Relative Hopf Modules—Equivalences and Freeness Criteria

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INTRODUCTION

Let $A$ be a Hopf algebra over a field $k$ and let $B \subseteq A$ be a Hopf subalgebra. The notion of right $(A, B)$-Hopf modules was introduced by me [1] and the category of those modules was studied to prove that $A$ is a faithfully flat $B$-module, if $A$ is either commutative or cocommutative. Recently Kadford [9] used this notion to know when $A$ is a free (or projective) $B$-module.

In this paper we generalize the notion of relative Hopf modules in two directions, and apply it to obtain many freeness or projectivity criteria for $A$ over $B$. First, we note that right $(A, B)$-Hopf modules are defined, if only $B$ is a right coideal subalgebra of $A$, which means that $B$ is such a subalgebra that $\Delta(B) \subseteq B \otimes A$. Dually, we can define left $(\pi(A), A)$-Hopf modules, where $\pi: A \to \pi(A)$ is a surjection of coalgebras and left $A$-modules.

In Section 1, we show that the category of right $(A, B)$-Hopf modules is equivalent to some comodule category, and the category of left $(\pi(A), A)$-Hopf modules is equivalent to some module category, with a little assumption on flatness. The theorem of Sweedler, which states that the category of right $(A, A)$-Hopf modules is equivalent to the category of $k$-vector spaces, follows from this. These equivalences are applied in the commutative case to prove that there is a 1-1 correspondence $B \leftrightarrow \bar{A}$ between right coideal subalgebras over which $A$ is a faithfully flat module and quotient Hopf algebras over which $A$ is a faithfully coflat left (or equivalently right) comodule. This means that if $G$ is an affine $k$-group scheme and $H \subseteq G$ a closed subgroup scheme, then the dual $k$-sheaf of left cosets $\overline{H \backslash G}$ [7, Chap. III, Sect. 3, 7.2] is affine if and only if the affine ring $O(G)$ is a faithfully coflat left or (and right $O(H)$-comodule. In the cocommutative case, it follows that there is a 1-1 correspondence $B \leftrightarrow \bar{A}$ between Hopf subalgebras and quotient left $A$-module coalgebras over which $A$ is a faithfully coflat comodule.

In Section 2, using the equivalences, we give the following freeness and projectivity criteria for relative Hopf modules. Let $B$ be a right coideal subalgebra
of $A$. Then each right $(A, B)$-Hopf module is, in particular $A$ is, a projective $B$-module, if either

1. $B$ is contained in the center of $A$, and $A$ is a faithfully flat $B$-module,
2. $B$ is a Hopf subalgebra contained in the center, and $A$ has a co-commutative coradical, or
3. $B$ is a Hopf subalgebra and $A$ is commutative.

Thus commutative Hopf algebras are projective over Hopf subalgebras. This generalizes Radford [9, Theorem 3] who proved this in case the coradical $A_c$ is a Hopf subalgebra and $B_0 = B \cap A_0$ is finitely generated. We prove that each right $(A, B)$-Hopf module is a free $B$-module, if either

4. (1), and $B^c$ is contained in the radical of $A$.
5. (1), $B/\text{Rad}(B)$ is Artinian, and $A/\text{Rad}(B).A$ is commutative,
6. $B$ is a Hopf subalgebra containing the coradical of $A$,
7. $B$ is a Hopf subalgebra and $A$ is pointed, or
8. $B$ is a locally finite Hopf subalgebra, and $A$ is commutative.

Case (5) generalizes [9, Theorem 1] which proved the same in case $B$ is finite dimensional. Radford [8] proved that $A$ is a free $B$-module with either assumption (6) or (7). His method is available for relative Hopf modules.

In Section 3, we fix a central right coideal subalgebra $D$ of $A$, and consider a correspondence $B \leftrightarrow \tilde{B}$ between intermediate right coideal subalgebras $D \subset B \subset A$ and right coideal subalgebras $B \subset \tilde{A}$, where $\tilde{A} = A/AD^c$ is a quotient Hopf algebra of $A$. If $A$ over $D$ is faithfully flat, there exists such a 1–1 correspondence. If $B$ and $\tilde{B}$ correspond to each other, it is shown that the category of right $(A, B)$-Hopf modules is equivalent to the category of right $(\tilde{A}, \tilde{B})$-Hopf modules.

In Section 4, we assume $A$ is commutative and prove the following results:

**Theorem.** If $k$ is perfect, $A$ is commutative reduced, and $B \subset A$ is a pointed Hopf subalgebra, then $A$ is a free $B$-module.

**Theorem.** If $A$ is a commutative pointed Hopf algebra, there is a 1–1 correspondence $V \leftrightarrow V^*\Lambda$ between right coideal subalgebras over which $A$ is free, and Hopf ideals of $A$.

In terms of group schemes, the last theorem means the fact that: If $G$ is a representationally solvable affine $k$-group scheme and $H \subset G$ is a closed subgroup scheme, then the quotient sheaf (with respect to the faithfully flat topology) $H^\Lambda G$ is always affine, and the projection $G \rightarrow H^\Lambda G$ is a free map.
Examples of commutative Hopf algebras which are not free over some Hopf subalgebras are known in [11]. We give a much simpler example of a torus which is not free over certain quotient tori, in the final section.

All vector spaces are over a fixed field \( k \). We follow the notation of [12] throughout.

1. Equivalences of Relative Hopf Modules

We use the sigma notation. Thus, if \( C \) is a coalgebra, we write \( \Delta(c) = \sum (c) e(i) \otimes e(j) \), \( c \in C \). If \( P \) (resp. \( Q \)) is a right (resp. left) \( C \)-comodule, the structure is written as

\[
x \mapsto \sum_{(x)} x_{(0)} \otimes x_{(1)} \left( \text{resp. } y \mapsto \sum_{(y)} y_{(-1)} \otimes y_{(0)} \right)
\]

for \( x \in P \) (resp. \( y \in Q \)). The co-tensor product \( P \Box_C Q \) denotes the subspace

\[
\left\{ \sum_{i} x_i \otimes y_i \in P \otimes Q \mid \sum_{i} x_{i(0)} \otimes x_{i(1)} \otimes y_i = \sum_{i} x_i \otimes y_{i(-1)} \otimes y_{i(1)} \right\}.
\]

We denote by \( \mathcal{M}^C \) and \( \mathcal{M} \) the categories of right \( C \)-comodules and left \( C \)-comodules, respectively.

A right \( C \)-comodule \( P \) is called (faithfully) coflat, if the functor \( P \Box_C ? \) is (faithfully) exact. It is known [3, A.2.1] that this is equivalent to the fact that \( P \) is an injective (cogenerator) of \( \mathcal{M}^C \).

Let \( A \) be a fixed Hopf algebra over \( k \). For each subalgebra \( B \subset A \) and quotient coalgebra \( \pi: A \to \pi(A) \), let

\[
\mathcal{M}_B = \text{the category of right } B \text{-modules},
\]

\[
\mathcal{M}_L = \text{the category of left } B \text{-modules},
\]

\[
\mathcal{M}^\pi = \text{the category of right } \pi(A) \text{-comodules},
\]

\[
\mathcal{M} = \text{the category of left } \pi(A) \text{-comodules}.
\]

For each \( P \in \mathcal{M}^\pi \) and \( Q \in \mathcal{M} \) we write \( P \Box_\pi Q \) for \( P \Box_{\pi(A)} Q \).

\( B \) is a right coideal subalgebra, if \( \Delta(B) \subset B \otimes A \). Then, we define an Abelian category \( \mathcal{M}_B^A \). The objects are vector spaces \( M \) over \( k \) equipped with a right \( A \)-comodule structure \( (M, \omega) \) and a right \( B \)-module structure such that \( \omega(mb) = \sum m_{(0)} b_{(0)} \otimes m_{(1)} b_{(1)} \) for \( m \in M \) and \( b \in B \). Morphisms are right \( A \)-comodule maps which are right \( B \)-comodule maps. \( \mathcal{M}_B^A \) is the category of right \( (A, B) \) Hopf modules.

