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Asymptotical periodicity for analytic triangular maps of type less than 2^∞

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ABSTRACT

We prove that if F is an analytic triangular map of type less than 2^∞ in the Sharkovsky ordering, then all points are asymptotically periodic for F. The same is true if, instead of being analytic, F is just continuous but has the property that each fibre contains finitely many periodic points. Improving earlier counterexamples in Kolyada (1992) [16] and Balibrea et al. (2002) [3], we also show that this need not be the case when F is a C^∞ map. Finally we remark that type less than 2^∞ and closedness of periodic points are equivalent properties in the C^1 setting for triangular maps.

1. Introduction

A *triangular* map is a continuous map F from the square $I^2 = [0,1] \times [0,1]$ into itself given by $F(x, y) = (f(x), g(x, y))$, that is, its first component depends only on the first coordinate x. Shortly after the western readership became aware of the now classic Sharkovsky theorem on coexistence of periods for interval maps, it was already pointed out by Kloeden that the theorem is also satisfied by triangular maps [15]. However it was only at the suggestion of Sharkovsky and Kolyada in the late eighties, in particular after the seminal papers [18,16], when specialists in topological dynamics began to study these maps in depth. Unfortunately, one of the main reasons fuelling this interest (the possibility that triangular maps could share a variety of key dynamical properties with interval maps, thus becoming a bridge from the relatively tractable interval setting into the much more complicated two-dimensional realm) soon proved unfounded.

To begin with, Kolyada's paper itself shows that, contrary to the interval case, a triangular map of type 2^∞ in the Sharkovsky ordering may have positive topological entropy, and also that a triangular map of type 1 may have some nonasymptotically periodic points (see, respectively, [16, Theorems 9 and 3]). Here recall that $w \in I^2$ is *periodic* for F if there is a (minimal) integer $r \geqslant 1$ (the *period* of w) such that $F^r(w) = w$, and we say that z is *asymptotically periodic* if there is a periodic point w such that $\|F^n(z) - F^n(w)\| \to 0$ as $n \to \infty$ (with $\|\cdot\|$ standing for a fixed norm in $\mathbb{R}^2$).

Some additional situations (among many others) when interval and triangular dynamics differ: neither zero topological entropy implies $R(F) = UR(F)$ [9], nor $R(F) = UR(F)$ implies zero topological entropy [24] (here $R(F)$ and $UR(F)$ denote, respectively, the set of recurrent and uniformly recurrent points of F); there are triangular maps with periodic points of all periods, and also triangular maps with homoclinic orbits, having no infinite ω-limit sets containing periodic points [10, 4]. While triangular maps of type 2^∞ have in particular deserved a lot of attention (see [19] for a recent example), their topological classification remains still completely unclear.

Of course Sharkovsky's theorem is not the only intersection point of interval and triangular dynamics. For instance, types less and greater than 2^∞ similarly imply, respectively, zero and positive entropy (the first result is implicitly stated in [16, p. 759], the second is a consequence of [16, Theorem 8]), and if F is a triangular map, then the set $\mathrm{Per}(F)$ of periodic points is closed if and only if it equals the set $\Omega(F)$ of nonwandering points [19, Proposition 3], see also [11] (we remark that the

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earlier paper [8] contains an incorrect proof of this result). In order to enlarge this intersection range the natural idea of assuming additional smoothness properties for the map F suggests itself, particularly taking into account how successful this approach has proven in the interval setting. (We emphasise that although the cream of the results here is mostly measure-theoretical in nature, the key and starting point of smooth one-dimensional dynamics is purely topological: nonexistence of wandering intervals. See [20] for a comprehensive, if somewhat outdated, account of the work in this fascinating field.)

Surprisingly, almost no attempt has been made to investigate the topological dynamics of smooth triangular maps. To the best of our knowledge just two results are available in this regard. In [1] is shown that if a C^1 triangular map (satisfying an additional hyperbolicity condition) has a closed set of periodic points, then $\Omega(F) = \mathrm{Per}(f)$, but we have just indicated that the same statement is true for continuous maps. On the other hand, improving a Kolyada example we mentioned earlier (see also [2]) a C^∞ triangular map of type 2^∞ with positive entropy is constructed in [17, Theorem 5.8].

In fact, the existence of such a map hardly raises an eyebrow, for even in the unidimensional setting it is well known that C^∞ differentiability alone does not prevent the appearance of several important pathologies (wandering intervals, for instance; see also [22,23,12]).

The next step ahead is analyticity and we deal with it in this paper. Namely we prove that, similarly to the interval case, the dynamics of analytic triangular maps of type less than 2^∞ is remarkably simple. Our main result works as well for continuous triangular maps with finitely many periodic points.

Theorem A. *Let F be a triangular map of type less than 2^∞. Assume that either F is analytic, or each fibre $\{x\} \times I$ contains finitely many periodic points of F. Then all points from I^2 are asymptotically periodic for F.*

Here by saying that F is *analytic* we mean that both f and g are *real* analytic, that is, for every $(x_0, y_0) \in I^2$ there are respective neighbourhoods $U(x_0)$ and $V(x_0, y_0)$ of x_0 and (x_0, y_0) in which f and g may be represented by respective convergent power series in the variables x and x, y.

Recall that, as we explained before, this statement need not be true if F is just continuous (a refinement of Kolyada's example in [16] can be found in [3]). The second result of this paper provides a C^∞ counterexample to Theorem A and, at the same time, a much easier construction than those in [16] and [3].

