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ON PICARD GROUPS OF ALGEBRAIC FIBRE SPACES

R. FOSSUM and B. IVERSEN
Mathematical Institute, University of Aarhus, Aarhus, Denmark

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0. Introduction

The purpose of this paper is to study the Picard group of algebraic fibre spaces in terms of the Picard group of the fibre and the base. Satisfactory results are obtained in case of fibrations which are locally trivial in the Zariski topology and have rational fibres.

We have divided the material in five sections:

1. Krull schemes.
2. Fibre spaces.
3. Principal homogeneous spaces:
4. Isogenies.
5. Chevalley groups.

Section 1 contains an elementary result on faithfully flat morphisms with integral fibres between Krull schemes.

Section 2 contains notably the following result: Let k denote an algebraically closed field. For an algebraic variety Y over k, let $U_k(Y)$ denote the group $\Gamma(Y, \mathcal{O}_Y^*)/k^*$, where $*$ stands for multiplicative units. Let $E \to V$ denote a fibre space with fibre F and which is locally trivial in the Zariski topology. If V and F are smooth and F is rational, then we have an exact sequence

$$0 \to U_k(V) \to U_k(E) \to U_k(F) \to \mathrm{Pic}(V) \to \mathrm{Pic}(E) \to \mathrm{Pic}(F) \to 0.$$

In Section 3, more precise results are obtained in case of a principal homogeneous space. We get the corollary that a smooth connected linear group G with the property that all its principal homogeneous spaces are locally trivial in the Zariski topology has vanishing Picard group.

In Section 4 on isogenies we obtain the following two complementary results:

(i) (Proposition 4.2) *Let $G' \to G$ be a surjective morphism between smooth connected linear algebraic groups whose kernel D' is diagonalizable. Then there exists an exact sequence*

$$0 \to X(G) \to X(G') \to X(D') \to \mathrm{Pic}(G) \to \mathrm{Pic}(G') \to 0$$

where $X(\cdot)$ *stands for character group.*

(ii) (Proposition 4.3). *Let G be a smooth connected linear algebraic group. Then there exists a morphism* $\bar{G} \to G$ *as in* (i) *whose kernel is finite (and diagonalizable) with* $\mathrm{Pic}(\bar{G}) = 0$.

Applications of these results are given.

In Section 5, we prove the following:

For a Chevalley group G over $\mathbf{Z}$ *("groupe deployé reductif"), the homomorphism* $\mathrm{Pic}(G) \to \mathrm{Pic}(G \times_{\mathbf{Z}} \mathbf{Q})$ *is an isomorphism.*

The group $\mathrm{Pic}(G \times_{\mathbf{Z}} \mathbf{Q})$ has been computed in [9] in terms of the system of roots and coroots. In particular we have, for a simple Chevalley group G, that $\mathrm{Pic}(G)$ takes the following values, depending on the type of the root system [9, C.2]:

A_n	B_n	C_n	D_{2m}	D_{2m+1}
$\mathbf{Z}/(n+1)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2) \times \mathbf{Z}/(2)$	$\mathbf{Z}/(4)$
E_6	E_7	E_8	F_4	G_2
$\mathbf{Z}/(3)$	$\mathbf{Z}/(2)$	0	0	0 .

1. Krull schemes

A scheme X is called a *Krull scheme* if it is integral and satisfies the following conditions. (Let K_X denote the function field of X and $X^{(1)}$ the set of points of X which are distinct from the generic point of X and have the generic point as their only generization).

K.1. If $p \in X^{(1)}$, then $\mathcal{O}_{X,p}$ is a discrete valuation ring. Let v_p denote the corresponding valuation of K_X.

K.2. If $t \in K_X^*$, then $v_p(t) = 0$ for all but finitely many $p \in X^{(1)}$.

K.3. If U is an open set of X, and $t \in K_X^*$ is such that $v_p(t) \geq 0$ for all $p \in U \cap X^{(1)}$, then $t \in \Gamma(U, \mathcal{O}_X)$.