Let \( \pi: A \to \pi(A) \) be a quotient left \( A \)-module coalgebra. (This means that \( \ker(\pi) \) is a coideal and a left ideal.) We can dually define the category \( \mathcal{M}^\pi \) of left \( \pi(A), A \)-Hopf modules. The objects are left \( \pi(A) \)-comodules \( (N, \omega) \).
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which are left $A$-modules so that $\omega(an) = \sum a_{(1)n_{(-1)}} \otimes a_{(\omega)n_{(\omega)}}$, where $\omega(n) = \sum n_{(-1)} \otimes n_{(\omega)}$, for $a \in A, n \in N$. Morphisms are left $\pi(A)$-comodule maps which are $A$ linear.

**Proposition 1.** If $B \subset A$ is a right coideal subalgebra, $A; AB^-$ (where $B^- = B \cap \text{Ker}(e)$) is a quotient left $A$-module coalgebra of $A$. Let $\pi_B; A \rightarrow A, AB^-$ be the projection. If $\pi; A \rightarrow \pi(A)$ is a quotient left $A$-module coalgebra, $B_\pi = \{a \in A \mid \sum \pi(a_{(1)}) \otimes a_{(\omega)} = \pi(1) \otimes a\}$ is a right coideal subalgebra of $A$.

**Proof.** If $a, b \in B_\pi$, $\sum \pi(a_{(1)})b_{(1)} \otimes a_{(\omega)}b_{(\omega)} = \sum a_{(1)}\pi(b_{(1)}) \otimes a_{(\omega)}b_{(\omega)} = \sum a_{(1)}\pi(l) \otimes a_{(\omega)}b = \sum \pi(a_{(1)}) \otimes a_{(\omega)}b = 1 \otimes ab$, hence $ab \in B_\pi$. The others are clear. Q.E.D.

In the following, let $B \subset A$ be a right coideal subalgebra.

If $S$ is a vector space, then $S \otimes A \in \mathcal{M}_B A$, where

$$\omega(s \otimes a) = s \otimes \Delta(a)$$

and

$$(s \otimes a)b = s \otimes ab$$

for $s \in S, a \in A, b \in B$.

Let $\pi = \pi_B$. If $S \in \mathcal{M}_B$, then it is easy to see that $S \subset_\pi A$ is an $A$-subcomodule and a $B$-submodule of $S \otimes A$. Let $\Psi(S) = S \subset_\pi A$. Then

$$\Psi; \mathcal{M}_B \rightarrow \mathcal{M}_B A, \quad \Psi(S) = S \subset_\pi A,$$

is a functor.

The functor $\Psi$ has the following left adjoint. Let $M \in \mathcal{M}_B A$. It is a right $\pi(A)$-comodule through $\pi$. $MB^-$ is a $\pi(A)$-subcomodule, hence $M = M; MB^-$ is a right $\pi(A)$-comodule. Let $\Phi(M) = \overline{M}$ be this comodule. Then

$$\Phi; \mathcal{M}_B A \rightarrow \mathcal{M}_B, \quad \Phi(M) = M; MB^-,$$

is obviously a functor.

We exhibit an adjoint relation $\Phi \rightarrow \Psi$. There is a $1-1$ correspondence $f \leftrightarrow F$ between the $\pi(A)$-comodule maps $f; M \rightarrow S$ and the $A$-comodule maps $F; M \rightarrow S \subset_\pi A$ defined as follows. $F$ is induced by

$$M \rightarrow M \otimes A \rightarrow S \otimes A$$

and $f = (f \otimes e) \cdot F$. It is easy to see that $F$ is right $B$-linear if and only if $f(MB^-) = 0$. Thus we have a natural isomorphism

$$\mathcal{M}_B(S, M) \simeq \mathcal{M}_B(M, S \subset_\pi A).$$

This gives the adjunction $\Phi \rightarrow \Psi$. 
For each $M \in \mathcal{M}^A$ and $N \in \mathcal{M}$, we have the fundamental isomorphism
\[ \xi: M \otimes N \cong M \otimes N, \quad \xi(m \otimes n) = \sum m_{(1)} \otimes m_{(2)} n. \]

If $M \in \mathcal{M}_B^A$, this induces
\[ \xi: M \otimes_B N \cong \Phi(M) \otimes N. \]

Hence in particular
\[ \xi: A \otimes_B N \cong \pi(A) \otimes N. \]

If $N$ is a flat left $B$-module, applying $S \boxtimes \pi$ where $S \in \mathcal{M}^e$, we get by [2, 1.3]
\[ \Psi(S) \otimes_B N \cong S \diamond N. \]

**Theorem 1.** Let $B \subseteq A$ be a right coideal subalgebra and let $\pi = \pi_B$. Suppose there is a left $A$-module $N$ which is a faithfully flat left $B$-module. Then $\Phi$ and $\Psi$ establish an equivalence between $\mathcal{M}_B^A$ and $\mathcal{M}^e$. We have $B = B_\pi$ and $A$ is a faithfully coflat left $\pi(A)$-comodule.

Note that $\Phi(B) = \pi(k)$ implies $B = B_\pi$ and $\Psi$ faithfully exact implies the last statement.

**Proof.** Let $M \in \mathcal{M}_B^A$ and $S \in \mathcal{M}^e$. Suppose $f: M \to S$ in $\mathcal{M}^e$ corresponds to $F: M \to S \boxtimes \pi A$ in $\mathcal{M}_B^A$. Then we have a commutative diagram:
\[
\begin{array}{ccc}
M \otimes_B N & \cong & M \otimes N \\
\downarrow f \otimes 1 & & \downarrow f \otimes 1 \\
(S \boxtimes \pi A) \otimes_B N & \cong & S \diamond N.
\end{array}
\]

Since $N$ is left faithfully flat over $B$, $f$ is an isomorphism if and only if $F$ is. This implies that the adjunctions
\[
m \mapsto \sum m_{(0)} \otimes m_{(1)},
\]
are isomorphisms. Hence $\Phi$ and $\Psi$ are equivalences.

In case $B = A$, the above result reduces to [12, Theorem 4.1.1].

We dualize the above theorem in the following.

**Proposition 2.** Let $\pi: A \to \pi(A)$ be a quotient left $A$-module coalgebra, and let $B = B_{\pi}$. If $N \in A^{\mathcal{M}}$, let $N^\pi = \{ n \in N : \sum n_{(-1)} \otimes n_{(0)} = \pi(1) \otimes n \}$ which
is a left B-module. If $T \in \mathfrak{B}$, $A \otimes_B T$ is a left $(\pi(A), A)$-Hopf module, where the $\pi(A)$-comodule structure is given by

$$\omega(a \otimes t) = \sum \pi(a_{(1)}) \otimes a_{(2)} \otimes t$$

for $a \in A$, $t \in T$. The functor

$$T \mapsto A \otimes_B T, \quad \mathfrak{B} \to \mathcal{A}^\pi,$$

is a left adjoint to

$$N \mapsto N^e, \quad \mathcal{A}^\pi \to \mathfrak{B}.$$

The proof is easy and omitted.

**Theorem 2.** With the notation as above, suppose there is a right $A$-comodule which is faithfully coflat as a right $\pi(A)$-comodule. Then the above functors are equivalences, we have $\pi = \pi_B$, and $A$ is a faithfully flat right B-module. In particular, if $A$ is a faithfully coflat right $\pi(A)$-comodule, this is the case.