Theorem B. *There is a C^∞ triangular map of type 1 having some nonasymptotically periodic point.*

At the present stage is hard to predict up to what extent analytic triangular maps may mimic the essential topological dynamics of continuous interval maps, but in our opinion this is a very promising researching field. In particular, we conjecture that an analytic triangular map has positive topological entropy if and only if has type greater than 2^∞. Observe that one of the key facts complicating things here is that an analytic triangular map of type 2^∞ *cannot* have a closed set of periodic points. Indeed the same is true even for C^1 maps (but not, of course, for continuous maps: just take an interval map f of type 2^∞ whose set of periodic points is closed—as that, for instance, in [7]—and consider $F(x, y) = (f(x), y)$). Such is the content of the last result of the paper:

Proposition C. *If F is a C^1 triangular map, then $\mathrm{Per}(F)$ is closed if and only if F has type less than 2^∞.*

The structure of the paper is simple enough: after some preparatory work in Section 2, we prove our theorems in Sections 3 and 4.

2. On interval maps of type 1 with finitely many fixed points

In this section we obtain some properties of interval maps of type 1 in the Sharkovsky ordering that will prove their usefulness in the next section. The main result is Proposition 2.7, which provides a convenient description of the dynamical structure of these maps (when the number of fixed points is finite) in terms of simpler pieces (so-called "atoms" and "molecules"). It will be instrumental to prove Proposition 2.8, which in turn will become a key tool in the next section. Proposition 2.7 was implicitly proved in [5] but the hypotheses and the definitions of atom and molecule used there are slightly different. Hence we have included a full proof for the convenience of the reader.

First we should recall precisely what the *Sharkovsky ordering* is. In this ordering (denoted by $\succ$) the set of positive integers (together with the additional symbol "2^∞") is ordered as follows:

$$\begin{aligned} &3 \succ 5 \succ 7 \succ \cdots \succ 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ \cdots \succ 4\cdot 3 \succ 4\cdot 5 \succ 4\cdot 7 \cdots \succ \cdots \\ &\quad \succ 2^n\cdot 3 \succ 2^n\cdot 5 \succ 2^n\cdot 7 \succ \cdots \succ \cdots \succ 2^\infty \succ \cdots \succ 2^n \succ \cdots \succ 4 \succ 2 \succ 1. \end{aligned}$$

Sharkovsky's theorem states that if h is either a (continuous) interval map or a triangular map, then there is some t in $\mathbb{Z}^+ \cup \{2^\infty\}$ (the *type* of f) such that the set of periods corresponding to periodic points of h is exactly $\{s \in \mathbb{Z}^+ \colon\ t \succeq s\}$. In particular, a map of type 1 is that having the property that their only periodic points are fixed points.

In what follows, if $B \subset I$, then $\operatorname{Int} B$, $\operatorname{Cl} B$ and $\operatorname{Bd} B$, the *interior*, the *closure* and the *boundary* of B, always refer to the topology of I.

Let h be an interval map (denoted $h \in C(I)$). A compact set $B \subset I$ is *invariant* for h if $h(B) \subset B$. A compact interval $J \subset I$ is *shrinking* for h if $h(J) \subset \operatorname{Int} J$. The *immediate basin of attraction* of a (possibly degenerate) invariant subinterval J of I is the largest (possibly degenerate) interval U containing J with the property that, for all $x \in U$, $\operatorname{dist}(f^n(x), J) \to 0$ as $n \to \infty$.

Notice that in the degenerate case $J = \{q\}$, when q is a fixed point of h, the interval U may just consist of the point q, or q may belong to $\operatorname{Bd} U$. If this is not the case, that is, U is a neighbourhood of q, then we say that q is an *attractor*. We say that q is *left-repelling* (respectively, *right-repelling*) if there is $\epsilon > 0$ such that $h(x) < x$ (respectively, $h(x) > x$) $x \in (q - \epsilon, q)$ (respectively, $x \in (q, q + \epsilon)$).

The next statement is a particular case of [6, Proposition 15, p. 100].

Proposition 2.1. *Let $h \in C(I)$ and let q be an attractor of h. Then it admits shrinking neighbourhoods as small as required.*

We say that an interval $[a, b]$ is *trivial* for h if $h(a) = a$, $h(b) = b$, $h((a, b)) = (a, b)$, there are no fixed points in (a, b), and neither a nor b are attractors.

Definition 2.2. Let $h \in C(I)$ and $J \subset I$.

- (i) J is an *atom of level* 0 of h if J is either a trivial interval or the closure of the immediate basin of attraction of an attractor.
- (ii) J is a *molecule of level* 0 of h if it is the connected union of a finite number of atoms of level 0 and if it is maximal with respect to this property.
- (iii) J is an *atom of level n* ($\geqslant 1$) of h if it is the closure of the immediate basin of attraction of a molecule of level $n - 1$.
- (iv) J is a *molecule of level n* ($\geqslant 1$) of h if it is the connected union of a finite number of atoms of level n and if it is maximal with respect to this property.

We emphasise that, by definition, all atoms and molecules of a map h are compact intervals.

A proof of the proposition below can be found, for instance, in [6, pp. 121–122].

Proposition 2.3. *Let $h \in C(I)$ be of type* 1 *and let $x \in I$. If $h(x) > x$ (respectively, $h(x) < x$), then $h^n(x) > x$ (respectively, $h^n(x) < x$) whenever $n \leqslant 1$.*

Proposition 2.4. *Let $h \in C(I)$ be of type* 1 *and let q be an isolated fixed point of h. Then it is an attractor if and only if it is neither left- nor right-repelling.*

Proof. If q is an attractor, then (regardless h is or not of type 1) it is neither left- nor right-repelling. Indeed, assume for instance that $h(x) > x$ but $h^n(x) \to q$ for every $x \in (q, q + \epsilon]$. Let n_0 be such that $h^{n_0}(q + \epsilon) \leqslant q + \epsilon$ and use that $[q, q + \epsilon] \subset h^n([q, q + \epsilon])$ for every n to find $y \in (q, q + \epsilon]$ such that $h^{n_0}(y) \geqslant q + \epsilon \geqslant y$. Then there is $z \in [y, q + \epsilon]$ such that $h^{n_0}(z) = z$, a contradiction.

