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Abstract

This paper addresses a question that was raised at the ICOAE 2014, 3-5 July, Chania, Island of Crete, Greece: how do the volatility skews for the BRICS countries generated by the Risk Neutral Historic Distribution model compare to those generated by using GARCH models? More precisely, in this paper a comparison is made between the volatility skews of the BRICS countries generated by using the RNHD model and those generated by using E-GARCH and GJR-GARCH models. The effect of different interest rates on the implied volatility skews of European call options is also considered.

Keywords: Borsa 100 index; BRICS; CSI 300; E-GARCH model; FTSE/JSE Top 40; GJR-GARCH model; index; IBrX; INDEXCF; JSE; option pricing; securities exchange; S&P BSE SENSEX; volatility skew.

1. Introduction

Some exchanges publish the data that they use to construct their volatility skews, while other exchanges do not. Exchanges use different methods to determine the volatility skews that they use for pricing and hedging. Volatility skew data is therefore not freely available and stakeholders need to resort to models to generate volatility skews pertaining to an underlying security on such exchanges.

In view of the fact that different exchanges use different methods to obtain their volatility skew data and some exchanges publish this data, while others do not, the question that arises is: how does one compare the volatility skews of (for example) an index on an exchange with the volatility skew of an index on another exchange? The only way that this could be addressed in a meaningful manner, is if the same model is used to generate the volatility skew on each exchange.

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Hunzinger et al. [1] addressed this problem for the BRICS indexes. They used a non-parametric model, referred to as the RNHD model, considered by Stutzer [2], Buchen & Kelly [3], Zou & Derman [4], Duan [5] and Araújo & Maré [6]. In this model an empirical distribution is constructed from a sample of the underlying asset prices. Then a risk-neutral distribution is obtained from the constructed empirical distribution, subject to the condition that the expected return equals the risk-free rate, by using the relative entropy principle. The risk-neutral distribution is used to construct option prices, which are in turn used to find the implied volatility skews.

This method was implemented in [1] to generate the volatility skew of the FTSE/JSE Top 40 index of the JSE. Moreover, Hunzinger et al. [1] compared the shapes of the RNHD model generated volatility skews on the following indexes of the BRICS securities exchanges:

- IBrX index - top 100 stocks traded on the Bovespa, Brazil.
- INDEXCF index - 50 most liquid Russian stocks on the Moscow exchange.
- S&P BSE Sensex index - is a cap weighted index on the Bombay Stock Exchange, India.
- CSI 300 index - which consists of 300 A-shares stocks listed on the Shanghai and Shenzhen Stock Exchange, China.
- FTSE/JSE Top 40 index - it includes the 40 largest companies by market capitalization on the JSE, South Africa.

The topic of this paper is to address the question that was raised at the ICOAE 2014, 3-5 July, Chania, Island of Crete, Greece, namely: how do the RNHD model generated volatility skews for the BRICS countries, as in [1], compare to those generated by using GARCH models? More precisely, a comparison is made between the volatility skews of the BRICS countries generated by using the RNHD model and those generated by using E-GARCH and GJR-GARCH models.

For notation and terminology not explained in the text, the reader may consult Hull [7] and Kotzé et al. [8, 9].

2. Model specification

2.1. RNHD model

A description of the risk neutral historical distribution (RNHD) model to compute the implied volatility of option prices is given by Hunzinger et al. [1]. Some of the details presented in [1] are repeated here for the convenience of the reader.

The model is based on the principle of relative entropy, with which a change of measure transforms a real world returns distribution of an asset into a risk-neutral returns distribution, which can then be used to find the price of an option by discounting the future expected cash flow under the risk-neutral measure by the risk-free rate.

For a given set of historical closing prices $M$ we obtain the daily $T$-period returns using

\[ \text{return}_i = \frac{\text{index}_{i+T}}{\text{index}_i} \quad \text{for} \quad i = 1, ..., M - T, \]

where $T$ is the maturity of the option to be valued.

We construct a historical return distribution by splitting the returns into bins of equal length and calculating the cumulative sum of the number of observations that lie in each bin. We divide this sum by the total number of observations to obtain the historical return distribution $P$.

To obtain the risk-neutral historical probability distribution $Q$, we apply the relative entropy principle which states that there exists a distribution $Q$ such that $E_Q[S_T] = S_0 \exp[rT]$, and

\[ E_Q \left[ \log \left( \frac{Q(x)}{P(x)} \right) \right] = \min \left\{ E_R \left[ \log \left( \frac{R(x)}{P(x)} \right) \right] : R \text{ is a distribution and } E_R[S_T] = S_0 \exp(rT) \right\}, \]

where $S_0$ is the spot price of the index, $S_t$ is the price at time $t$ and $r$ is the risk-free rate.