**Proof.** For each $M \in \mathcal{A}^A$ and $N \in \mathcal{A}^\pi$, the fundamental isomorphism

$$\xi : M \otimes N \simeq M \square N$$

induces

$$\xi : M \otimes N^e \simeq M \square_e N,$$

hence

$$\xi : M \otimes B \simeq M \square_e A$$

in particular. If $M$ is a coflat right $\pi(A)$-comodule, we have an isomorphism

$$(M \square_e A) \otimes_B T \simeq M \square_e (A \otimes_B T)$$

for each $T \in \mathfrak{B}$. The proof is dual to [2, Proposition 1.3]. Hence $\xi$ induces

$$M \otimes T \simeq M \square_e (A \otimes_B T).$$

If $g : T \to N^e$ in $\mathfrak{B}$ corresponds to $G : A \otimes_B T \to N$ in $\mathcal{A}^\pi$, we have a commutative diagram

$$\begin{array}{ccc}
M \otimes T & \simeq & M \square_e (A \otimes_B T) \\
\downarrow f \otimes g & & \downarrow f \square g \\
M \otimes N^e & \simeq & M \square_e N.
\end{array}$$
Hence, if $M$ is a faithfully coflat right $\pi(A)$-comodule, then $g$ is an isomorphism if and only if $G$ is. This proves that the adjunctions $A \otimes_B N^* \rightarrow N$ and $T \rightarrow (A \otimes_B T)^*$ are isomorphisms. Thus $\mathcal{A} \mathcal{M} \approx \mu \mathcal{M}$. Since $\pi(A) \in \mathcal{A} \mathcal{M}$, $\pi(A) \simeq A \otimes_B \pi(k)$ (where $\pi(k) = \pi(A)^*$) means $\pi = \pi_B$. Since $M \otimes ? \simeq M \sqcup \pi(A \otimes_B ?)$ is faithfully exact so is $A \otimes_B ?$, hence $A$ is a faithfully flat right $B$-module. Q.E.D.

Theorems 1 and 2 give rise to a 1-1 correspondence $B \mapsto \pi_B$ and $\pi \mapsto B^*$ between some class of right coideal subalgebras and some class of quotient left $A$-module coalgebras. In the commutative case and in the cocommutative case this correspondence has a nice description.

**Theorem 3.** Let $A$ be a commutative Hopf algebra over $k$. The maps $B \mapsto \pi_B$ and $\pi \mapsto B^*$ give rise to a bijection between the set of right coideal subalgebras $B \subset A$ over which $A$ is a faithfully flat module and the set of quotient Hopf algebras $\pi: A \rightarrow \pi(A)$, where $A$ is a left (or equivalently right) faithfully coflat $\pi(A)$-comodule.

**Proof.** If $B \subset A$ is a right coideal subalgebra, then $AB^+ \subset A$ is a Hopf ideal. Hence $\pi_B: A \rightarrow A/AB^-$ is a Hopf quotient. If $A$ is a faithfully flat $B$-module, $A$ is a faithfully coflat left $\pi_B(A)$-comodule and $B^{\pi_B} = B$ by Theorem 1. Using the antipode, we see that $A$ is also a right faithfully coflat $\pi_B(A)$-comodule. Let $\pi: A \rightarrow \pi(A)$ be a quotient left $A$-module coalgebra. Since $A$ is commutative, $\pi(A)$ is a quotient bialgebra of $A$. If $A$ is a faithfully coflat right $\pi(A)$-comodule, then $\pi = \pi_B$ with $B = B^*$ and $A$ is a faithfully flat $B$-module by Theorem 2. In particular $\pi(A)$ is a Hopf quotient. This proves the claim.

Q.E.D.

**Theorem 4.** Let $A$ be a cocommutative Hopf algebra over $k$. There is a 1-1 correspondence $B \mapsto \pi_B$ between the Hopf subalgebras of $A$ and the quotient left $A$-module coalgebras of $A$ over which $A$ is a faithfully coflat comodule.

**Proof.** If $B \subset A$ is a Hopf subalgebra, then $A$ is a left and right faithfully flat $B$-module by [1, Theorem 3.1]. Hence $A$ is a faithfully coflat $\pi_B(A)$-comodule and $B = B^{\pi_B}$ by Theorem 1. Let $\pi: A \rightarrow \pi(A)$ be a quotient left $A$-module coalgebra where $A$ is a faithfully coflat $\pi(A)$-comodule. Let $B = B^*$ which is a subbialgebra of $A$. Let $S: A \rightarrow A$ be the antipode and let $a \in B$. Then

$$\sum \pi(a_{(1)}) \otimes S(a_{(2)}) = \sum \pi(1) \otimes S(a),$$

hence

$$\pi(S(a)) = S(a) \pi(1) = \sum S(a_{(2)}) \pi(a_{(1)})$$

$$= \sum \pi(S(a_{(2)}) a_{(1)}) = \epsilon(a) \pi(1),$$
and

\[ \sum \pi(S(a_{(i)})) \otimes S(a_{(i)}) = \pi(1) \otimes S(a). \]

Thus \( S(B) \subset B \) and \( B \) is a Hopf subalgebra. Since \( \pi = \pi_B \) by Theorem 2, we are done. Q.E.D.

2. Freeness Criteria for Relative Hopf Modules

In this section, using the equivalences of Section 1, we examine when each right \((A, B)\)-Hopf module is a free \( B \)-module, where \( B \) is a right coideal subalgebra of a fixed Hopf algebra \( A \). The proof of the following lemma is easy and omitted.

**Lemma 1.** Let \( C \) be a \( k \)-coalgebra with coradical filtration \( \{C_n\} \) and let \((S, \omega)\) be a right \( C \)-comodule. Let \( S_n = \omega^{-1}(S \otimes C_n) \). Then the \( S_n \) are subcomodules of \( S \), \( S = \bigcup_n S_n \), and \( S_n/S_{n-1} \) are \( C_0 \)-comodules (hence semisimple \( C \)-comodules) for each \( n \geq 0 \).

Let \( \text{cent}(A) \) be the center of \( A \). The following is a generalization of [9, Theorem 3].

**Theorem 5.** Let \( A \) be a Hopf algebra over \( k \) and let \( B \subset A \) be a right coideal subalgebra. Suppose \( B \subset \text{cent}(A) \) and \( A \) is a faithfully flat \( B \)-module. Then each \( M \in \mathcal{M}^A \) is a projective \( B \)-module. In particular \( A \) is a projective \( B \)-module.

**Proof.** Since the category \( \mathcal{M}^A \) is isomorphic to \( \mathcal{M}^\pi \) where \( \pi = \pi_B \) by Theorem 1, it follows from Lemma 1 that each \( M \in \mathcal{M}^A \) has a filtration \( \{M_n\} \) such that \( M = \bigcup_{n \geq 0} M_n \) and each \( M_n/M_{n-1} \) is a semisimple object (we put \( M_{-1} = 0 \)). Suppose

\[ M_n/M_{n-1} = \bigoplus_{\lambda \in \Lambda} N_\lambda \]

is the direct sum of simple objects in \( \mathcal{M}^A \). Then \( \Phi(N_\lambda) = \overline{N}_\lambda \) is finite dimensional. Hence

\[ N_\lambda \otimes_B A \cong \overline{N}_\lambda \otimes A \]

is a finite free right \( A \)-module. Hence \( N_\lambda \) is a finitely generated projective \( B \)-module by [5, Proposition 12, p. 53]. Hence the \( M_n/M_{n-1} \) are projective \( B \)-modules. It follows that \( M_n \cong M_{n-1} \oplus (M_n/M_{n-1}) \) as \( B \)-modules. Hence

\[ M \cong \bigoplus_{n \geq 0} M_n/M_{n-1} \]

is a projective \( B \)-module. Q.E.D.
If $A$ is commutative or if $A$ has a cocommutative coradical, $A$ is a faithfully flat left and right module over its arbitrary Hopf subalgebra [1, Theorems 3.1, 3.2].

**Corollary 1.** Let $A$ be a Hopf algebra over $k$ and let $B \subset A$ be a Hopf subalgebra. If either

(a) $A$ is commutative, or  
(b) $A$ has a cocommutative coradical and $B \subset \text{cent}(A)$,

then $A$ is a projective $B$-module.

**Theorem 6.** Let $A$ and $B$ be as in Theorem 5. If $B^+ \subset \text{Rad}(A)$ (the Jacobson radical of $A$), each $M \in \mathcal{M}_B^A$ is a free $B$-module, hence $A$ is a free $B$-module.