Conversely, assume that q is neither left- nor right-repelling. We show that there is a neighbourhood U of q such that $(h^n(x))$ tends to q for every x in U. Let ϵ be sufficiently small so that q is the only fixed point of $(q - \epsilon, q + \epsilon)$, and let $\epsilon_1 \leqslant \epsilon$ be such that $|h(x) - q| < \epsilon$ whenever $|x - q| < \epsilon_1$. We show that $U = (q - \epsilon_1, q + \epsilon_1)$ is appropriate for our purposes.

Assume that $x \in (q - \epsilon_1, q)$ (the other case is similar). Then $x \leqslant h^n(x)$ for each n by Proposition 2.3 and the hypothesis on q. If additionally $h^n(x) \leqslant q$ for each n, then $(h^n(x))$ tends to the only fixed point q in $(q - \epsilon, q + \epsilon)$ (because h is of type 1). If k is the first integer such that $h^k(x) > q$, then we have

$$h^n(x) \in \left[h^{k-1}(x), h^k(x)\right] \subset (q - \epsilon, q + \epsilon)$$

whenever $n \geqslant k$ (by Proposition 2.3, the hypothesis on q and the definition of ϵ_1). Then $(h^n(x))$ tends to the only fixed point q in $(q - \epsilon, q + \epsilon)$ again. □

Lemma 2.5. *Let $h \in C(I)$. Then*:

- (i) *the boundary of an atom is invariant by h*;
- (ii) *two different atoms of h of the same level have at most one common* (*fixed*) *point.*

Proof. Both statements follow easily from the definition of atom (notice, regarding (ii), that different molecules of the same level must be disjoint). □

Lemma 2.6. *Let $h \in C(I)$ be a type* 1 *with finitely many fixed points. Then it has a positive* (*finite*) *number of atoms of level* 0.

Proof. The finiteness is an immediate consequence of Lemma 2.5(ii) and the fact that every atom of level 0 contains at least one fixed point.

Then it suffices to show that h has at least one attractor. Let $0 \leqslant q_1 < \cdots < q_n \leqslant 1$ be the fixed points of h. The statement is trivial in the case $n = 1$: all orbits are attracted to the unique fixed point of h. Assume $n > 1$. If $h(x) < x$ for each $x \in (q_1, q_2)$, then q_1 must be an attractor (if $q_1 = 0$ this is clear, while if $q_1 > 0$, then $h(x) > x$ for every $0 \leqslant x < q_1$ and Proposition 2.4 applies). Proceeding in this way and assuming that none of the points $q_1, \ldots, q_{n-1}$ is an attractor we get $h(x) > x$ for each $x \in (q_i, q_{i+1})$ $(i = 1, \ldots, n-1)$. Hence q_n is an attractor by an argument similar to that used for q_1. $\square$

Proposition 2.7. *Let $h \in C(I)$ be a type* 1 *map with finitely many fixed points. Then I is an atom of level l of h for some $l \geqslant 0$.*

Proof. Let Q_m be the set of fixed points not belonging to the interior of some atom of level m. It suffices to show that $\operatorname{card} Q_m > \operatorname{card} Q_{m+1}$ unless $Q_m = \emptyset$, then $Q_l = \emptyset$ must hold for some l. Notice that h maps the boundary of every atom to itself. Hence, if this boundary is nonempty, then it must contain some fixed point. By Lemma 2.6, h must have an atom of level l. Since this atom has empty boundary (otherwise its boundary would contain a fixed point), it must be the whole interval I.

Thus assume $Q_m \neq \emptyset$. Let $J_i = [p_i, q_i]$ $(i = 1, \ldots, r)$ be the atoms of level m of h. If two of these intervals are not disjoint, then they must have exactly one common (fixed) point by Lemma 2.5(ii), thus belonging to Q_m but not to Q_{m+1}. Hence we can assume that the intervals J_i are pairwise disjoint. Moreover, after reordering if necessary, we can assume that J_i is to the left of J_{i+1} for every $1 \leqslant i < r$. We show that one of the points p_i or q_i is fixed and belongs to the interior of some atom of level $m+1$ of h.

Suppose initially that $p_1 > 0$ and that p_1 is a fixed point. Then $h(x) \geqslant x$ for every $x \in [0, p_1]$: if not, then $[0, p_1)$ contains an attractor (see the proof of Lemma 2.6), which is impossible. So, if $\epsilon > 0$ is small enough, $h(x) > x$ for every $x \in [p_1 - \epsilon, p_1)$. Hence the orbits of all points from $[p_1 - \epsilon, p_1]$ either eventually fall into J_i or converge to p_1. This means that $[p_1 - \epsilon, p_1]$ is included in the atom of level $m+1$ containing J_1 and we are done.

Now assume that either $p_1 = 0$ or p_1 is not a fixed point. Since $Q_m \neq \emptyset$, J_1 cannot be the whole interval I so $\operatorname{Bd} J_1 \neq \emptyset$ (recall that “boundary” refers to the topology of I). Since $\operatorname{Bd} J_1$ is invariant (Lemma 2.5(i)), it must contain some fixed point. Thus $q_1 < 1$ and $h(q_1) = q_1$. Moreover, we can assume that $r \geqslant 2$, because otherwise we can argue as before to finish the proof. Next, assume that p_2 is fixed. Then $h(x) \geqslant x$ for every $x \in [q_1, p_2]$. Otherwise, either (q_1, p_2) contains an attractor, or the orbits of all points from $[q_1, q_1 + \epsilon]$ approach to J_1 if ϵ is sufficiently small; the first thing is impossible, the second one means that $[q_1, q_1 + \epsilon]$ is included in the atom of level $m+1$ containing J_1. Now, as in the previous paragraph, $h(x) \geqslant x$ for every $x \in [q_1, p_2]$ implies that there is an atom of level $m+1$ containing p_2 in its interior.