Let X be a Krull scheme. Put $U(X) = \Gamma(X, \mathcal{O}_X^*)$ and let $\mathrm{Div}(X)$ denote the free abelian group on the points of $X^{(1)}$. We let $\mathrm{div}: K_X^* \to \mathrm{Div}(X)$ denote the map defined by $t \mapsto \Sigma v_p(t)p$, the sum being over all $p \in X^{(1)}$. This makes sense in virtue of K.2. The kernel of div is $U(X)$ in virtue of K.3. The cokernel of div is denoted by $\mathrm{Cl}(X)$. In résumé, we have an exact sequence

$$0 \to U(X) \to K_X^* \to \mathrm{Div}(X) \to \mathrm{Cl}(X) \to 0.$$

A morphism $f: X \to Y$ of schemes is called a *Krull morphism* if X and Y are Krull schemes, the generic point of X is mapped onto the generic point of Y and a point of $X^{(1)}$ is mapped onto the generic point of Y or a point of $Y^{(1)}$ (cf. [4, §1, condition (PDE)]).

Let $f: X \to Y$ be a Krull morphism and $p \in X^{(1)}$ such that $q = f(p) \in Y^{(1)}$. Then we put $e_f(p) = v_p(f_p(t_q))$ where t_q denotes a local parameter for $\mathcal{O}_{Y,q}$.

We let $\mathrm{Div}(f): \mathrm{Div}(Y) \to \mathrm{Div}(X)$ denote the linear map which to $q \in Y^{(1)}$ assigns $\Sigma e_f(p)p$, the sum being over all $p \in X^{(1)}$ for which $f(p) = q$. We leave it to the reader to establish the commutativity of the square

$$\begin{array}{ccc} K_Y^* & \xrightarrow{\ f\ } & K_X^* \\ \big\downarrow \mathrm{div} & & \big\downarrow \mathrm{div} \\ \mathrm{Div}(Y) & \xrightarrow{\ \mathrm{Div}(f)\ } & \mathrm{Div}(X) \end{array}$$

which allows us to define the homomorphism $\mathrm{Cl}(f): \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$. It is also left to the reader to exhibit Cl and Div as contravariant functors from the Krull category to the category of abelian groups.

Let $f: X \to Y$ be a Krull morphism with generic fibre W. Then W is a Krull scheme and the canonical map $i: W \to X$ is a Krull morphism.

For the proof we may assume that Y is affine. Then W obviously satisfies K.1 and K.2. As for the verification of K.3, we may assume that X is affine (as a consequence of K.3 applied to X). The result now follows from the general lemma:

Let A denote a Krull domain, S a multiplicative subset of A not containing 0. *Then* $S^{-1}A$ *is a Krull domain* [4, §1, No.4, Proposition 6].

Proposition 1.1. *Let $f: X \to Y$ be a morphism between Krull schemes. Suppose f is faithfully flat with integral fibres. Then f is a Krull morphism, the generic fibre W of f is a Krull scheme and we have an exact sequence*

$$0 \to U(Y) \to U(X) \to U(W)/K_Y^* \to \mathrm{Cl}(Y) \to \mathrm{Cl}(X) \to \mathrm{Cl}(W) \to 0.$$

Proof. We are first going to prove that f is a Krull morphism. For this we may assume X and Y affine. So let f be represented $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Then f faithfully flat implies that $A \to B$ is injective.

The result now follows from [4, §1, No. 10, Proposition 15]. From this and the remarks preceding Proposition 1.1 it follows that W is a Krull scheme and that the canonical map $W \to X$ is a Krull morphism.

Next we are going to prove that $f: X \to Y$ satisfies the condition:

> For all $q \in Y^{(1)}$ there exists precisely one $p \in X^{(1)}$ such that $f(p) = q$. Moreover, for that p we have $e_f(p) = 1$.

By the same argument we used in proving that W is a Krull scheme, we may make the base change $\mathrm{Spec}(\mathcal{O}_{Y,q}) \to Y$, i.e., we may assume that Y is the spectrum of the discrete valuation ring A. Let t be a local parameter for A. Then $f(t) \in \Gamma(X, \mathcal{O}_X)$ is not a unit. The map f is surjective, whence the stalk of $f(t)$ at a point $x \in X$ which is mapped onto q is a non-unit. By K.3 we can find a $p \in X^{(1)}$ such that $v_p(t) > 0$. Obviously $f(p) = q$. The rest is now clear.