**Proof.** Let $M \in \mathcal{M}_B^A$. We may assume $M = \Phi(M)$ is finite dimensional. Let $\{u \mid u \in X\}$ be a $k$-basis for $M$ where $X \subset M$. Notice that $AB^+$ is a Hopf ideal contained in $\text{Rad}(A)$. Hence we have an automorphism of the right $\pi(A)$-modules $M \otimes \pi(A)$ where $\pi = \pi_B$,

$$\xi : M \otimes \pi(A) \cong \bar{M} \otimes \pi(A),$$

given by $\xi(m \otimes \pi(a)) = \sum \bar{m}_{(0)} \otimes \bar{m}_{(1)} \pi(a)$. Since $X$ is a basis for $M \otimes \pi(A)$ over $\pi(A)$, $\xi(X)$ is, too. Since $AR^+ \subset \text{Rad}(A)$, it follows by [4, Proposition 2.12, p. 90] that $\xi(X)$ is a basis for $\bar{M} \otimes A$ over $A$, where

$$\xi : M \otimes_B A \cong \bar{M} \otimes A$$

is an isomorphism of right $A$-modules. This implies that $X$ is a basis for the right $A$-module $M \otimes_B A$, hence a basis for the $B$-module $M$. Q.E.D.

**Definition 1.** Let $\Omega$ be the class of (unitary) rings $R$ such that

$$R^n \cong R^m$$

as right $R$-modules implies $n = m$ (cf. [4, Definition 5.1, p. 190]).

All commutative rings are contained in $\Omega$. $\Omega$ also contains all Artinian rings. A ring $R$ belongs to $\Omega$ if there is a two-sided ideal $\mathcal{A}$ such that $R/\mathcal{A} \in \Omega$. (Note that $0 \notin \Omega$.) Hence, if $R/\text{Rad}(R)$ is Artinian or commutative, then $R \in \Omega$. If a ring $R$ contains a subring $S$ such that $S \in \Omega$ and $R$ is a finite free right $S$-module, then $R \in \Omega$. For example, if $S$ is a commutative ring, then all faithfully flat $S$-algebras which are finitely generated $S$-modules are contained in $\Omega$ (we can reduce to the case where $S$ is a field).

The following is a generalization of [9, Theorem 1].

**Theorem 7.** Let $A$ and $B$ be as in Theorem 5. Assume $\bar{B} = B/\text{Rad}(B)$ is
Artinian. Hence there is a set of orthogonal idempotents $e_1, \ldots, e_m$ of $B$ such that $1 = e_1 + \cdots + e_m$ and each $Be_i$ is a field. Let $\tilde{A} = A/\text{Rad}(B)$. If $\tilde{A}e_i \in \Omega$ for each $1 \leq i \leq m$, $A$ is a free $B$-module. In fact each $M \in \mathcal{M}_B$ is.

**Proof.** Let $M \in \mathcal{M}_B$ and assume $[\tilde{M} : k] = n$. Let $\tilde{M} = M/M \text{Rad}(B)$. We need only to show $\tilde{M} \simeq B^n$ as $B$-modules. Let $1 \leq i \leq m$ and $\tilde{M}e_i \simeq (Be_i)^r$ as $Be_i$-vector spaces. Since $\xi$ induces

$$
\tilde{M} \otimes_B A \simeq \tilde{M} \otimes A \simeq B^n \otimes_B A \quad \text{and} \quad \tilde{M}e_i \otimes_B A \simeq (\tilde{A}e_i)^r,
$$

it follows by $\tilde{A}e_i \in \Omega$ that $r = n$. Hence $\tilde{M} = \tilde{M}e_1 \oplus \cdots \oplus \tilde{M}e_m \simeq B^n$ as $B$-modules. Q.E.D.

Let $A_0$ be the coradical of $A$.

**PROPOSITION 3.** If $B$ is a Hopf subalgebra of $A$ containing $A_0$, then each $M \in \mathcal{M}_B$ is a free $B$-module.

**Proof.** It is enough to modify the proof of [8, Corollary 1] as follows. For each right $A$-comodule $M$ with structure $\omega: M \to M \otimes A$, let $M^{(0)} = \omega^* (M \otimes B)$, or $M \square_A B$, which is the largest $B$-subcomodule of $M$. Define $M^{(i)} / M^{(i-1)} = (M/M^{(i-1)})^{(0)}$ by induction. This is called the filtration of $M$ with respect to $B$. Since $B \supset A_0$, it follows that $M = \bigcup_n M^{(n)}$. If $M \in \mathcal{M}_B$, each $M^{(n)} / M^{(n-1)} \in \mathcal{M}_B$. By the theorem of Sweedler, each $M^{(n)} / M^{(n-1)}$ is a free $B$-module. Hence $M$ is a free $B$-module. Q.E.D.

Let $G(A)$ be the set of group-like elements of $A$, i.e., the set of $x \in A$ with $\Delta(x) = x \otimes x$, $\epsilon(x) = 1$.

**PROPOSITION 4.** Let $B$ be a Hopf subalgebra of $A$. If $G(A)B$ contains $A_0$, then each $M \in \mathcal{M}_B$ is a free $B$-module. In particular if $A$ is pointed, this holds.

**Proof.** We have only to modify the above proof. Let $\{M^{(n)}\}$ be the filtration of $M$ with respect to $G(A)B$. Then $M = \bigcup_n M^{(n)}$ and $M^{(n)}/M^{(n-1)} \in \mathcal{M}_B$. Since $G(A)B$ is the direct sum of $gB$, $g \in S$, for some subset $S \subset G(A)$ [8, Lemma 5], it follows that $M^{(n)} / M^{(n-1)}$ is the direct sum of submodules $N_g^{(n)}$, $g \in S$, where $N_g^{(n)} \in \mathcal{M}_B$. Since $\mathcal{M}_B \simeq \mathcal{M}_B$, each $N_g^{(n)}$ is $B$-free as before. Q.E.D.

A commutative Hopf algebra is *locally finite* if it is the union of finite-dimensional Hopf subalgebras.

Let $A$ be a commutative Hopf algebra, and let $B \subset A$ be a Hopf subalgebra.

**LEMMA 2.** Suppose for each finitely generated Hopf subalgebra $B' \subset B$ and each $M' \in \mathcal{M}_{B'}$, $M'$ is a free $B'$-module. Then each $M \in \mathcal{M}_B$ is a free $B$-module.
Proof. Let \( M \in \mathcal{M}_B^A \). To claim that \( M \) is \( B \)-free, we can assume \( M = VB \), where \( V \) is a simple \( A \)-subcomodule of \( M \) [9, Proposition 1]. By assumption \( VB' \) is \( B' \)-free for each finitely generated \( B' \subseteq B \). Let \( n \) be the minimum of the \( B' \)-ranks of \( VB' \) where \( B' \subseteq B \) ranges all finitely generated Hopf subalgebras. Take such a \( B' \subseteq B \) that \( VB' \) is of rank \( n \). Then for each finitely generated \( B' \subseteq B'' \subseteq B \), the surjection
\[
VB' \otimes_{B'} B'' \to VB''
\]
should be isomorphic, hence going to \( \lim \)
\[
VB' \otimes_{B'} B \cong VB = M.
\]
Thus \( M \) is a free \( B \)-module. Q.E.D.

**Proposition 5.** If \( A \) is a commutative Hopf algebra and \( B \) a locally finite Hopf subalgebra of \( A \), then each \( M \in \mathcal{M}_B^A \) is a free \( B \)-module.

This follows from Theorem 7.

3. Intermediate Coideal Subalgebras

In this section let \( A \) be a Hopf algebra over \( k \), and fix a right coideal subalgebra \( D \subseteq A \) contained in the center. Since \( AD^+ \) is a Hopf ideal of \( A \), \( \overline{A} = A/AD^+ \) is a quotient Hopf algebra.

Let \( D \subseteq B \subseteq A \) be a right coideal subalgebra. Then \( \overline{B} = B/BD^+ \) is a \( k \)-algebra and \( \Delta: B \to B \otimes A \) induces a right \( \overline{A} \)-comodule structure map
\[
\overline{\Delta}: \overline{B} \to \overline{B} \otimes \overline{A},
\]
which is also an algebra map. Thus \( \overline{B} \) is a right \( \overline{A} \)-comodule algebra. If \( A \) is a flat \( D \)-module, \( B \) is a right coideal subalgebra of \( A \). Indeed \( \overline{B} \subseteq \overline{A} \) follows by \( \overline{B} \otimes A \cong B \otimes_D A \subseteq A \otimes_D A \cong \overline{A} \otimes \overline{A} \).