Thus we can assume that p_2 is not fixed. Then, reasoning as in the beginning of the above paragraph, we obtain that $q_2 < 1$ and q_2 is fixed. After repeating the previous reasoning we reach the desired conclusion after a finite number of steps, or conclude that $q_r < 1$ and q_r is fixed. But in this last case we apply to q_r the reversed argument to that we applied to p_1 in the third paragraph of the proof, to conclude that for a small positive ϵ the interval $[q_r, q_r + \epsilon]$ is included in the atom of level $m+1$ containing J_r. $\square$

As a consequence of a Sharkovsky's result (see, e.g., [6, Corollary 13, p. 76]), if U is a neighbourhood of the set of fixed points of a map h of type 1, then the set of points of an arbitrary orbit of h lying outside U is uniformly bounded. The proposition below can be seen as a refinement of this result and is the key tool to prove Theorem A.

In what follows we denote $V_q(\epsilon) = [q - \epsilon, q + \epsilon] \cap I$, $V_q^-(\epsilon) = [q - \epsilon, q] \cap I$, $V_q^+(\epsilon) = [q, q + \epsilon] \cap I$. If q is an attractor for h, then we fix for every $\epsilon > 0$ a shrinking interval $S_q(\epsilon)$ contained in $V_q(\epsilon)$ (Proposition 2.1). We decompose the set Q of fixed points of h into the set of attractors A and the set of nonattractors R, and denote $U(\epsilon) = \bigcup_{q \in A} S_q(\epsilon) \cup \bigcup_{q \in R} V_q(\epsilon)$.

Proposition 2.8. *Let $h \in C(I)$ be a type* 1 *map having finitely many fixed points. Let $\epsilon > 0$ be small enough. Then there are a number $k = k(\epsilon)$ and, for each left-repelling* (*respectively, right-repelling*) *fixed point q, a shrinking interval $I_{q,\epsilon}^-$* (*respectively, $I_{q,\epsilon}^+$*) *not intersecting $V_q(2\epsilon)$, such that*:

(i) *If $y \in I \setminus U(\epsilon)$, then there is $m \leqslant k$* (*depending on y*) *such that $h^m(y) \in U(\epsilon)$.*
(ii) *If additionally there are a left-repelling* (*respectively, right-repelling*) *fixed point q and a point $z \in V_q^-(\epsilon)$* (*respectively, $z \in V_q^+(\epsilon)$*) *such that $h^j(z) = y$ for some j, then $h^k(y) \in I_{q,\epsilon}^-$* (*respectively, $h^k(y) \in I_{q,\epsilon}^+$*).

Proof. The statement (i) follows from the above-mentioned Sharkovsky's result. Alternatively, let $y \in K = \operatorname{Cl}(I \setminus U(\epsilon))$. Since the orbit of y is attracted by some fixed point, there are a number k_y and a small neighbourhood $W(y)$ of y such that $h^{k_y}(x) \in U(\epsilon)$ for every $x \in W(y)$. Use the compactness of K to find a finite covering $W(y_1), \ldots, W(y_r)$ of K. Then $k' = \max\{k_1, \ldots, k_r\}$ does the job. Of course the number k' does depend on ϵ, but here it is of no consequence whether ϵ is small or not.

We next show that if $q \in R$ (when q must be left- and/or right-repelling by Proposition 2.4) and $\epsilon > 0$ is small enough, then there are an interval $I_{q,\epsilon}^{\pm}$ and a number $k_q^{\pm}$ having (when k is replaced by $k_q^{\pm}$) the required properties in (ii). This suffices to finish the proof because if J is a shrinking interval and $h^m(y) \in J$ for some y, then $h^n(y) \in J$ for every $n \geqslant m$

because J is invariant. Hence, if k is larger than the number k' from the paragraph above and all numbers $\{k_q^{\pm}\}_{q\in R}$, then it is adequate for our purposes.

Say, for instance, that q is right-repelling. Since ϵ is very small, we have $h(x) > x$ for every $q < x \leqslant q + 2\epsilon$. Moreover, there is a number $l \geqslant 0$ such that $(q, q + 2\epsilon]$ does not intersect any atom of level less than l and there is an atom $T = [q, t]$ of level l containing $[q, q + 2\epsilon] = V_q^+(2\epsilon)$ (Proposition 2.7).

We next prove the proposition in the following particular cases:

(a) T is the closure of the immediate basin of attraction of an attractor p;
(b) T has level $l \geqslant 1$ and if $M = [r, s]$ is the molecule of level $l - 1$ contained in T, then either $s < t$, or $s = t$ is not a fixed point, or $h(s) = s = t$ is not right-repelling.

