = (x_1, \ldots, x_m) \) and \( Y_n = (y_1, \ldots, y_n) \).

Definition (2.2). A polynomial \( F \) in the \( m+n \) variables \( X_m \) and \( Y_n \) will be called supersymmetric if \( F \) is symmetric with respect to \( X_m \) and with respect to \( Y_n \), and the substitution \( x_m = t, y_n = -t \) in \( F \) yields a polynomial independent of \( t \).

It follows immediately from the definition in (1.3) that the super Schur polynomials \( S_\lambda(X_m/Y_n) \) are supersymmetric.

Note that if the polynomial \( F \) is supersymmetric, then the polynomial \( F_{x_m = y_n = 0} \) obtained by substitution of \( x_m = 0 = y_n \) in \( F \) is supersymmetric in \( X_{m-1} \) and \( Y_{n-1} \). Moreover, if \( F \) is supersymmetric, then the polynomial \( F_{x_m = y_n = 0} \) is equal to zero if and only if \( F \) is divisible by the product \( P(X_m, Y_n) := \prod_i \prod_j (x_i + y_j) \).

Proposition (2.3). (1) Every polynomial \( F_\lambda(X_m/Y_n) \) is supersymmetric.

(2) The family of polynomials \( F_\lambda(X_m/Y_n) \), where \( \lambda \) ranges over the partitions contained in the \( (m, n) \)-hook, forms a basis for the \( \mathbb{Z} \)-module of supersymmetric polynomials in \( X_m \) and \( Y_n \).

Proof. (1) The polynomial \( F_\lambda(X_m/Y_n) \) is the result of symmetrizing, and hence symmetric in \( X_m \) and \( Y_n \).

Denote by \( N_\lambda \) the numerator in the fraction appearing in the definition of \( F_\lambda \). Then \( F_\lambda \) is the quotient obtained by dividing the following sum
\[
\sum_w \text{sign}(w) w(N_\lambda)
\]
by the product \( A(X_m) A(Y_n) \). Clearly, the substitution \( x_m = t, y_n = -t \) in the latter product yields a polynomial of degree \( m + n - 2 \) in \( t \). Therefore,
to prove that $F_\lambda$ is supersymmetric, it suffices to prove for each term of the sum (2.3.1) that the substitution $x_m = t$, $y_n = -t$ in it yields a polynomial of degree less than or equal to $m + n - 2$ in $t$. Equivalently, it suffices to prove for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$ the following assertion: The substitution $x_i = t$, $y_j = -t$ in $N_\lambda$ yields a polynomial of degree less than or equal to $m + n - 2$ in $t$.

The proof of the latter assertion is divided in two cases. Assume first that $j \leq \lambda_i$, that is, $(i, j)$ belongs to the diagram $D^{m,n}_\lambda$. Then $N_\lambda$ contains the factor $x_i + y_j$. Consequently the substitution $x_i = t$, $y_j = -t$ in $N_\lambda$ yields the zero polynomial.

Assume next that $j > \lambda_i$, or, equivalently, that $i > \lambda'_j$, where $\lambda'$ is the conjugate partition of $\lambda$. Then, in particular, $\lambda_i \leq m$ and $\lambda'_j \leq n$. Hence, in the notation of Definition (1.2), $\mu_i = 0$ and $\nu_j = 0$. Therefore, the factors in $N_\lambda$ that contain either $x_i$ or $y_j$ are

$$(x_i + y_1), \ldots, (x_i + y_{\lambda_i}), (x_1 + y_j), \ldots, (x_{\lambda_i} + y_j), x_1^{m-i}, y_j^{n-j}.$$ 

The number of factors is the degree of their product. Hence the degree is equal to

$$\lambda_i + \lambda'_j + (m - i) + (n - j). \quad (2.3.2)$$

Consequently, the substitution $x_i = t$, $y_j = -t$ in $N_\lambda$ yields a polynomial of degree equal to (2.3.2) in $t$. By assumption, $j > \lambda_i$ and $i > \lambda'_j$. Hence the degree (2.3.2) is less than or equal to $m + n - 2$. Thus (1) has been proved.