Anyway, we can define the category of right \((\overline{A}, B)\)-Hopf modules \( \mathcal{M}_{\overline{B}}^B \) just as before.

The categories \( \mathcal{M}_D^A \) and \( \mathcal{M}_{\overline{B}}^B \) are subcategories of \( \mathcal{M}_B^A \) and \( \mathcal{M}_{\overline{B}}^B \). The functors \( \Phi: \mathcal{M}_D^A \to \mathcal{M}_{\overline{B}}^B \) and \( \Psi: \mathcal{M}_{\overline{B}}^B \to \mathcal{M}_D^A \) induce the following set of functors.

If \( M \in \mathcal{M}_B^A \), let \( \overline{M} = M/MD^+ \). The structure maps \( M \otimes B \to M \) and \( M \to M \otimes A \) induce \( \overline{M} \otimes \overline{B} \to \overline{M} \) and \( \overline{M} \to \overline{M} \otimes \overline{A} \) through which \( \overline{M} \in \mathcal{M}_{\overline{B}}^B \).

Let \( \Phi: M \mapsto \overline{M}, \mathcal{M}_B^A \to \mathcal{M}_{\overline{B}}^B \).

If \( S \in \mathcal{M}_{\overline{B}}^B \), \( S \otimes A \in \mathcal{M}_B^A \) with respect to the structure
\[
\omega(s \otimes a) = s \otimes \Delta(a)
\]
and

\[(s \otimes a)b = \sum s\delta_{(i)} \otimes ab_{(i)}\]

for \(s \in S, a \in A, b \in B,\) and \(S \triangleleft A \subseteq S \otimes A\) is a subobject. Let \(\Psi: S \to S \triangleleft A, \mathcal{H}_B^A \to \mathcal{H}_B^A.\)

If a right \(\mathcal{H}_B^A\)-comodule map \(f: M \to S\) corresponds to a right \(\mathcal{H}_B^A\)-comodule map \(F: M \to S \triangleleft A\) \((F(M) = \sum f(m_{(0)}) \otimes m_{(1)})\), a direct calculation shows that \(F\) is right \(B\)-linear if and only if \(f\) is right \(B\)-linear. Since \(SD^+ = 0\), if this is the case, \(f\) induces an \(\mathcal{H}_B^A\)-map: \(M \to S\). This implies that \(\Phi\) is a left adjoint to \(\Psi\).

Notice that the adjunctions

\[M \to \Psi(\Phi(M)) \quad \text{and} \quad \Phi(\Psi(S)) \to S,\]

where \(M \in \mathcal{H}_B^A\) and \(S \in \mathcal{H}_B^A\), do not depend on \(B\). Hence (a) of Theorem 8 follows by Theorem 1. The 1-1 correspondence between subobjects of \(A\) in \(\mathcal{H}_B^A\) and subobjects of \(\mathcal{H}_B^A\) induces the correspondence described in (b) and (c).

**Theorem 8.** Suppose there is a left \(A\)-module which is a faithfully flat \(D\)-module.

(a) If \(D \subseteq B \subseteq A\) is a right coideal subalgebra, then \(B\) is a right coideal subalgebra of \(\mathcal{H}_B^A\), \(\Phi\), and \(\Psi\) give rise to an equivalence of categories \(\mathcal{H}_B^A \approx \mathcal{H}_B^A\). and we have \(B = B \triangleleft A\).

(b) If \(B \subseteq A\) is a right coideal subalgebra, then \(B \triangleleft A\) is a right coideal subalgebra of \(A\) containing \(D\), and we have \(B = B \triangleleft A\).

(c) There is a 1-1 correspondence \(B \leftrightarrow \mathcal{H}_B^A\) between right coideal subalgebras of \(A\) containing \(D\) and right coideal subalgebras of \(A\).

Let \(C = A; AB^- = A; AB^-\). There are the following three sets of adjoint pairs of functors

\[
\begin{align*}
\Phi_1: \mathcal{H}_B^A &\to \mathcal{H}_B^A \\
\Phi_2: \mathcal{H}_B^A &\to \mathcal{H}_C^A \\
\Phi_3: \mathcal{H}_B^A &\to \mathcal{H}_B^A
\end{align*}
\]

and

\[
\begin{align*}
\Psi_1: \mathcal{H}_B^A &\to \mathcal{H}_B^A \\
\Psi_2: \mathcal{H}_C^A &\to \mathcal{H}_B^A \\
\Psi_3: \mathcal{H}_C^A &\to \mathcal{H}_B^A
\end{align*}
\]

where \(\Phi_1(M) = M; MD^-\), \(\Phi_2(M) = M; MB^-\), \(\Phi_3(N) = N; NB^-\), \(\Psi_1(N) = N \triangleleft A\), \(\Psi_2(S) = S \triangleleft C\), and \(\Psi_3(S) = S \triangleleft C\) for \(M \in \mathcal{H}_B^A\), \(N \in \mathcal{H}_B^A\), \(S \in \mathcal{H}_C^A\). It is easy to see \(\Phi_2 \circ \Phi_1 = \Phi_2\) and \(\Psi_3 = \Psi_1 \circ \Psi_2\). Thus with the assumption of Theorem 8, \((\Phi_2, \Psi_2)\) is identified with \((\Phi_3, \Psi_3)\). In particular, if one of these is an equivalence, so is the other. Note that \(\Psi_3\) (resp. \(\Psi_3\)) faithfully exact implies that \(A\) (resp. \(\mathcal{H}_B^A\)) is a left faithfully coflat \(C\)-comodule.
Corollary 2. Let $D \subseteq B \subseteq A$ be a right coideal subalgebra, and let $C = A/AB^r$. Assume there is a left $A$-module which is a faithfully flat $D$-module.

(a) Assume either there is a left $A$-module which is a faithfully flat left $B$-module or there is a left $\tilde{A}$-module which is a faithfully flat left $\tilde{B}$-module. Then $A$ and $\tilde{A}$ are faithfully coflat left $C$-comodules, and we have

$$B = k \square_c A, \quad \tilde{B} = k \square_c \tilde{A},$$

where $k$ denotes the image of $k \subseteq A$ in $C$.

(b) Assume $C$ is a Hopf quotient of $A$, and $A$, $\tilde{A}$, and $C$ have bijective antipodes. If $A$ (resp. $\tilde{A}$) is a left faithfully flat $B$-module (resp. $B$-module), then $\tilde{A}$ (resp. $A$) is a right faithfully flat $\tilde{B}$-module (resp. $B$-module).

(c) If $A$ is commutative, $A$ over $B$ is faithfully flat if and only if $\tilde{A}$ over $\tilde{B}$ is.

Proof. (a) Follows from Theorem 1. (b) In this case, $A$ (resp. $\tilde{A}$) is a faithfully coflat left $C$-comodule if and only if it is as a right $C$-comodule. The claim will follow from (a) and Theorem 2. (c) If $A$ is commutative, $C$ is a Hopf quotient, and $A$, $\tilde{A}$, and $C$ have bijective antipodes. The claim follows from (b). Q.E.D.