To begin with, notice that taking if necessary a smaller ϵ we can assume that if $\delta > 2\epsilon$ is given by $h(V_q^+(2\epsilon)) = V_q^+(\delta)$, then $q + \delta < p$ (in case (a)) or $q + \delta < r$ (in case (b)). Now it suffices to find a shrinking interval $I_{q,\epsilon}^+$ neighbouring, respectively, p and M, and not intersecting $V_q^+(2\epsilon)$. Indeed, in such a case we can define k_q^+ as follows. First observe that if $x \in [q + \epsilon, q + \delta]$, then there is m such that $h^m(x) \in I_{q,\epsilon}^+$ and hence $h^{m+1}(x) \in \operatorname{Int} I_{q,\epsilon}^+$. Then we can repeat the reasoning at the beginning of the proof and use the invariance of $I_{q,\epsilon}^+$ to find the number k_p^+ with the property that $h^{k_p^+}([q + \epsilon, q + \delta]) \subset I_{q,\epsilon}^+$. Finally realise that if $y \in I \setminus U(p)$ has a preimage in $V_q^+(\epsilon)$, then it also has a preimage in $[q + \epsilon, q + \delta]$, hence $h^{k_q^+}(y) \in I_{q,\epsilon}^+$.

Now we must explain how to construct the shrinking interval $I_{q,\epsilon}^+$. In case (a) this is simple enough: just use Proposition 2.1. In case (b) several possibilities must be considered separately.

Assume $h(s) \in \operatorname{Int} M$. Then the existence of $I_{q,\epsilon}^+$ is clear unless $h(r) = r$ (in fact *every* closed neighbourhood of M close enough to it is shrinking). If r is fixed, then $h(x) > x$ for every $q < x < r$ because of the definition of M and, again, every small neighbourhood of M is shrinking.

If $h(s) = r$, then $h(r) < s$ because h is of type 1. Now if $h(r) > r$, then we reason in similar vein as before, while if $h(r) = r$, then we use again that r is not left-repelling to prove that every interval $[r - \epsilon', s + \epsilon'']$ is shrinking provided that both ϵ' and ϵ'' are very small and $h([s, s + \epsilon'']) \subset (r - \epsilon', r + \epsilon')$.

This leaves $h(s) = s$ as the only pending possibility. But if s is not right-repelling (which is in particular the case when $s < t$), then we can again easily find $I_{q,\epsilon}^+$. This finishes the proof in the case (b).

Thus we may assume that either T is trivial (when by definition t is right-repelling), or t is both the right endpoint of T and the maximal molecule it contains (and t is right-repelling). Now we repeat the previous argument for the right-repelling fixed point t; we see that after a finite number of step a sequence $q_0 < q_1 < \cdots < q_j$ of points arises where $q_0 = q$, $q_1 = t$, all points q_i except q_j are right-repelling fixed points, all intervals $[q_{i-1}, q_i]$ are atoms (of possibly different levels) of h and the atom $[q', t'] := [q_{j-1}, q_j]$ has similar properties (for q' and t') to those described in cases (a) or (b) for q and t. Clearly, $I_{q,\epsilon}^+ := [q + \delta, t' + \epsilon']$ if ϵ' is sufficiently small (or $I_{q,\epsilon}^+ := [q + \delta, t']$ if $t' = 1$) is the shrinking interval we are looking for: the definition of k_q^+ involves no differences at all. $\square$

3. Proof of Theorem A

Before going to the proof of Theorem A, some simplifications are in order. First of all, if F is an analytic triangular map of type 2^n for some nonnegative integer n, then F^{2^n} is also analytic and triangular, and has type 1. If all orbits for F^{2^n} converge to fixed points, then all orbits for F are asymptotically periodic. Hence it is not restrictive to assume that F is of type 1. Similarly, if F is a triangular map of type 2^n such that every fibre $\{x\} \times I$ contains finitely many periodic points of F, the same is true for the map F^{2^n}, and again we can assume that F has type 1 without loss of generality.

Now we fix a point $(x_0, y_0) \in I^2$ and write $(x_n, y_n) = F^n(x_0, y_0)$ for every n. We must show that $((x_n, y_n))$ converges to some fixed point (p_0, q_0) of F. To begin with, F is triangular, so $x_n = f^n(x_0)$ for every n. Since f is of type 1, there is a fixed point p_0 of f such that $x_n \to p_0$. Denote $g_x(y) = g(x, y)$ for every x, y. Since p_0 is fixed for f, $F(p_0, y) = (p_0, g_{p_0}(y))$ for every y, that is, the fibre $\{p_0\} \times I$ is invariant by F. Recall that F is of type 1, hence $h = g_{p_0}$ is of type 1 as well. Finally, if F is analytic, then $g(x, y)$ is analytic, so h is analytic as well.

Now two possibilities arise: either h has finitely many fixed points, or h is analytic and has infinitely many fixed points. We remark that in the second case h is the identity map. In fact, let $\rho(y) = h(y) - y$. Then ρ is an analytic map having infinitely many zeros c_n, $n = 1, 2, \ldots$. We can assume that the sequence $(c_n)_n$ converges, say to c. Then c is not only a zero of ρ, but also, due to Rolle's theorem, of all derivatives of ρ. The analyticity of ρ then implies that ρ equals zero in a whole maximal subinterval J of I. By continuity, J is closed. By analyticity, both endpoints of J admit neighbourhoods where ρ vanishes. Thus we get $J = I$ by the maximality of J, that is, $\rho \equiv 0$.

We consider these two cases separately.

Case 1. h has finitely many fixed points.

Fix $\epsilon > 0$. It suffices to show that there are a fixed point q of h and a number n_0 such that $y_n \in V_q(2\epsilon)$ whenever $n \geqslant n_0$ (because if ϵ is small enough, then the point q cannot depend on ϵ; here we use that h has finitely many fixed points). We can assume that the distance between consecutive fixed points of h is greater than 3ϵ.