From the preceding remarks, it is immediate to verify that we have the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_Y^* & \longrightarrow & K_X^* & \longrightarrow & K_W^*/K_Y^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Div}(Y) & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Div}(W) & \longrightarrow & 0\,. \end{array}$$

The snake lemma will give us the long exact sequence in question if we can identify the kernel of the verical arrow to the right with $U(W)/K_Y^*$. This, however, is accomplished by the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_Y^* & \longrightarrow & K_Y^* & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U(W) & \longrightarrow & K_W^* & \longrightarrow & \mathrm{Div}(W) & \longrightarrow & 0 \end{array}$$

and the snake lemma.

Remark 1.2. Let $f: X \to Y$ be a Krull morphism which satisfies:

> For all $q \in Y^{(1)}$ there exists precisely one $p \in X^{(1)}$ such that $f(p) = q$. Moreover, for that p we have $e_f(p) = 1$.

Then Proposition 1.1 still holds as it follows from its proof.

2. Fibre spaces

Throughout this section, k denotes an algebraically closed field. For an algebraic variety Y over k, we let $U_k(Y)$ denote the group $\Gamma(Y, \mathcal{O}_Y^*)/k^*$.

The key to the geometric case is the following lemma of Rosenlicht [12].

Lemma 2.1. *For varieties X and Y over k, the canonical map $U_k(X) \oplus U_k(Y) \to U_k(X \times Y)$ is an isomorphism.*

Proof (F. Oort). The injectivity is obvious. To prove the surjectivity, we may assume that Y is an open subset of a normal projective variety $\widetilde{Y}$ and that X is normal. Let now $u : X \times Y \to \mathrm{G}_m$ be a morphism. Then u extends to a rational function $\widetilde{u}$ on $X \times \widetilde{Y}$. Let D be the divisor of $\widetilde{u}$. Then D is of the form $p_2^* E$, where E is a divisor on $\widetilde{Y}$: If H is an irreducible component of the support of D, then $\overline{p_2(H)}$ equals all of $\widetilde{Y}$ or has codimension 1 (since p_2 is flat). We can now rule out the first possibility since $H \cap X \times Y$ is empty.

Now let x_0 and x be points of X. The rational function on $\widetilde{Y}$,

$$y \mapsto \widetilde{u}(x, y)\,\widetilde{u}(x_0, y)^{-1},$$

is constant on $\widetilde{Y}$: its divisor is $E - E = 0$ and $\widetilde{Y}$ is complete. Whence we can write

$$\widetilde{u}(x, y) = \widetilde{u}(x_0, y)\,v(x),$$

where v is a nowhere vanishing function on X. This v is obviously regular.

Corollary 2.2 (Rosenlicht [12]). *Let G be a smooth connected algebraic group over k and T a torus. Then any morphism of varieties $f: G \to T$ with $f(e) = e$ is a morphism of algebraic groups.*

Proof. We may assume $T = \mathbf{G}_m$. We can find morphisms h_1 and h_2 from G to $\mathbf{G}_m$ such that $f(g_1, g_2) = h_1(g_1)h_2(g_2)$. Modifying h_1 and h_2 with a constant, we may assume $h_1(e) = h_2(e) = e$. Substitute $g_i = e$ to see that $h_i = f$.

Proposition 2.3. *Let $f: E \to V$ denote a fibre space with fibre F, locally trivial in the Zarisky topology. Let $j: F \to E$ induce an isomorphism between F and a fibre of f. If V and F are smooth and F is rational, then there exists an exact sequence*

$$0 \to U_k(V) \xrightarrow{U(f)} U_k(E) \xrightarrow{U(j)} U_k(F) \to \mathrm{Pic}(V) \xrightarrow{\mathrm{Pic}(f)} \mathrm{Pic}(E)$$
$$\xrightarrow{\mathrm{Pic}(j)} \mathrm{Pic}(F) \to 0.$$