Let $A \supseteq B \supseteq B' \supseteq D$ be right coideal subalgebras. If $M \in \mathcal{M}^A$, then $M \otimes B$ has the structure of a right $A$-comodule,

$$\alpha(m \otimes b) = \sum m_{(0)} \otimes b_{(0)} \otimes m_{(1)}b_{(1)}$$

for $m \in M$, $b \in B$. $M \otimes B$ is a right $(A, B)$-Hopf module, together with the obvious $B$-module structure. If $M \in \mathcal{M}^A_{B'}$, $M \otimes_{B'} B$ is a quotient object of $M \otimes B$ in $\mathcal{M}^A_B$. Thus we have a functor

$$\alpha: M \mapsto M \otimes_{B'} R, \quad \mathcal{M}^A_{B'} \mapsto \mathcal{M}^A_B.$$

We also have a similar functor

$$\beta: S \mapsto S \otimes_{B'} \tilde{B}, \quad \mathcal{M}^\tilde{A}_{B'} \mapsto \mathcal{M}^\tilde{A}_B.$$

Corollary 3. With the assumption of Theorem 8, we have a commutative diagram of functors

$$\mathcal{M}^A_{B'} \cong \mathcal{M}^\tilde{A}_{B'},$$

$$\begin{array}{c}
\alpha \\
\downarrow
\end{array} \quad \begin{array}{c}
\beta \\
\downarrow
\end{array}$$

$$\mathcal{M}^A_B \cong \mathcal{M}^\tilde{A}_B.$$

The proof is easy.

As an application of the theory of this section, we can add to the freeness criteria for relative Hopf modules in Section 2. We start with a simple lemma.
Lemma 3. Let \( D \) and \( B \) be as in Theorem 8(a). Denote by \( \mathcal{M} \rightarrow \bar{\mathcal{M}} \) the equivalence \( \mathcal{M}_B^A \cong \mathcal{M}_{\bar{B}}^{\bar{A}} \). If each \( \mathcal{M} \in \mathcal{M}_B^A \) such that \( \bar{M} = \bar{V} \bar{B} \) for some simple \( A \)-subcomodule \( V \subset \bar{M} \) is a free \( B \)-module, then each \( \mathcal{M} \in \mathcal{M}_B^A \) is a free \( B \)-module.

This can be proved in the same way as [9, Proposition 1].

Theorem 9. Let \( A \) be a commutative Hopf algebra, and let \( D \subset B \subset A \) be right coideal subalgebras. Assume \( D \) is finite dimensional, \( A \) over \( D \) is faithfully flat, and \( B = B/BD^- \) is a group-like Hopf algebra. Then each \( \mathcal{M} \in \mathcal{M}_B^A \) is a free \( B \)-module.

Proof. By the above lemma, we can assume \( \bar{M} = \bar{V} \bar{B} \) for some simple \( \bar{A} \)-subcomodule \( \bar{V} \subset \bar{M} \). Since \( \bar{B} = k[G(B)] \) and \( \bar{V} \) is finite dimensional, it follows from [9, Proposition 2] that there are a finite Hopf subalgebra \( \mathcal{B} \subset \bar{B} \) and an object \( \mathcal{N} \in \mathcal{M}_B^\mathcal{B} \), such that \( \bar{M} \cong \mathcal{N} \otimes_{\mathcal{B}} \bar{B} \) in \( \mathcal{M}_B^\mathcal{B} \). By Theorem 8(c), there is a unique right coideal subalgebra \( D \subset B' \subset B \) \((\mathcal{A}(B') \subset B' \otimes B)\) such that \( B' = \mathcal{B}' \). Let \( N \in \mathcal{M}_B^\mathcal{A} \) be such that \( \bar{N} \cong \mathcal{N} \). Then by Corollary 3, \( N \otimes_{\mathcal{B}'} B' \in \mathcal{M}_B^{A'} \) corresponds to \( \mathcal{N} \otimes_{\mathcal{B}} \bar{B} \), hence \( \mathcal{M} \cong N \otimes_{\mathcal{B}} \bar{B} \) in \( \mathcal{M}_B^{A'} \). We have only to prove that \( N \) is \( B' \)-free. Since \( \bar{B}' = \mathcal{B}' \) is finite dimensional over \( k \), \( B' \) is a finitely generated projective \( D \)-module, hence finite dimensional. Since \( A \) over \( B' \) is faithfully flat, so is \( A \) over \( B' \) by Corollary 2(c). It follows by Theorem 7 that \( N \) is a free \( B' \)-module.

Q.E.D.

4. Commutative Hopf Algebras

The category of affine \( k \)-group schemes is antiequivalent to the category of commutative Hopf algebras over \( k \). Closed subgroup scheme (resp. quotient group scheme) corresponds to Hopf ideal (resp. Hopf subalgebra). Let \( G \) be an affine \( k \)-group scheme corresponding to the Hopf algebra \( A = O(G) \), and let \( H \subset G \) be a closed subgroup scheme corresponding to a Hopf ideal \( I \subset A \). The left coset dur \( k \)-sheaf \( H \backslash G \) is the cokernel in \( M_k^E \) [7, Chap. III, Sect. 1, 3.4] of

\[
H \times G \xrightarrow{\text{projection}} G.
\]

This is affine if \( H \) is normal [7, Chap. III, Sect. 3, 7.2], but not in general. It is affine if and only if there is a right coideal subalgebra \( B \subset A \) such that \( A \) is faithfully flat over \( B \) and \( B^+A = I \).

Thus Theorem 3 can be restated in terms of group schemes as follows.

Theorem 10. Let \( G \) be an affine \( k \)-group scheme and let \( H \subset G \) be a closed subgroup scheme. The dur \( k \)-sheaf of left cosets \( H \backslash G \) [7, Chap. III, Sect. 3, 7.2] (or
equivalently the sheaf of right cosets $G^\wedge_H$ is affine if and only if the affine ring $O(G)$ is a faithfully coflat left or right $O(H)$-comodule.

A morphism of affine $k$-schemes $X \to Y$ is free (resp. projective) if the affine ring $O(X)$ is a free (resp. projective) $O(Y)$-module. The following is simply a restatement of Theorem 5 in the commutative case.

**Theorem 11.** Let $G$ be an affine group scheme over a field $k$ and let $H \subset G$ be a closed subgroup scheme. If $H \cap G$ is affine, the projection $G \to H \cap G$ is projective. If $G''$ is a quotient group scheme of $G$, the projection $G \to G''$ is projective.

We consider when commutative Hopf algebras are free over Hopf subalgebras.

**Proposition 6.** Let $A$ be a commutative Hopf algebra over $k$ and let $B, C \subset A$ be Hopf subalgebras. Then $B \otimes_{BC} C \cong BC$. (Two Hopf subalgebras are linearly disjoint over their intersection.)

**Proof.** Let $G$ be a group and let $N, K \subset G$ be normal subgroups. We have a Cartesian (pullback) diagram

$$
\begin{array}{ccc}
G:N \cap K & \longrightarrow & G:N \\
\downarrow & & \downarrow \\
G:K & \longrightarrow & G:NK,
\end{array}
$$

where each arrow is the projection. The same is true of affine $k$-group schemes [7, Chap. III, Sect. 1, No. 21]. The above is simply a restatement of this fact in terms of Hopf algebras. Q.E.D.

With the above notation, $A$ is a free $B$-module if $A$ over $BC$ and $C$ over $B \cap C$ are free. The same argument was used in [8, Proposition 1].

In the following let $A$ be a commutative Hopf algebra and let $B \subset A$ be a Hopf subalgebra.

**Lemma 4.** The Hopf algebra $A/B\Delta A$ is irreducible if and only if $B \supset A_0$ (the coradical of $A$).

**Proof.** The coradical filtration of $A$ induces the structure of a filtered coalgebra on $A/B^+A$. Hence $A_0/B^+A \supset (A/B^+A)_0$. This proves the "if" part. To prove the inverse, let $G$ be the affine $k$-group scheme associated with $A$ and let $N \subset G$ be the normal closed subgroup scheme associated with $A/B^+A$. Let $V$ be a simple right $A$-comodule or equivalently a simple left $k$-G-module. Since $V$ is simple, $V = V^N$, hence $V$ is a $k$-G$^\wedge N$-module, or equivalently $V$ is a right $B$-comodule. This means $A_0 \subset B$. Q.E.D.
The lemma means that \( A/\mathfrak{g} A \) corresponds to the unipotent radical. Thus, roughly speaking, \( A \) represents a reductive group if and only if \( \mathfrak{g} A \) generates \( A \) as an algebra.

In view of the lemma, Radford [8, Corollary 1], or Proposition 3, has proved the following.