Apply Proposition 2.8 to h and ϵ to find the corresponding number k and (for all the left- and/or right-repelling fixed points q) the corresponding shrinking intervals $I_q^{\pm} := I_{q,\epsilon}^{\pm}$. Find a small number $0 < \epsilon' < \epsilon$ such that the closed ϵ'-neighbourhoods T_q of $S_q(\epsilon)$ (when q is an attractor) and $J_q^{\pm}$ of $I_q^{\pm}$ (when q is left- and/or right-repelling) are shrinking. Moreover, we can assume in the last case

$$J_q^{\pm} \cap V_q(2\epsilon) = \emptyset. \tag{1}$$

With the help of Proposition 2.8 we can now describe how the sequence (y_n) moves along the interval I. Recall that $y_{n+1} = g_{x_n}(y_n)$ and realise that, because of the uniform continuity of the map $g(x, y)$, the sequence of maps (g_{x_n}) converges uniformly to h. In particular, there is n_1 such that if $n \geqslant n_1$, then the following properties are satisfied:

(i) all intervals T_q and $J_q^{\pm}$ are shrinking for g_{x_n};
(ii) if $0 \leqslant l \leqslant k$, then $\|(g_{x_{n+l}} \circ \cdots \circ g_{x_n}) - h^{l+1}\|_\infty < \epsilon'$;
(iii) if q is not an attractor, $y_n \in V_q(2\epsilon)$ and $y_{n+1} < q - 2\epsilon$, then q is left-repelling for h and y_n has a preimage for some iterate or h in $V_q^-(\epsilon)$; similarly to the right of q.

In fact, properties (i) and (ii) are immediate; to get property (iii) we also use that q is the only fixed point of h in $V_q(3\epsilon)$.

We are ready to find the point q and the number n_0. We start from y_{n_1}; according to Proposition 2.8 and (ii), there are $0 \leqslant l \leqslant k$ and $q_1 \in Q$ such that $y_{n_1+l} \in T_{q_1}$ (if q_1 is an attractor) or $y_{n_1+l} \in V_{q_1}(2\epsilon)$ (if it is not). In the first case we have $y_n \in V_{q_1}(2\epsilon)$ for every $n \geqslant n_1 + l$ by (i), which finishes the proof after writing $q = q_1$ and $n_0 = n_1 + l$. If q_1 is not an attractor, then $y_n \in V_{q_1}(2\epsilon)$ for every $n \geqslant n_1 + l$ is still possible, but it also may happen that $y_{n'} \notin V_{q_1}(2\epsilon)$ for some minimal number $n' > n_1 + l$. Now we apply (iii) to find a preimage of $y_{n'}$ for h in $V_{q_1}^{\pm}(\epsilon)$, and then Proposition 2.8 to get $h^k(y_{n'}) \in I_{q_1}^{\pm}$. Then $y_{n'+k} \in J_{q_1}^{\pm}$ by (ii) and, indeed, $y_n \in J_{q_1}^{\pm}$ for every $n \geqslant n' + k$ by (i). In particular, by (1), the sequence $(y_n)_{n \geqslant n'+k}$ never visits $V_{q_1}(2\epsilon)$.

Next we repeat the previous reasoning starting from $n_2 = n' + k$ to find a fixed point $q_2 \neq q_1$ and either a number n_0 such that $y_n \in V_{q_2}(2\epsilon)$ for every $n \geqslant n_0$, or a number n_3 such that $(y_n)_{n \geqslant n_3}$ never visits $V_{q_1}(2\epsilon) \cup V_{q_2}(2\epsilon)$. Hence, after using the argument finitely many times, we either get the desired q and n_0, or find a number m such that $(y_n)_{n \geqslant m}$ stays away from $\bigcup_{q \in Q} V_q(2\epsilon)$. In the latter case we use again Proposition 2.8 and (ii) to arrive at a contradiction.

This concludes the proof of Theorem A in Case 1.

Case 2. h is the identity map.

Here the analyticity of both $f(x)$ and $g(x, y)$ is essential. Firstly, by the definition, $g(x, y)$ is the restriction to I^2 of an analytic map defined in a region O containing I^2. For simplicity, we keep using the notation $g(x, y)$ to refer to this extension. Let $u : O \to \mathbb{R}$ be defined by $u(x, y) = g(x, y) - y$. Since u is analytic, the topological structure of the set $C = \{(x, y) \in O\colon\ u(x, y) = 0\}$ of zeros of u is prescribed by Lojasiewicz's theorem (a simplified statement of the theorem adequate for our purposes, together with a proof, can be found, for instance, in [14, Theorem 4.3]): either C is the whole domain O, or every point of C is locally the vertex of an r-star for some nonnegative integer r depending on the point (in fact it can be proved that r is even, but this is of no consequence here). By an *r-star*, $r \geqslant 1$, we mean a compact connected topological space X homeomorphic to $\{z \in \mathbb{C}\colon\ z^r \in I\}$. The homeomorphism maps 0 to a point $v \in X$ (a *vertex* of the star), which is unambiguously defined except in cases $r = 1$ (when X is homeomorphic to I and v is one of the endpoints of X) and $r = 2$ (when X is homeomorphic to I and v is *not* one of the endpoints of X). A *0-star* is just a single point, its vertex being the point itself.

If $C = O$, then $g(x, y) = y$ for every $(x, y) \in I^2$ and Theorem A trivially follows. Hence we can assume that the other possibility holds. Realise that the segment $\{p_0\} \times I$ is contained in C, and that except for finitely many points $(p_0, s_1), \ldots, (p_0, s_k)$, all points of the segment are locally vertexes of 2-stars in C. In particular, if $\epsilon > 0$ is fixed, then there is $\delta = \delta_\epsilon > 0$ such that if J is one of the components of $I \setminus \bigcup_{i=1}^k V_{s_i}(\epsilon)$, then $u(x, y)$ does not vanish either in $(p_0, p_0 + \delta) \times J$ or in $(p_0 - \delta, p_0) \times J$. Notice that the sign of $u(x, y)$ in these two sets need not be the same.