The assertion (2) will be proved by induction. The assertion holds when $m = 0$ or $n = 0$. Indeed, if $n = 0$, then the condition on $\lambda$ is that its length is less than or equal to $m$. Moreover, the polynomial $F_\lambda(X_m/Y_n)$ for $n = 0$ is the polynomial $F_\lambda(X_m)$ where the sum is over all permutations of $X_m$. By Lemma (1.5)(2), the polynomials $F_\lambda(X_m)$, where the length of $\lambda$ is less then or equal to $m$, form a basis of the $\mathbb{Z}$-module of symmetric polynomials in $X_m$. Similarly, if $n = 0$, then the condition on $\lambda$ is that the conjugate partition $\lambda'$ has length less than or equal to $n$, and the polynomial $F_\lambda(X_m/Y_n)$ for $n = 0$ is the polynomial $F_\lambda(Y_n)$.

Assume next that $m > 0$ and $n > 0$. Denote by $\mathcal{H}_{m,n}$ the set of partitions contained in the $(m, n)$-hook. Divide the set $\mathcal{H}_{m,n}$ into the subset $\mathcal{H}_{m-1,n-1}$ consisting of partitions contained in the $(m-1, n-1)$-hook and its complement $\mathcal{H}^0$. Clearly, a partition $\lambda$ in $\mathcal{H}_{m,n}$ belongs to $\mathcal{H}^0$ if and only if the part $D^{m,n}_\lambda$ of its diagram consists of all points in the rectangle $1 \leq i \leq m$, $1 \leq j \leq n$. It follows that partitions $\lambda$ in $\mathcal{H}^0$ correspond bijectively to pairs
(µ, ν), where µ and ν are partitions of lengths at most m and n, respectively. Clearly, if λ ∈ ℳ_0 is the partition corresponding to (µ, ν), then

\[ F_λ(X_ν/Y_ν) = P(X_m, Y_n) F_µ(X_m) F_ν(Y_n), \]

(2.3.3)

where \( P(X_m, Y_n) \) is the product considered in Definition (2.2). Moreover, by Lemma (1.5)(2), the products \( F_µ(X_m) F_ν(Y_n) \), where µ and ν are partitions of lengths at most m and n, respectively, form a basis of the \( \mathbb{Z} \)-module of polynomials that are symmetric in \( X_m \) and in \( Y_n \). Therefore, the polynomials \( F_λ(X_m/Y_n) \), where \( λ \in ℳ_0 \), form a basis of the \( \mathbb{Z} \)-module of polynomials that are symmetric in \( X_m \) and in \( Y_n \) and divisible by \( P(X_m, Y_n) \). Clearly, the latter set of polynomials is equal to the set of supersymmetric polynomials for which the substitution \( x_m = 0 = y_n \) yields the zero polynomial.

To finish the inductive proof of (2), let \( F \) be a given supersymmetric polynomial. Substitute \( x_m = 0 = y_n \) in \( F \). The resulting polynomial is supersymmetric in \( X_{m-1} \) and \( Y_{n-1} \), and may therefore uniquely be written as a linear combination of the \( F_λ(X_{m-1}/Y_{n-1}) \), where \( λ \) belongs to \( ℳ_{m-1,n-1} \). Subtract the same linear combination of the \( F_λ(X_m/Y_n) \) from \( F \). By construction, the substitution \( x_m = 0 = y_n \) in the difference yields 0. Moreover, the difference is supersymmetric. Therefore, as proved above, the difference is uniquely a linear combination of the \( F_λ(X_m/Y_n) \), where \( λ \) belongs to \( ℳ_0 \). Hence \( F \) is uniquely a linear combination of the \( F_λ(X_m/Y_n) \), where \( λ \) belongs to \( ℳ_{m,n} \). Thus (2) has been proved.

Thus the proposition is proved.

3. The Formula and Its Consequences

**Theorem (3.1).** For every partition \( λ \) the following equation holds:

\[ S_λ(X_m/Y_n) = F_λ(X_m/Y_n). \]

**Proof.** The two sides of the equation specialize when \( m \) and \( n \) varies. Indeed, the assertion for \( F_λ \) holds by Lemma (1.7)(2), and the assertion for \( S_λ \) follows immediately from the definition in (1.3). Therefore, it suffices to prove that the equation holds when \( m \) is large. Thus, without loss of generality, it will be assumed that the weight \( |λ| \) of the partition \( λ \) is less than or equal to \( m \).