Proof. Let $v \in V$ denote the point onto which $f \cdot j$ maps F and let E_v denote the fibre of f at v ($j: F \xrightarrow{\sim} E_v$). The proof is based on the existence of specialization maps (let g denote the generic point of V)

$$s_v : U(E_g)/K_V^* \to U(E_v)/k^*, \qquad t_v : \mathrm{Pic}(E_g) \to \mathrm{Pic}(E_v).$$

Put $E_{(v)} = E \times_V \mathrm{Spec}(\mathcal{O}_{V,v})$, and let $j_v : E_v \to E_{(v)}$ denote the map induced by the inclusion of E_v in E. Applying Proposition 1.1 to $p_2 : E_{(v)} \to \mathrm{Spec}(\mathcal{O}_{V,v})$, and noting that $\mathrm{Cl}(\mathrm{Spec}(\mathcal{O}_{V,v})) = 0$, we get an exact sequence

$$U(\mathcal{O}_{V,v}) \to U(E_{(v)}) \to U(E_g)/K_V^* \to 0$$

and an isomorphism

$$\mathrm{Pic}(E_{(v)}) \xrightarrow{\sim} \mathrm{Pic}(E_g).$$

Noting that the composite $U(\mathcal{O}_{V,v}) \to U(E_{(v)}) \to U_k(E_v)$ is zero, we can factor the last map through $U(E_g)/K_V^*$. The result is $s_v : U(E_g)/K_V^* \to U_k(E_v)$. Similarly t_v : $\mathrm{Pic}(E_g) \to \mathrm{Pic}(E_v)$ is obtained by composing the inverse of the isomorphism

$\mathrm{Pic}(E_{(v)}) \to \mathrm{Pic}(E_g)$ obtained above and $\mathrm{Pic}(j_v)$.

It is clear that we have commutative diagrams

$$\begin{array}{ccc} U_k(E) & \longrightarrow & U(E_g)/K_V^* \\ & \searrow & \downarrow s_v \\ & & U_k(E_v) \end{array} \qquad \begin{array}{ccc} \mathrm{Pic}(E) & \longrightarrow & \mathrm{Pic}(E_g) \\ & \searrow & \downarrow t_v \\ & & \mathrm{Pic}(E_v) \end{array}$$

where the unlabelled arrows are the obvious ones.

Applying Proposition 1.1 to $f: E \to V$, we see that the specialization maps are isomorphisms in this case. In order to prove this it suffices to treat the case $p_1: V \times F \to V$. We have $E_g = K_V \times F$. We remark that the composite of the map $U_k(F) \to U(F \times K_V)/K_V^*$ induced by $p_2 : K_V \times F \to F$ and s_v is the identity. A similar remark applies to t_v. Whence it suffices to prove the next lemma.

Lemma 2.4. *Let F denote a normal variety over the algebraically closed field k and let L denote a finite type field extension of k. Then $U(F)/k^* \to U(F \times_k L)/L^*$ is an isomorphism. If moreover L is a purely transcendental extension of k, then $\mathrm{Cl}(F) \to \mathrm{Cl}(F \times_k L)$ is an isomorphism.*

Proof. The well-known fact that the tensor product of two integral domains over an algebraically closed field is an integral domain ensures that $F \times_k L$ is an integral scheme and that the fibres of $p_1 : F \times_k L \to F$ are integral.

According to [3, §1, No.7], $F \times_k L$ is a Krull scheme. The projection p_1 is faithfully flat, so we may apply Proposition 1.1 to get an exact sequence

$$0 \to U(F) \to U(F \times_k L) \to U(K_F \otimes_k L)/K_F^* \to \mathrm{Cl}(F) \to \mathrm{Cl}(F \times_k L)$$
$$\to \mathrm{Cl}(K_F \otimes_k L) \to 0.$$

Let us remark that if A and B are integral domains over k, then the canonical map $U_k(A) \oplus U_k(B) \to U_k(A \otimes_k B)$ is an isomorphism as follows by applying Lemma 2.1 to the finite type subalgebras of A and B.

This remark implies that $U(F \times_k L) \to U(K_F \otimes_k L)/K_F^*$ is surjective. Modifying it slightly we also get that $U(F)/k^* \to U(F \times_k L)/L^*$ is surjective.