At the beginning of the proof we simplified the problem to assume that F is of type 1. Now observe that F^2 is also a triangular map of type 1 whose base map, f^2, has an additional property: if p is a fixed point of f^2 (or, equivalently, a fixed point of f), then $(f^2)'(p) = f'(p)f'(f(p)) = (f'(p))^2 \geqslant 0$. Thus, replacing if necessary F by its square, there is no loss of generality in assuming that the derivative of f at all its fixed points is nonnegative. In particular $f'(p_0) \geqslant 0$. At this point we exploit the analyticity of f to ensure that the sequence (x_n) converging to p_0 is eventually monotone (say decreasing). Observe finally that $g(p_0, y) = y$ for every y implies that the sequence (y_n) satisfies $y_{n+1} - y_n \to 0$. Hence, in order to prove that (y_n) converges, it suffices to prove that it has finitely many accumulation points.

If for some $\epsilon > 0$ and all sufficiently large numbers n the points y_n belong to the same component K of $I \setminus \bigcup_{i=1}^k V_{s_i}(2\epsilon)$, things are even easier. Indeed, since (x_n) is eventually decreasing and we can discard the trivial case when (x_n) eventually equals p_0, there is a number n_0 such that $(x_n, y_n) \in (p_0, p_0 + \delta_\epsilon) \times K$ for every $n \geqslant n_0$. Say that $u(x, y) > 0$ in $(p_0, p_0 + \delta_\epsilon) \times K$. Then $y_{n+1} = g(x_n, y_n) > y_n$ for every $n \geqslant n_0$, and the convergence of (y_n) follows.

Thus we can assume that if we fix $\epsilon > 0$ and K is a fixed component of $I \setminus \bigcup_{i=1}^k V_{s_i}(2\epsilon)$, then (y_n) does not eventually stay in K. We next show that in fact there is n_1 such that $y_n \notin K$ for every $n \geqslant n_1$. To prove it we can assume that $y_{n_0} \in K$

for some sufficiently large number n_0 so that $x_n \in (p_0, p_0 + \delta_\epsilon)$ and $|y_{n+1} - y_n| < \epsilon$ for every $n \geqslant n_0$. Say that $g(x, y) > y$ for every $(x, y) \in (p_0, p_0 + \delta_\epsilon) \times K$. Notice that the same is true for every $(x, y) \in (p_0, p_0 + \delta_\epsilon) \times J$, where J denotes the component of $I \setminus \bigcup_{i=1}^{k} V_{s_i}(\epsilon)$ containing K. Recall that there is no number l such that $y_n \in K$ for every $n \geqslant l$. We claim that if n_1 is the first number greater that n_0 such that $y_{n_1} \notin K$, then $y_n \notin K$ for every $n \geqslant n_1$.

We prove the claim. To begin with, we have $y_{n_0} < y_{n_0+1} < \cdots < y_{n_1}$, so y_{n_1} belongs to the right component R of $J \setminus K$. Now observe that $y_n < y_{n+1}$ whenever $y_n \in R$, and also that if y_n is to the right of R, then y_{n+1} cannot belong to J (because R has length ϵ and $|y_{n+1} - y_n| < \epsilon$). This proves the claim.

We have shown that if ϵ is given, then the points y_n belong to the union set $\bigcup_{i=1}^{k} V_{s_i}(2\epsilon)$ provided that n is large enough. Hence (y_n) can only accumulate at the points $s_1, \ldots, s_k$ as we desired to show.

4. Proofs of Theorem B and Proposition C

Proof of Theorem B. Consider the autonomous system of differential equations defined by

$$\begin{aligned} x' &= P(x) = \frac{2}{\pi(1 + \tan^2(\pi x/2))}, \\ y' &= Q(x, y) = \sin\bigl(\pi \tan(\pi x/2)\bigr) \sin(\pi y), \end{aligned} \tag{2}$$

in the xy-domain $(-1, 1) \times \mathbb{R}$. It is easy to calculate explicitly all (maximal) solutions of (2) (because the change of variable $x = 2/\pi \arctan u$ transforms it into the system $u' = 1$, $y' = \sin(\pi u)\sin(\pi y)$), which turn out to be defined for every $t \in \mathbb{R}$. For instance, those satisfying $(x(0), y(0)) = (0, r)$ with $0 < r < 1$ (whose associate trajectories—their images on $\mathbb{R}^2$—cover the open rectangle $(-1, 1) \times (0, 1)$) are given by

$$\begin{aligned} x(t) &= \frac{2}{\pi} \arctan t, \\ y(t) &= \frac{1}{\pi} \arccos\left(\frac{k_r e^{2\cos(\pi t)} - 1}{k_r e^{2\cos(\pi t)} + 1}\right), \end{aligned}$$

where

$$k_r = \frac{1 + \cos(\pi r)}{e^2(1 - \cos(\pi r))} \in (0, \infty).$$

We emphasise that the set of limit points of these solutions as $t \to \infty$ are the (nondegenerate) vertical segments $\{1\} \times [r, b_r]$, with

$$b_r = \frac{1}{\pi} \arccos\left(\frac{k_r - e^2}{k_r + e^2}\right).$$

Observe also that the trajectories corresponding to the solutions starting from $(0, 0)$ and $(0, 1)$ are, respectively, the horizontal segments $(-1, 1) \times \{0\}$ and $(-1, 1) \times \{1\}$.