The RNHD $Q$ is calculated by applying the theory of Lagrange multipliers, which states that there exists a $\psi$ such that

\[ Q(x) = \frac{P(x) \exp(\psi x)}{\int_{-\infty}^{\infty} P(x) \exp(\psi x)}, \]
and \( \psi \) is determined numerically (see Cover and Thomas [10]). Under the risk-neutral measure \( Q \) the price of an option is then calculated by discounting the future expected payoff by the risk-free rate. The price of a European call option with strike \( K \) and maturity \( T \) is then given by

\[
C_0 = \sum_t Q(x) \max(S_0e^{rt} \text{return}_t - K, 0).
\]

### 2.2. GARCH models

To compute the implied volatility of option prices we perform a Monte-Carlo simulation using two different GARCH models, namely the GJR-GARCH model and the E-GARCH model. The asset return dynamics in the real world measure \( P \) are given by

\[
\ln \left( \frac{S_{t+1}}{S_t} \right) = r + \lambda \sqrt{h_{t+1}} \varepsilon_t + \frac{1}{2} h_{t+1} \varepsilon_t^2 + \varepsilon_t,
\]

where \( r \) is the one period continuously compounded risk free interest rate, and \( \varepsilon_t \) is a sequence of independent standard normal variables with respect to measure \( P \) and \( h_t \) is given by either a GJR-GARCH process, given by

\[
h_{t+1} = \beta_0 + h_t [\beta_1 + \beta_2 \varepsilon_t^2 + \beta_3 \max(0, -\varepsilon_t)^2],
\]
or a E-GARCH process, given by

\[
h_{t+1} = \exp \left( \beta_0 + \beta_1 \ln(h_t) + \beta_2 (|\varepsilon_t| + \gamma \varepsilon_t) \right),
\]

\( \lambda \) is a constant unit risk premium, and \( h_{t+1} \) is the conditional variance of the asset return. From Duan [5] the above processes, in a locally risk-neutral measure \( Q \), can be written as

\[
\ln \left( \frac{S_{t+1}}{S_t} \right) = r - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}} \zeta_{t+1},
\]

where \( \zeta_t \) is a sequence independent standard normal variables with respect to measure \( Q \). In the risk-neutral measure \( Q \) the conditional variance process of the GJR-GARCH model is given by

\[
h_{t+1} = \beta_0 + h_t \left[ \beta_1 + \beta_2 (\zeta_t - \lambda)^2 + \beta_3 \max(0, -\zeta_t + \lambda)^2 \right],
\]

and the conditional variance process of the E-GARCH model in the \( Q \) measure is given by

\[
h_{t+1} = \exp \left( \beta_0 + \beta_1 \ln(h_t) + \beta_2 [\zeta_t - \lambda] + \beta_3 (\zeta_t - \lambda) \right),
\]

where \( \zeta_t = \varepsilon_t + \lambda \) is a random innovation term which becomes a standard normal random variable under the measure \( Q \). In order to compute the price of an option with maturity \( T \), the terminal stock price \( S_T \) needs to be computed. The terminal stock price in the risk-neutral measure \( Q \) can be found by

\[
S_T = S_t \exp \left[ (T-t)r - \frac{1}{2} \sum_{t=1}^T h_i + \sum_{t=1}^T \sqrt{h_i} \zeta_t \right],
\]

where \( h \) is defined by either the GJR-GARCH or the E-GARCH process. To compute the price of a European call option, a Monte-Carlo simulation is performed by simulating \( n \) number of terminal stock prices with the above specifications, and discounting the average of the future expected payoff in the risk-neutral measure \( Q \) by the risk-free interest rate, i.e.

\[
C = e^{-r(T-t)} \frac{1}{n} \sum_{i=1}^n \max(S_{T_i} - K, 0).
\]

The implied volatility is found by matching the price of the call option computed through the Monte-Carlo simulation with the volatility, using the Black-Scholes model.
3. Model parameter values

The central bank rates of the various BRICS nations are used as a proxy for the risk-free rate, and are given in table 1, along with the relevant spot index levels on the 28th May 2014.

Table 1: The central bank rate for the various BRICS countries and the relevant countries’ index spot levels on the 28th May 2014. (Source: [13, 14, 15])

<table>
<thead>
<tr>
<th>BRICS nation</th>
<th>Central bank rate</th>
<th>Index</th>
<th>Current index level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brazil</td>
<td>11%</td>
<td>IBrX</td>
<td>21695.41</td>
</tr>
<tr>
<td>Russia</td>
<td>7%</td>
<td>INDEXCF</td>
<td>1381.50</td>
</tr>
<tr>
<td>India</td>
<td>8%</td>
<td>S&amp;P BSE Sensex</td>
<td>24556.09</td>
</tr>
<tr>
<td>China</td>
<td>6%</td>
<td>CSI 300</td>
<td>2169.35</td>
</tr>
<tr>
<td>South Africa</td>
<td>5.5%</td>
<td>FTSE/JSE Top 40</td>
<td>44732.26</td>
</tr>
</tbody>
</table>

The fitted parameters for the GARCH models are given in table 2 and 3, and were found by using maximum likelihood estimators.