The long exact sequence above now gives the short exact sequence

$$0 \to \mathrm{Cl}(F) \to \mathrm{Cl}(F \times_k L) \to \mathrm{Cl}(K_F \otimes_k L) \to 0.$$

Now if F is rational of dimension n, we have

$$K_F \otimes_k L \cong k(X_1, ..., X) \otimes_k L$$

which is a factorial ring, that is, $\mathrm{Cl}(K_F \otimes_k L) = 0$.

Corollary 2.5. *Let X and T be smooth varieties over k, $(L_t)_{t \in T}$ an algebraic family of line bundles on X. If X is rational, then $(L_t)_{t \in T}$ is constant.*

Proof. For a $t \in T$, let $i_t : X \to X \times T$ denote the map given by $i_t(x) = (x, t)$. Let L be a line bundle on $X \times T$ such that $i_t^* L = L_t$. Put $L_0 = L_{t(0)}$ for some fixed $t(0) \in T$. Consider the sheaf $L \otimes p_1^* L_0^{-1}$. It follows from Proposition 2.4 that $L \otimes p_1^* L_0^{-1}$ is the pullback of an invertible sheaf on T along p_2.

In [10, §13], Mumford implicitly establishes the following result:
Let the smooth connected linear algebraic group G act on the normal, projective variety V. Then there exists an exact sequence

$$0 \to X(G) \to \mathrm{Pic}^G(V) \to \mathrm{Pic}(V) \to \mathrm{Pic}(G).$$

Using precisely the same technique, it is easy to establish the same sequence when V instead of normal and projective is smooth and rational. Namely, G acts trivially on $\mathrm{Pic}(V)$ in this case, as follows from Corollary 2.5.

Let us also remark that the special case of Proposition 2.3 where V is quasi-projective and E is a vector bundle can be found in [5, 4 –35]. This was the departure of our investigations. Various cases of exactness of the middle part of the exact sequence of Proposition 2.3 are established by Raynaud in [11, p. 106].

3. Principal homogeneous spaces

Let H be a smooth, connected algebraic group over k and $f : E \to V$ a principal homogeneous (right) H-space locally trivial in the Zariski topology.

Let $c : X(H) \to \mathrm{Pic}(V)$ denote the map which to a character $\chi \in X(H)$ associates the line bundle $L(\chi) \to V$ obtained as follows. The space $L(\chi)$ is the orbit space for the action of H on $E \times \mathbf{G}_a$ given by $(y, x)h = (y, h, \chi(h^{-1})x)$. The first projection $E \times \mathbf{G}_a \to E$ induces $L(\chi) \to V$.

Now let e be a point of E and let $i_e : H \to E$ denote the map given by $i_e(h) = eh$. Let us remark that, by Corollary 2.2, we may identify $U_k(H)$ and $X(H)$.

Proposition 3.1. *With the notation above, if H is a linear algebraic group and V is smooth, then the following sequence is exact*

$$0 \to U_k(V) \xrightarrow{U(f)} U_k(E) \xrightarrow{U(i_e)} X(H) \xrightarrow{c} \mathrm{Pic}(V) \xrightarrow{\mathrm{Pic}(f)} \mathrm{Pic}(E)$$
$$\xrightarrow{\mathrm{Pic}(i_e)} \mathrm{Pic}(H) \to 0.$$

Proof. Let us first recall that a smooth connected linear algebraic group is rational. [1, 15.8]. The exactness of the two rows is contained in Proposition 2.3. We can now proceed by identifying c with the corresponding map of Proposition 2.3. We think, however, that it is more illustrative to proceed directly. It is then convenient to work with principal homogeneous $\mathbf{G}_m$-bundles instead of line bundles. So for a given $\chi \in X(H)$, let $L^*(\chi)$ denote the orbit space for the action of H on $E \times \mathbf{G}_m$ given by $(y, x)h = (yh, \chi(h^{-1})x)$. The projection $p_1 : E \times \mathbf{G}_m \to E$ will induce a map

$L^*(\chi) \to V$ and the action of G_m on the second factor will induce the structure of principal homogeneous G_m-space. Recall that $L^*(\chi)$ is trivial if and only if it has a global section.