**Proposition 7.** If \( A/\mathfrak{g} A \) is irreducible, \( A \) is a free \( B \)-module.

Next we shall apply Theorem 9 to prove the following:

**Theorem 12.** Let \( k \) be perfect, let \( A \) be a commutative reduced Hopf algebra over \( k \), and let \( B \subset A \) be a pointed Hopf subalgebra. Then \( A \) is a free \( B \)-module.

**Proof.** Let \( A_0 \) be the coradical of \( A \), and let \( (A_0) \) be the Hopf subalgebra generated by \( A_0 \). Since \( A \) over \( (A_0)B \) is free by Proposition 3, we have only to prove that \( (A_0) \) over \( (A_0) \cap B \) is free by Proposition 6. Assume \( A = (A_0) \). We claim that each \( M \in \mathcal{A}_A^A \) is a free \( B \)-module. By Lemma 2, we can assume \( B \) is finitely generated. Since we can assume \( M \) is of the form \( M = V^B \) for some simple \( A \)-subcomodule \( V \subset M \), we can also assume \( A \) is finitely generated. Let \( k \) be the algebraic closure of \( k \). Since \( k \) is perfect, \( k \) is also reduced and \( k \otimes A_0 = (k \otimes A)_0 \). Let \( G \) be the affine algebraic \( k \)-group scheme corresponding to \( A \), and let \( G^0 \) be the connected component. Then \( G \) is smooth. Hence the unipotent radical \( N \) of \( G^0 \otimes k \) is normal in \( G \otimes k \). Since \( C \otimes k \) has no normal unipotent closed subgroup scheme, it follows that \( N = \{ e \} \). Thus \( G^0 \otimes k \) is reductive. Hence the derived group \( [G^0 \otimes k, G^0 \otimes k] \) is semisimple by [6, Proposition 14.2, p. 325], and has no solvable quotient. Since \( (G^0 \otimes k, [G^0 \otimes k, G^0 \otimes k]) \) is a torus, it follows that each solvable quotient of \( G^0 \otimes k \) is a torus. Let \( H \) be the affine \( k \)-group scheme associated with \( B \). This means that \( H^0 \subset G^0 \) is a torus. Let \( \mathfrak{h} \) be the affine ring of \( H^0 \). Since \( \mathfrak{h} \) is pointed and \( \mathfrak{h} \) is generated by the group-likes, it follows that \( \mathfrak{g} = \mathfrak{h}[G(\mathfrak{g})] \), where \( G(\mathfrak{g}) \) denotes the group-like elements. Let \( D \subset B \) be the unique Hopf subalgebra such that \( \mathfrak{g} = B^D \). Then \( D \) is finite dimensional over \( k \), since it corresponds to \( H \cap H^0 \). Since \( \mathfrak{g} \) is a group-like Hopf algebra, it follows from Theorem 9 that each \( M \in \mathcal{A}_A^A \) is a free \( B \)-module. Q.E.D.

In the following we consider Hopf ideals and right coideal subalgebras of a pointed commutative Hopf algebra.

Let \( p = \text{Max}(1, \text{char}(k)) \). A polynomial \( f(X) = \sum \alpha_i X^i \) with coefficients in \( k \) is a \( p \)-polynomial if \( \alpha_i = 0 \) unless \( i \) is a power of \( p \). (In particular, \( \alpha_p = 0 \).) Let \( a \) be an element of a group. \( f(X) \) is a \( (p, a) \)-polynomial if it is a \( p \)-polynomial and \( \alpha_i = 0 \) and \( \alpha_j = 0 \) mean \( a^i = a^j \).

Let \( A \) be a commutative pointed Hopf algebra and let \( B \subset A \) be a Hopf subalgebra. Assume \( z \in A \) satisfies \( A(z) = z \otimes a \doteq \sum z - u \), where \( a \in G(\mathfrak{g}) \) and \( u \in B \). By [10, Lemma 4], either \( z \) is algebraically independent over \( B \),
or there is a unique monic \((p, a)\)-polynomial \(f(X) \in \mathbb{k}[X]\) of degree \(n\) such that \(f(x) \in B\) and \(B[x]\) is a free \(B\)-module with basis \(\{1, z, \ldots, z^{n-1}\}\). In the latter case, we have

\[
\Delta(f(x)) = f(x) \otimes a^n + 1 \otimes f(x) + f(u).
\]

**Lemma 5.** Let \(B[x]\) be as above and let \(I \subset B[x]\) be a Hopf ideal such that \(B \cap I = 0\). There is a unique right coideal subalgebra \(L \subset B[x]\) such that \(L + B[x] = I\) and \(B[x]\) is a free \(L\)-module. Then \(BL\) is a Hopf subalgebra of \(B[x]\) and there is a natural isomorphism \(B \otimes L \cong BL\).

**Proof.** We need only to show the existence. We can view \(B \subset B[x]/I\) as a Hopf subalgebra. Let \(B[z] = B[x]/I\) with \(z\) the image of \(x\). If \(z\) is algebraically independent over \(B\), \(I = 0\) and we have nothing to prove. Otherwise, let \(g(X)\) be the minimal monic \((p, a)\)-polynomial such that \(g(z) \in B\). Let \(y = g(x) - g(z) \in B[x]\). Then

\[
\Delta(y) = y \otimes a^m + 1 \otimes y,
\]

where \(m = \deg(g(X))\). Hence \(B[y]\) is a Hopf subalgebra of \(B[x]\), and \(B[x]\) over \(B[y]\) is a free module. We claim that \(B \otimes k[y] \cong B[y]\). Since \(I - k[y] \subset B\) and \(\Delta(k[y]) \subset k[y] \otimes B[x]\), this will imply the lemma. If \(y\) is algebraically independent over \(B\), we have nothing to prove, again. Let \(h(X)\) be the minimal monic \((p, a^m)\)-polynomial for \(y\). Applying the projection \(B[x] \to B[z]\), \(x \mapsto z\), we conclude that \(h(y) = 0\). If \(l = \deg(h(X))\), the \(k\)-algebra \(k[y]\) has a basis \(\{1, y, \ldots, y^{l-1}\}\). Hence \(B \otimes k[y] \cong B[y]\). Q.E.D.

**Theorem 13.** Let \(A\) be a pointed commutative Hopf algebra over \(k\) and let \(I \subset A\) be a Hopf ideal. There is a unique right coideal subalgebra \(V \subset A\) such that \(I = V \cdot A\) and \(A\) is a free \(V\)-module.

**Proof.** Consider the set of triples \((B, W, \Gamma)\) where \(B \subset A\) is a Hopf subalgebra, \(W \subset B\) a right coideal subalgebra \((\Delta(W) \subset W \otimes B)\), and \(\Gamma \subset B\) a free \(W\)-basis for \(B\). Define an order \(\leq\) on the set by \((B, W, \Gamma) \leq (B', W', \Gamma')\) if \(B \subset B'\), \(W \subset W'\), \(\Gamma \subset \Gamma'\). By Zorn's lemma, we can take a maximal element \((B, W, \Gamma)\) such that \(A_0 \subset B\). (The theorem is obviously true of \(A_0\).) Assume \(B \neq A\). There is a \(z \in A - B\) such that \(\Delta(z) = z \otimes a + 1 \otimes z + u\) for some \(a \in G(B)\) and \(u \in B \otimes B\). To deduce a contradiction we may assume \(A = B[z]\). The projection \(A \to A/I\) is the composite of

\[
A = B[z] \to A/(I \cap B)A = \overline{B}[z] \to A/I,
\]

where \(\overline{B} = B/I \cap B\), and \(\overline{z}\) denotes the image of \(z\). Let

\[
L = k \Box_{A/I} \overline{B}[\overline{z}],
\]
which is a right coideal subalgebra of $B[z]$ such that $L^+B[z] = I/(I \cap B)A$ and $B \otimes L \cong BL$, by Lemma 5. There is an integer $m$ or $m = \infty$ such that $\{z^i \mid 0 \leq i < m\}$ is a free basis for $B[z]$ over $BL$. Thus it is a free basis for $B[z] \otimes A$ over $BL \otimes A$. Since $W^+B = I \cap B$ and $W^+A = (I \cap B)A$, we have

$$\xi : A \otimes_w A \cong B[z] \otimes A.$$  

Since $A$ is $W$-free, applying $k \Box_{A/I}$ we have

$$\xi : V \otimes_w A \cong L \otimes A,$$

where $V = k \Box_{A/I} A$ which is a right coideal subalgebra of $A$. We have

$$\xi(z) = z \otimes a + 1 \otimes z \mod B \otimes A,$$

hence

$$\xi(z^i) = z^i \otimes a^i - g_i(z \otimes i),$$

where $g_i(X)$ is a polynomial of degree $<i$ with coefficients in $B \otimes A$, for each $i$.