Table 2: E-GARCH model parameters for the different indexes.

<table>
<thead>
<tr>
<th>Country</th>
<th>Index</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brazil</td>
<td>IBrX</td>
<td>-0.1546</td>
<td>0.9818</td>
<td>0.1333</td>
<td>-0.0827</td>
</tr>
<tr>
<td>South Africa</td>
<td>FTSE/JSE Top 40</td>
<td>-0.1144</td>
<td>0.9873</td>
<td>0.0998</td>
<td>-0.1066</td>
</tr>
<tr>
<td>India</td>
<td>S&amp;P BSE Sensex</td>
<td>-0.0626</td>
<td>0.9925</td>
<td>0.1293</td>
<td>-0.0441</td>
</tr>
<tr>
<td>Russia</td>
<td>INDEXCF</td>
<td>-0.1581</td>
<td>0.9803</td>
<td>0.1411</td>
<td>-0.0538</td>
</tr>
<tr>
<td>China</td>
<td>CSI 300</td>
<td>-0.1581</td>
<td>0.9803</td>
<td>0.1411</td>
<td>-0.0538</td>
</tr>
</tbody>
</table>

Table 3: GJR-GARCH model parameters for the different indexes.

<table>
<thead>
<tr>
<th>Country</th>
<th>Index</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brazil</td>
<td>IBrX</td>
<td>3.4564E-06</td>
<td>0.9192</td>
<td>0.0129</td>
<td>0.0980</td>
</tr>
<tr>
<td>Alsi</td>
<td>FTSE/JSE Top 40</td>
<td>1.3777E-06</td>
<td>0.9302</td>
<td>NA</td>
<td>0.1172</td>
</tr>
<tr>
<td>India</td>
<td>S&amp;P BSE Sensex</td>
<td>2.0065E-06</td>
<td>0.9176</td>
<td>0.0089</td>
<td>0.1340</td>
</tr>
<tr>
<td>Russia</td>
<td>INDEXCF</td>
<td>4.9150E-06</td>
<td>0.9037</td>
<td>0.0444</td>
<td>0.0650</td>
</tr>
<tr>
<td>China</td>
<td>CSI 300</td>
<td>4.9150E-06</td>
<td>0.9037</td>
<td>0.0444</td>
<td>0.0650</td>
</tr>
</tbody>
</table>

4. GARCH model interpretation

Conventional wisdom among GARCH researchers is that a major restriction on GARCH models is the non-negativity constraint on the coefficients. An important benefit of using an E-GARCH model is that there are no non-negativity constraints. Furthermore, according to Francq and Zakoian [11], the GJR-GARCH model is flexible because it allows past lags to display asymmetries, this gives an indication of the leverage effect. According to Brooks [12], in order for a leverage effect to exist, the coefficient of the threshold term of the GJR-GARCH model needs to be positive. In addition, because of the negative relationship between volatility and returns, the coefficient of the threshold term of the E-GARCH model will be negative.

If one considers the Akaike information criterion (AIC) and Bayesian information criterion (BIC) of the GARCH models and the GJR-GARCH models for all the countries included it is evident that the GJR-GARCH model is a better fit for all the countries. The Augmented Dickey Fuller test indicates that the returns are stationary at a five percent level of significance. Additionally, the ARCH-LM test shows evidence of ARCH effects at a five percent level of significance. Furthermore, all the asymmetry terms of the GJR-GARCH models are statistically significant and have
the correct sign, this is evidence of leverage effects. Moreover, the asymmetry terms of the E-GARCH models are negative and statistically significant; this is also evidence of the existence of volatility feedback or leverage effects. This implies that rise in volatility is greater after a negative shock when compared to the rise in volatility after a positive shock (see Brooks [12]).