Proof that $U_k(E) \to X(H) \to \mathrm{Pic}(V)$ *is exact*: Let $\chi \in X(H)$. A prolongation of χ to all of E is easily seen to define a global section in $L^*(\chi)$. Conversely, suppose that $L^*(\chi)$ admits a global section over V. The corresponding section over E of the pull-back of $L^*(\chi)$ along f (which we may identify with $E \times G_m$) has the form $y \mapsto (y, t(y))$, where $t : E \to G_m$ satisfies

$$t(yh) = t(y)\,\chi(h^{-1}).$$

Substitute $y = e$ to obtain the desired prolongation of χ.

Proof that $X(H) \to \mathrm{Pic}(V) \to \mathrm{Pic}(E)$ *is exact*: Let D be a principal homogeneous G_m-bundle on V. If $D = L^*(\chi)$ for some $\chi \in X(H)$, it obviously has a section over E. Conversely, suppose f^*D has a section s, say. Interpret s as a map from E to D. Consider the map $a : E \times H \to G_m$ defined by $s(gh) = s(y)a(y, h)$. According to Lemma 2.1 and Corollary 2.2, a has the form $a(y, h) = b(y)\,\chi(h)$, where $b \in U(E)$ and $\chi \in X(H)$. Recapitulating, $s(yh) = s(y)b(y)\chi(h)$. Substituting $h = e$, one sees that b is constant 1. The map $(y, z) \mapsto s(y)\chi(z)$ will now induce an isomorphism $D \cong L(\chi)$.

We are now going to give some general remarks concerning Proposition 3.1.

If H is a smooth connected solvable linear algebraic group, then all principal homogeneous spaces (locally trivial in the faithfully flat topology) are locally trivial in the Zariski topology [8, IV, §4, 3.7]. Moreover $\mathrm{Pic}(H) = 0$ in that case [8, IV, §4, 3.8].

If G is a smooth connected linear algebraic group and P a parabolic subgroup, then $G \to G/P$ is locally trivial in the Zariski topology. The case in which G is reductive can be found in [2, 4.13]. The general case follows immediately by applying the above remark to $G \to G/R_u(G)$, where $R_u(G)$ is the unipotent radical.

The exact sequence we obtain,

$$0 \to X(G) \to X(P) \to \mathrm{Pic}(G/P) \to \mathrm{Pic}(G) \to \mathrm{Pic}(P) \to 0,$$

generalizes the one obtained in [9, Corollary 3].

Corollary 3.2. *Let G be a smooth connected linear algebraic group with the property that all principal homogeneous G-spaces (in the étale topology) are locally trivial in the Zariski topology (a "special" group in the terminology of [5, exposé 1]). Then* $\mathrm{Pic}(G) = 0$.

Proof. Choose a closed immersion $G \to \mathrm{Gl}_n$. The fibration $\mathrm{Gl}_n \to \mathrm{Gl}_n/G$ is locally trivial in the Zariski topology by assumption. Proposition 3.1 gives the exact sequence

$$\mathrm{Pic}(\mathrm{Gl}_n/G) \to \mathrm{Pic}(\mathrm{Gl}_n) \to \mathrm{Pic}(G) \to 0.$$

But $\mathrm{Pic}(\mathrm{Gl}_n) = 0$, whence $\mathrm{Pic}(G) = 0$.

A semi-simple group G with $\mathrm{Pic}(G) = 0$ is simply connected as will be seen in Corollary 4.5 below. For a complete classification of special semi-simple groups see [5, Exposé 5].

4. Isogenies

First the group-theoretical part.

Lemma 4.1. *Let G denote a smooth connected linear algebraic group and D a normal diagonalizable subgroup (D need not be smooth). Then D is contained in any maximal torus of G.*

Proof. Let T be a maximal torus. Let T operate on D by inner conjugation. By rigidity of diagonalizable groups, this operation is trivial, that is, $D \subseteq Z_G(T)$, the fixed point scheme for inner conjugation of T on G. Now $Z_G(T)$ is smooth by [8, §5, 2.8] and connected by [1, 11.12]. In conclusion, D is contained in the Cartan subgroup C corresponding to T. But C is nilpotent [1, 11.7] and therefore a product of T and a unipotent group. Since D is diagonalizable we get $D \subseteq T$.