We want to extend (2) to a smooth system defined on the whole plane with the same trajectories as those of (2) and the rest of points being singular (stationary) points of the new system. To do this we use C^∞ maps $\lambda_n : \mathbb{R} \to [0, 1]$ satisfying $\lambda_n(x) = 1$ whenever $|x| \leqslant (n-1)/n$ and $\lambda_n(x) = 0$ whenever $|x| \geqslant n/(n+1)$, and find positive numbers ϵ_n small enough so that all partial derivatives up to the order n of the maps $\epsilon_n \lambda_n(x)(P(x), Q(x, y))$ are bounded by $1/2^n$ (this is possible by the periodicity of $Q(x, y)$ in the second variable). Let $\lambda(x) = \sum_n \epsilon_n \lambda_n(x)$ and consider the system

$$\begin{aligned} x' &= A(x), \\ y' &= B(x, y) \end{aligned} \tag{3}$$

defined by

$$\bigl(A(x), B(x, y)\bigr) = \begin{cases} \lambda(x)(P(x), Q(x, y)) & \text{if } x \in (-1, 1), \\ (0, 0) & \text{otherwise.} \end{cases}$$

Due to the way the numbers ϵ_n have been chosen, both $A(x)$ and $B(x, y)$ are C^∞ maps. Moreover $\lambda(x)$ is a nonnegative C^∞ map vanishing exactly at $\mathbb{R} \setminus (-1, 1)$, so the trajectories of (3) in $(-1, 1) \times \mathbb{R}$ are exactly the same as those of (2).

Let $\phi(t, x_0, y_0)$ be the flow of (3), that is, $\phi(t, x_0, y_0) = (x(t), y(t))$ where $(x(t), y(t))$ is the solution of (3) such that $x(0) = x_0$, $y(0) = y_0$. It is well known that ϕ is a C^∞ map satisfying (after replacing x_0, y_0 by the more usual variables x, y) $\phi(t + s, x, y) = \phi(t, \phi(s, x, y))$ for every $t, s, x, y \in \mathbb{R}$. We conclude that $F(x, y) = \phi(1, x, y)$ is a C^∞ map whose iterates are given by $F^n(x, y) = \phi(n, x, y)$ for every nonnegative integer n. Moreover, the restriction of F to I^2 (we call it F as well) maps the square into itself and, since the first component of (3) depends only on x, the first component of F depends only on x, that is, F is a triangular map.

The dynamics of F can be easily described in terms of that of ϕ. To begin with, the vertical line $\{1\} \times I$ consists of fixed points of F. Also, every orbit $(F^n(x,0))$ and $(F^n(x,1))$ converges, respectively, to the fixed points $(1,0)$ and $(1,1)$. We claim that the orbit of every point $(x,y) \in [0,1) \times (0,1)$ accumulates at a segment $\{1\} \times [r, b_r]$ (with r depending on (x,y)), which simultaneously shows that F is of type 1 and neither of the points (x,y) are asymptotically periodic, hence finishing the proof.

To prove the claim we fix such (x,y) and find $r \in (0,1)$ and $s > 0$ such that $\phi(s,0,r) = (x,y)$. Notice that the sequence $(\phi(s+n,0,r)) = (F^n(x,y))$ accumulates at a subset of $\{1\} \times I$. Now, because of the continuity of ϕ and the fact that all points from $\{1\} \times I$ are singular for ϕ, we can find for every given $\epsilon > 0$ a number $\delta < 1$ close enough to 1 so that if $\tilde{x} \in [\delta, 1]$, $\tilde{y} \in I$, then $\|\phi(u, \tilde{x}, \tilde{y}) - (\tilde{x}, \tilde{y})\| < \epsilon$ for every $0 \leqslant u \leqslant 1$. In particular,

$$\max_{u \in [0,1]} \bigl\|\phi(s+n+u,0,r) - \phi(s+n,0,r)\bigr\| = \max_{u \in [0,1]} \bigl\|\phi\bigl(u, \phi(s+n,0,r)\bigr) - \phi(s+n,0,r)\bigr\|$$

goes to 0 as $n \to \infty$, which means that $(\phi(s+n,0,r))$ (that is, $(F^n(x,y))$) accumulates exactly at $\{1\} \times [r, b_r]$, as we desired to show. □

Proof of Proposition C. One direction of the statement is obvious and only requires continuity: if F is of type less than 2^∞, then $\mathrm{Per}(F)$ is the set of fixed points of F^{2^n} for some appropriate nonnegative integer, hence it is a closed set.

Now assume that $\mathrm{Per}(F)$ is closed and write as usual $F(x,y) = (f(x), g(x,y))$. It is well known (see, e.g., [19, Lemma 9]) that both f and F have type at most 2^∞. Now we use differentiability for the first time: every C^1 interval map of type 2^∞ has an infinite ω-limit set [13] and such sets contain no periodic points but do contain some points in the closure of the set of periodic points [6, p. 131]. Then f has type less than 2^∞.

We must prove that F also has type less than 2^∞. Replacing if necessary F by an appropriate iterate F^{2^n}, we can assume that f has type 1. Now, if F has type 2^∞, then we can find a convergent sequence (p_r) of fixed points of f such that the corresponding fibre maps $g_{p_r}(y) = g(p_r, y)$ have strictly increasing types 2^{n_r}. If p is the limit fixed point of (p_r), then g_p is of type 2^∞ (because it is of type at most 2^∞ and, according to a result by Misiurewicz [21], if a C^1 interval map is of type 2^k for some nonnegative integer k, then the type of all maps C^1-close enough to it is at most 2^{k+1}). Thus, as in the above paragraph, the closure of set of periodic points of F in $\{p\} \times I$ contains some nonperiodic points, contradicting the hypothesis on $\mathrm{Per}(F)$. □

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