Therefore $\{\xi(z^i) \mid 0 \leq i < m\}$ is a free basis for $B[z] \otimes A$ over $BL \otimes A$. Since

$$\xi : (B \otimes_w V) \otimes_w A \cong (B \otimes L) \otimes A$$

and $B \otimes L \cong BL$, it follows that $\{z^i \mid 0 \leq i < m\}$ is a free basis for $A \otimes_w A$ over $(B \otimes_w V) \otimes_w A$, hence a basis for $A$ over $B \otimes_w V$. Thus

$$\Gamma = \{\gamma z^i : 0 \leq i < m, \gamma \in \Gamma\}$$

is a free basis for $A$ over $B$ containing $\Gamma$ ($\Gamma$ is a $W$-basis for $B$). By Theorem 1, we have $V/W^{-}V = L$. Since $L^{-}B[z] = I/W^{-}A$, and $V \supset W$, it follows that

$$V^{-}A = I.$$  

Thus $(B, W, \Gamma) \leq (A, V, \bar{\Gamma})$, a contradiction.  

Q.E.D.

**Corollary 4.** Let $A$ be a pointed commutative Hopf algebra. There is a 1-1 correspondence $V \leftrightarrow V^{-}A$ between right coideal subalgebras over which $A$ is free, and Hopf ideals of $A$.

Of course we can replace right by left.

An affine $k$-group scheme $G$ is representationally solvable (or trigonalizable) if each simple left $k$-module is 1-dimensional. This is equivalent to the fact that the Hopf algebra $O(G)$ is pointed. Thus the above theorem is equivalent to the following.

**Theorem 14.** Let $G$ be a representationally solvable affine $k$-group scheme and let $H \subset G$ be a closed subgroup scheme. Then $H^m G$ is affine and the projection $G \to H^m G$ is free.
5. An Example

We give an elementary example to show that the Hopf algebra of a torus is not necessarily free over Hopf subalgebras.

Let $G(A)$ be the group-like elements of a Hopf algebra $A$. The Hopf algebra $A$ is group-like if $A = k[G(A)]$. It is multillicative if $A \otimes k$ is a group-like Hopf $k$-algebra, where $k$ is the algebraic closure of $k$. It is known that if $A$ is multiplicative, $A \otimes k_s$ is already group-like, where $k_s$ is the separable closure of $k$.

Let $\Pi$ be the Galois group of $k_s/k$. It has a natural topology. A $\Pi$-group means a pair $(G, \phi)$ where $G$ is a group and $\phi: \Pi \times G \to G$ an action such that $\phi(x, gh) = \phi(x, g) \phi(x, h)$ for $x \in \Pi, g, h \in G$ and that $\Pi_s = \{x \in \Pi \mid \phi(x, g) = g\}$ is an open subgroup of $\Pi$ for each $g \in G$. They form a category.

If $A$ is a multiplicative Hopf algebra, let $X(A) = G(A \otimes k_s)$ the group-like elements of $A \otimes k_s$. The action $\Pi \times A \otimes k_s \to A \otimes k_s$, $(x, a \otimes \lambda) \mapsto a \otimes \phi(x, \lambda)$ makes $X(A)$ into a $\Pi$-group.

If $G$ is a $\Pi$-group, extend the action by $(x, g, h, \lambda) \mapsto (x, g, \lambda)$. By Galois theory, $k_s[G] \simeq k_s[G]^{\Pi}$ as $k_s$-Hopf algebras. It follows that $k_s[G]$ has a unique Hopf structure such that $k_s[G] \simeq k_s[G]^{\Pi}$ as $k_s$-Hopf algebras.

It is known [7, Chap. II, Sect. 5, 1.7] that $A \mapsto X(A)$ and $G \mapsto k_s[G]^{\Pi}$ give rise to an equivalence between the categories of multiplicative Hopf algebras and of $\Pi$-groups.

In particular there is a categorical equivalence between commutative multiplicative Hopf algebras and left $\mathbb{Z}[\Pi]$-modules. A $\mathbb{Z}[\Pi]$-module $M$ corresponds to a torus if $M \simeq \mathbb{Z}^n$ as $\mathbb{Z}$-modules.

There is a torus whose Hopf algebra is not free over Hopf subalgebras.

Let $k = \mathbb{R}$. Then $k_s = \mathbb{C}$ and $\Pi = \{1, \sigma\}$, where $\sigma(\alpha) = \bar{\alpha}, \alpha \in \mathbb{C}$. Make $\mathbb{Z}$ a $\Pi$-module by $\sigma(n) = -n$. Correspondingly, $\Pi$ acts on $\mathbb{C}[X, X^{-1}]$ by

$$\sigma\left(\sum_{i \in \mathbb{Z}} \lambda_i X^i\right) = \sum_{i \in \mathbb{Z}} \bar{\lambda}_i X^{-i}.$$  

Let $A = \mathbb{C}[X, X^{-1}]^{\Pi}$ and $B = \mathbb{C}[X^2, X^{-2}]^{\Pi}$. Then $A$ is a commutative cocommutative Hopf algebra and $B \subset A$ a Hopf subalgebra. We claim that $A$ is not a free $B$-module.

If it were a free $B$-module, the rank would be 2. Let $(f(X), g(X))$ be a basis. Since $\mathbb{C} \otimes_{\mathbb{R}} A = \mathbb{C}[X, X^{-1}]$ and $\mathbb{C} \otimes_{\mathbb{R}} B = \mathbb{C}[X^2, X^{-2}]$, $(f(X), g(X))$ and $(1, X^{-1})$ are two bases of $\mathbb{C}[X, X^{-1}]$ over $\mathbb{C}[X^2, X^{-2}]$. We can write

$$f(X) = a(X^2) + b(X^2) X^{-1},$$
$$g(X) = c(X^2) + d(X^2) X^{-1},$$
where \(a(X^2), b(X^2), c(X^2), d(X^2) \in \mathbb{C}[X^2, X^{-2}]\). By a simple calculation, \(a(X^2)\) and \(c(X^2)\) are \(\mathbb{R}\)-linear combinations of \(X^{2n} - X^{-2n}, iX^{2n} - iX^{-2n}, n \in \mathbb{Z}\), where \(i^2 = -1\), and \(b(X^2)\) and \(d(X^2)\) are \(\mathbb{R}\)-linear combinations of \(X^{2n} - X^{-2n}, iX^{2n} - iX^{-2n}, n \in \mathbb{Z}\). Hence \(a(X^2) d(X^2) - b(X^2) c(X^2)\) is also an \(\mathbb{R}\)-linear combination of \(X^{2n} - X^{-2n}, iX^{2n} - iX^{-2n}, n \in \mathbb{Z}\). Since the only units of \(\mathbb{C}[X^2, X^{-2}]\) are monomials, this implies that the determinant

\[
\begin{vmatrix}
  a(X^2) & b(X^2) \\
  c(X^2) & d(X^2)
\end{vmatrix}
\]

is not a unit of \(\mathbb{C}[X^2, X^{-2}]\). This is a contradiction.


Note added in proof. In the proof of Theorem 3, we mention quotient bialgebras of a commutative Hopf algebra. Nichols proves that they are necessarily Hopf quotients. Theorem 4 has a better description, since we can prove that any quotient left module coalgebra of a cocommutative Hopf algebra \(A\) is necessarily a faithfully coflat \(A\)-comodule.

REFERENCES