It is clear from the definition that the polynomial \( S_λ(X_m/Y_n) \) is supersymmetric. Therefore, by Proposition (2.3)(2), \( S_λ(X_m/Y_n) \) is a finite linear combination

\[ S_λ(X_m/Y_n) = \sum_{ν ∈ ℳ_{m,n}} a_ν F_ν(X_m/Y_n), \]

(3.1.1)
where \( a_v \in \mathbb{Z} \) and the sum is over all partitions \( v \) contained in the \((m, n)\)-hook. It has to be proven that \( a_\lambda = 1 \) and \( a_v = 0 \) for \( v \neq \lambda \). The left hand side of (3.1.1) has degree equal to the weight \( |\lambda| \). Therefore, for any term on the right hand side appearing with a non-zero coefficient \( a_v \), the weight of \( v \) is equal to the weight of \( \lambda \). In particular, the length of a partition \( v \) corresponding to a non-zero term in (3.1.1) is less than or equal to \( m \). Consequently, it may be assumed that the sum in (3.1.1) is over partitions of length less than or equal to \( m \).

Substitute \( y_1 = \cdots = y_n = 0 \) in Eq. (3.1.1). It follows that

\[
S_\lambda(X_m) = \sum_v a_v F_v(X_m),
\]

where the sum is over partitions \( v \) of length less than or equal to \( m \). The polynomial on the left hand side of Eq. (3.1.2) is the usual Schur polynomial. Therefore, by Lemma (1.5), (1) and (2), Eq. (3.1.2) implies that \( a_\lambda = 1 \) and \( a_v = 0 \) for \( v \neq \lambda \).

Thus the theorem has been proved.

**Corollary (3.2).** The polynomials \( S_\lambda(X_m/Y_n) \), where \( \lambda \) ranges over all partitions contained in the \((m, n)\)-hook, form a basis of the \( \mathbb{Z} \)-module of supersymmetric polynomials in \( X_m \) and \( Y_n \).

**Proof.** The assertion holds, because the parallel assertion for the polynomials \( F_\lambda \) holds by Proposition (2.3)(2).

**Corollary (3.3).** Let \( \lambda \) be a partition contained in the \((m, n)\)-hook such that \( \lambda_m \geq n \). Denote by \( \mu \) and \( v \) the partitions obtained from \( \lambda \) using the \((m, n)\)-hook as in Definition (1.2). Then the following factorization formula holds:

\[
S_\lambda(X_m/Y_n) = S_\mu(X_m) S_v(Y_n) \prod_{i=1}^m \prod_{j=1}^n (x_i + y_j).
\]

In particular,

\[
S_{(n, \ldots, n)}(X_m/Y_n) = \prod_{i=1}^m \prod_{j=1}^n (x_i + y_j),
\]

where the partition \((n, \ldots, n)\) has \( m \) non-zero parts.

**Proof.** The first formula holds, because the parallel formula for the polynomials \( F_\lambda \) is immediate from their definition, cf. Eq. (2.3.3). Clearly, the second equation is a special case.
COROLLARY (3.4). For every partition $\lambda$ the following equation holds:

$$S_\lambda(X_m/Y_n) = S_\lambda(Y_n/X_m).$$

Proof. The equation holds, because the parallel equation for the polynomials $F_\lambda$ is obvious from their definition.

Note (3.5). Corollary (3.2) is a refinement of both the main result of [S] and of Lemma 6.4 in [B-R]. The factorization formula was proven originally in [B-R, Theorem 6.20]. Recently, N. Bergeron and A. Garsia [B-G] and, independently, J. Van der Jeugt and V. Fack [J-F] have used the formula in (3.1) to give a new derivation of the Littlewood–Richardson rule describing the coefficients in products of Schur polynomials.

Note added in proof. Having become acquainted with the formula in (3.1) from a preliminary version of [P, Sect. 2], A. Lascoux gave still another proof of it based on the Schubert polynomial technique.

REFERENCES