Proposition 4.2. *Let $f: G' \to G$ be a surjective morphism between smooth connected linear algebraic groups whose kernel D' is diagonalizable. Then there exists an exact sequence*

$$0 \to X(G) \to X(G') \to X(D') \to \mathrm{Pic}(G) \to \mathrm{Pic}(G') \to 0.$$

Proof. Pick a maximal torus T of G. Then $T' = f^{-1}(T)$ is a maximal torus of G': Let T_1 be a maximal torus of G'. Then $f(T_1)$ is a maximal torus of G by [1, 11.14]. Whence, by the conjugacy theorem for maximal tori, we can find a conjugate T_2 of T_1 such that $f(T_2) = T$. Now T_2 contains D' by Lemma 4.1; therefore $T_2 = f^{-1}(T) = T'$.

Pick a Borel subgroup B of G, containing T. Then $B' = f^{-1}(B)$ is a Borel subgroup of G': Let B_1 be any Borel subgroup of G' containing T'. Now $f(B_1)$ is a Borel subgroup of G by [1, 11.14]. By the conjugacy theorem for Borel subgroups, we can find a conjugate B_2 of B_1 such that $f(B_2) = B$; $B_2 \supseteq D'$, whence $B_2 = B' = f^{-1}(B)$.

The exact, commutative diagram below and the snake lemma will conclude the proof:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & X(G) & \longrightarrow & X(G') & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & X(T) & \longrightarrow & X(T') & \longrightarrow & X(D') & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Pic}(G/B) & \xrightarrow{\simeq} & \mathrm{Pic}(G'/B') & \longrightarrow & 0 & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & \\
 & \mathrm{Pic}(G) & \longrightarrow & \mathrm{Pic}(G') & & & \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0. & & &
\end{array}$$

Proposition 4.3. *Let G be a smooth connected linear algebraic group. Then there exists an isogeny $\pi : \bar{G} \to G$ with diagonalizable kernel, wnere $\bar{G}$ is a smooth connected lienar algebraic group with* $\mathrm{Pic}(\bar{G}) = 0$.

Proof. Let T be a maximal torus of G and B a Borel subgroup of G containing T. The proof of [9, Theorem 3.7] shows that there exists an isogeny $\pi : \bar{G} \to G$ with diagonalizable kernel such that $\bar{B} = f^{-1}(B)$ (resp. $\bar{T} = f^{-1}(T)$) is a Borel subgroup (resp. a maximal torus) of $\bar{G}$ (by the proof of Proposition 4.2) and such that $X(\bar{T}) \to \mathrm{Pic}(\bar{G}/\bar{B})$ is surjective.

By virtue of Proposition 3.1, we have an exact sequence

$$X(\bar{T}) \to \mathrm{Pic}(\bar{G}/\bar{B}) \to \mathrm{Pic}(\bar{G}) \to 0.$$

The conjugation of Propositions 4.2 and 4.3 is very powerful. Let us illustrate this by a series of more or less well-known theorems.

Corollary 4.4. *If G is a smooth connected linear algebraic group, then* $\mathrm{Pic}(G)$ *is a finite group.*

Proof. Apply Proposition 4.2 to the isogeny $\pi : \bar{G} \to G$ of Proposition 4.3. Another proof using intersection theory can be found in [5, 5–21].

Let us now make a general remark on Proposition 3.1. The three groups in the first row are finitely generated and free [12, Theorem 1]. The last one in the second row is finite.

We are now going to generalize the construction of universal covering spaces. An isogeny $f : G' \to G$ is called *special* if G and G' are smooth connected linear groups and the kernel of f is finite and diagonalizable. A linear algebraic group G is called *simply connected* if any special isogeny $G' \to G$ is an isomorphism.

Corollary 4.5. *Let G be a smooth connected linear algebraic group with $X(G) = 0$. Then G is simply connected if and only if* $\mathrm{Pic}(G) = 0$.