<table>
<thead>
<tr>
<th>Country</th>
<th>Index</th>
<th>GARCH(1,1) AIC</th>
<th>GARCH(1,1) BIC</th>
<th>GJR-GARCH(1,1) AIC</th>
<th>GJR-GARCH(1,1) BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brazil</td>
<td>IBrX</td>
<td>-6.31766</td>
<td>-6.30573</td>
<td>-6.34823</td>
<td>-6.33331</td>
</tr>
<tr>
<td>Alsi</td>
<td>FTSE/JSE Top 40</td>
<td>-5.88987</td>
<td>-5.87794</td>
<td>-5.90504</td>
<td>-5.89013</td>
</tr>
<tr>
<td>India</td>
<td>S&amp;P BSE Sensex</td>
<td>-6.45092</td>
<td>-6.43898</td>
<td>-6.46519</td>
<td>-6.45028</td>
</tr>
<tr>
<td>Russia</td>
<td>INDEXCF</td>
<td>-5.99797</td>
<td>-5.98604</td>
<td>-6.00614</td>
<td>-5.99123</td>
</tr>
<tr>
<td>China</td>
<td>CSI 300</td>
<td>-6.60274</td>
<td>-6.59081</td>
<td>-6.62603</td>
<td>-6.61112</td>
</tr>
</tbody>
</table>

5. Volatility skews of the models

In our implementation of the above models we calibrate the GARCH processes to historical closing prices from the 2nd January 2009 to the 28th May 2014. The options which are priced are written on the 28th May 2014 and have a maturity of 10 days.

Fig. 1: The prices of European call options on the Russian INDEXCF index for the RNHD, E-GARCH and GJR-GARCH models.

Figure 1 shows that the call option price is model sensitive, with very similar E-GARCH - and GJR-GARCH model generated prices, which are typically greater than the RNHD model generated call option prices.

Figures 2, 3 and 4 provide a comparison of the BRICS countries’ volatility skews generated by the RNHD -, the E-GARCH - and the GJR-GARCH models for the European call option. These graphs emphasize the similarity in shapes between the corresponding volatility skews generated by the two GARCH models, and their differences in shapes with the corresponding RNHD model generated volatility skews. The RNHD model produces volatility skews which tend to increase as the moneyness increases, whereas the GARCH models produce smoother volatility skews. The RNHD model only calibrates one parameter to fit the forward distribution, namely $\psi$, instead of the four different parameters from the GARCH models given by $\beta_i$, $i = 0, 1, 2, 3$ which intuitively makes the GARCH models more accurate than the RNHD model.
Fig. 2: The RNHD model’s implied volatility skews for European call options on the BRICS nations indexes.

Fig. 3: The E-GARCH model’s implied volatility skews for European call options on the BRICS nations indexes.

Fig. 4: The GJR-GARCH model’s implied volatility skews for European call options on the BRICS nations indexes.
Fig. 5: The effect of different interest rates on the implied volatility skews of European call options on the Russian INDEXCF index obtained with the RNHD model.

Fig. 6: The effect of different interest rates on the implied volatility skews of European call options on the Russian INDEXCF index obtained with the E-GARCH model.

Fig. 7: The effect of different interest rates on the implied volatility skews of European call options on the Russian INDEXCF index obtained with the GJR-GARCH model.
6. The effect of interest rates on the models

Figures 5, 6 and 7 provide a comparison of different interest rates on the volatility skews of the Russian INDEXCF index generated by the RNHD-, the E-GARCH- and the GJR-GARCH models. The latter two models yield similar shaped volatility skews, whereas the former yields corresponding volatility skews different in shape to those of the GARCH models. The RNHD model is more sensitive to changes in interest rates than the GARCH models. The E-GARCH model produces negative sloping implied volatility skews for low interest rates. In general we can observe that a higher interest rate causes the volatility skews to becomes higher. This is explained by noting that the forward price of the underlying asset, \( S_T \), increases as the interest rate increases, since the expected return of the asset is equal to the risk free interest rate in the risk neutral measure. This in turn means that the distribution of the forward price of the asset is higher, which causes the price of an European call option to increase. Since an increase in the prices of options imply a higher volatility skew, we observe that the higher interest rates increase the implied volatility skew. If the interest rate is close to zero, then the forward price of the underlying asset under the risk neutral distribution is approximately equal to the spot price, and the distribution of the future expected asset price \( S_T \) constructed by the models becomes a function of only the volatility of the underlying’s returns. Since the GARCH models employ E-GARCH and GJR-GARCH processes to explicitly model the volatility of the underlying, and the RNHD model does not, the volatility skews produced for close to zero interest rates should be more accurate for the GARCH models than the RNHD model. Continuing on this path of thought, since we assume that the risk-free rate is constant, this argument holds for any interest rate, and as such one can assume that the GARCH models produce more accurate results than the RNHD model.

7. Conclusion

This paper compares the option prices produced by three different models, namely the RNHD, E-GARCH and GJR-GARCH models, and the implied volatility skews for these models are constructed. The two GARCH models produce similar results, but differ significantly from the RNHD model. The argument that the RNHD model only implicitly calibrates one parameter to the historical return time series, instead of the four parameters used by the GARCH models might imply that the GARCH models are more accurate than the RNHD model. The last section argues that for constant interest rates the GARCH models produce more accurate results than the RNHD model.

References