Proof. Suppose G is simply connected. By Proposition 4.3, we can find a special isogeny $\pi : \bar{G} \to G$ with $\mathrm{Pic}(\bar{G}) = 0$. Now π has to be an isomorphism. Conversely, suppose $\mathrm{Pic}(G) = 0$, and let $G' \to G$ be a special isogeny with kernel D'. Proposition 4.2 gives us the exact sequence

$$0 \to X(G') \to X(D') \to 0.$$

Now $X(G')$ is torsion free and $X(D')$ is finite, whence $X(D') = 0$, and therefore $D' = 0$ since D' is diagonalizable.

Corollary 4.6. *Let G be a smooth connected linear algebraic group with $X(G) = 0$. Then there exists a special isogeny $\widetilde{G} \to G$ with $\widetilde{G}$ simply connected (the universal* *covering space). Let $\pi_1(G)$ denote the kernel of $\widetilde{G} \to G$. Then*

$$\mathrm{Pic}(G) \xrightarrow{\sim} X(\pi_1(G)).$$

Proof. Let $\widetilde{G} \to G$ be a special isogeny with $\mathrm{Pic}(G) = 0$ and let $\pi_1(G)$ denote its kernel. Proposition 4.2 gives us the exact sequence

$$0 \to X(\widetilde{G}) \to X(\pi_1(G)) \to \mathrm{Pic}(G) \to 0,$$

where $X(\widetilde{G})$ is torsion free and $X(\pi_1(G))$ is finite; whence $X(\widetilde{G}) = 0$. According to Corollary 4.5, $\widetilde{G}$ is simply connected. Consequently, $\mathrm{Pic}(G) \xrightarrow{\sim} X(\pi_1(G))$.

Corollary 4.7. *Let G be a reductive algebraic group. Then there exists a central isogeny $G' \times T \to G$, where G' is semisimple and T is a torus.*

Proof. Let T denote the reduced connected centre of G and let G' denote the universal covering space of the semi-simple group G/T. It suffices to prove that $G' \to G/T$ can be factored through $G \to G/T$. Pulling $G \to G/T$ back along $G' \to G/T$, it suffices to prove:

Let H be a simply connected semi-simple group and $f: K \to H$ a morphism of a linear algebraic group. If Ker(f) *is diagonalizable, then f admits a section (as morphism of algebraic groups).*

Proof. Replacing K by K^0_{red}, we may assume that K is smooth and connected. Proposition 4.2 yields an exact sequence

$$0 \to 0 \to X(K) \to X(\mathrm{Ker}(f)) \to 0 \to 0.$$

Recall that $K/[K, K]$ is diagonalizable [1, 14.2]. Whence $\mathrm{Ker}(f) \to K/[K, K]$ is an isomorphism, i.e., the inclusion of Ker(f) in K has a retraction r, say. The map $K \to K$ defined by $x \to r(x^{-1})x$ is a group homomorphism and factors through $f: K \to H$ to give the required section.

5. Chevalley groups

By a *Chevalley group G* we understand a reductive group scheme over **Z** which has a maximal torus T which is diagonalizable over **Z** (see [7]).

Proposition 5.1. *Let G be a Chevalley group. Let $\mathcal{R} = (M, M^*, R, R \to M^*)$ be its system of roots and coroots ("deploiement de G relative a T" [7, 3.16]) and let S be a basis for $\mathcal{R}$. Then* Pic(G) *is isomorphic to the cokernel of $M \to \mathbf{Z}^S$, $m \to (\langle s^*, m\rangle)_{s\in S}$.*

Proof. The canonical map $G \to \mathrm{Spec}(\mathbf{Z})$ is smooth with non-empty integral fibres. It follows that G is a regular scheme. In particular it is a Krull scheme and we may identify Pic(G) and Cl(G). Let $G_{\mathbf{Q}}$ denote the base extension of G to $\mathbf{Q}$. The exact sequence of Proposition 1.1 gives us an exact sequence

$$\mathrm{Pic}(\mathbf{Z}) \to \mathrm{Pic}(G) \to \mathrm{Pic}(G_{\mathbf{Q}}) \to 0.$$

The group Pic($G_{\mathbf{Q}}$) is evaluated as above in [9, Appendix C] (see the remarks at the end of the introduction).

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