On the global solvability in Gevrey classes on the \( n \)-dimensional torus

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Abstract

Let \( P \) be a linear partial differential operator with coefficients in the Gevrey class \( G^s(T^n) \), where \( T^n \) is the \( n \)-dimensional torus and \( s \geq 1 \). We prove a necessary condition for the \( s \)-global solvability of \( P \) on \( T^n \). We also apply this result to give a complete characterization for the \( s \)-global solvability for a class of formally self-adjoint operators with nonconstant coefficients.

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In the last years many papers are concerned with the study of the global solvability and hypoellipticity of linear partial differential operators on compact manifolds, e.g., torus, in large scales of functional spaces (see, e.g., [3,7–10,18–20,23] and references listed therein). It is well known that the theory of global properties of differential operators is not well developed in comparison with the one of local properties. In particular, the global properties are open problems except for certain classes of operators. On the other hand, the local and global solvability/hypoellipticity are rather different in general. In fact, there are...
hyperbolic linear partial differential operators which are hypoelliptic on the torus and there are solvable linear partial differential operators whose adjoint operator have an infinite-dimensional kernel.

Motivated by these facts, in this paper we are interested in the problem of global solvability for linear partial differential operators in the setting of Gevrey classes $G^s$ of order $s \geq 1$ on the $n$-dimensional torus $T^n$. More precisely, we give a necessary condition for the global solvability in $G^s(T^n)$ in terms of a priori estimate analogous to the ones of [11,12] and [6] (see Section 2). The proof of this abstract result is of a functional–analytic nature and more difficult in comparison with the $C^\infty$-case because of the complicated inductive topology of $G^s(T^n)$. We also apply such a necessary condition to give a complete characterization for the Gevrey-solvability for a class of linear partial differential operators with variable coefficients, where diophantine properties of the coefficients play a crucial role (see Section 3).

1. Preliminaries

If $\Omega$ is an open subset of $\mathbb{R}^n$ we write $C^\infty(\Omega)$ for the space of all infinitely differentiable functions in $\Omega$; $C^\infty_0(\Omega)$ is the subspace of $C^\infty(\Omega)$ of all compactly supported functions and $\mathcal{D}'(\Omega)$, its topological dual, is the space of all distributions on $\Omega$. If $K$ is a regular compact set (i.e., $K$ is the closure of the set of its interior points) then we can also define $C^\infty(K)$ as the space of all infinitely differentiable functions in $\bar{K}$ whose partial derivatives of every order have bounded continuous extension on $\partial K$.

The Gevrey classes $G^s(\Omega)$ are defined as follows. Let $K$ be a regular compact subset of $\mathbb{R}^n$ and $\eta > 0$, $s \geq 1$; we denote

$$G^s(K, \eta) = \left\{ \phi \in C^\infty(K); |\phi; K, s, \eta| := \sum_{\alpha \in \mathbb{N}^n_0} |D^\alpha \phi|_{L^2(K)} \frac{\eta^{|\alpha|}}{(\alpha!)^s} < \infty \right\},$$

where

$$|\phi|_{L^2(K)} = \left( \int_K |\phi(x)|^2 \, dx \right)^{1/2}.$$

Then we define

$$G^s(K) = \operatorname{ind lim}_{\eta \to 0} G^s(K, \eta), \quad G^s(\Omega) = \operatorname{proj lim}_{K \subset \Omega} G^s(K).$$

In particular $G^1(\Omega) = A(\Omega)$ is the space of all real analytic functions in $\Omega$.

We emphasize that there are in fact many different but equivalent ways to define these spaces; see [14,15,21]. We also remark that the space $G^1(K)$ is an inductive limit of Banach spaces with compact linking maps and then it is a dual Fréchet–Schwartz space, i.e., a (DFS)-space (therefore, the strong dual of a reflexive Fréchet space).

The elements of the topological dual of these spaces are called ultradistributions for $s > 1$ and real analytic functionals for $s = 1$ [15,22]. In particular, denote by $\mathcal{E}^e_0(\Omega) =$
(\mathcal{G}^s(\Omega))' and \mathcal{E}'_s(K) = (\mathcal{G}^s(K))' both endowed with the corresponding strong topology; therefore, \mathcal{E}'_s(K) is a Fréchet–Schwartz space.

Next, denote by \( T^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n \) the \( n \)-dimensional torus. For each \( s \geq 1 \) let \( \mathcal{G}^s(T^n) \) be the space of all \( \mathcal{G}^s \)-functions on \( T^n \), which are identified with the \( \mathcal{G}^s \)-functions on \( \mathbb{R}^n \) that are \( 2\pi \)-periodic in each variable. Clearly, \( \mathcal{G}^s(T^n) \) is a closed subspace of \( \mathcal{G}^s(\mathbb{R}^n) \).

Moreover, put \( K_\pi = [-\pi, \pi] \) and \( \mathcal{G}^s(T^n, \eta) = \{ \phi \in \mathcal{G}^s(T^n) ; |\phi; K_\pi, s, \eta| = \sum_{\alpha \in \mathbb{N}^n_0} |D^\alpha \phi|_{L^2(K_\pi)} \eta^{|\alpha|} (\alpha!)^s < \infty \} \) for each \( \eta > 0 \), it holds that the inclusions maps \( \mathcal{G}^s(T^n, \eta) \hookrightarrow \mathcal{G}^s(T^n, \eta') \), \( \eta > \eta' > 0 \), are compact and that \( \mathcal{G}^s(T^n) = \text{ind lim } \eta \to 0 \mathcal{G}^s(T^n, \eta) \) when \( \mathcal{G}^s(T^n) \) is endowed with the topology induced on it by \( \mathcal{G}^s(\mathbb{R}^n) \). Consequently, it is also a dual Fréchet–Schwartz space, i.e., a (DFS)-space (hence the strong dual of a reflexive Fréchet space) and its topological dual \( \mathcal{E}'_s(T^n) = (\mathcal{G}^s(T^n))' \) equipped with the corresponding strong topology is a Fréchet–Schwartz space.

We denote as usual \( D^\alpha = D\alpha^1 \cdots D\alpha^n \) for every \( \alpha \in \mathbb{N}^n_0 \), where \( D_j = -i \partial / \partial x_j \) for \( j \in \mathbb{N} \), and consider the linear partial differential operator of order \( m \),

\[
P = P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,
\]

(1.1)

with \( (a_\alpha)_{|\alpha| \leq m} \subset \mathcal{G}^s(T^n) \). Clearly, \( P \) and its transposed operator \( ^tP \) are linear continuous maps both from \( \mathcal{G}^s(T^n) \) in \( \mathcal{G}^s(T^n) \) and from \( \mathcal{E}'_s(T^n) \) in \( \mathcal{E}'_s(T^n) \).

The operator \( P \) is said to be \( s \)-globally hypoelliptic (globally analytic hypoelliptic if \( s = 1 \)) in \( T^n \) if the conditions \( u \in \mathcal{E}'_s(T^n) \) and \( Pu \in \mathcal{G}^s(T^n) \) imply that \( u \in \mathcal{G}^s(T^n) \). The operator \( P \) is said to be \( s \)-globally solvable (globally analytic solvable if \( s = 1 \)) in \( T^n \) if for every \( f \in \{ v \in \mathcal{G}^s(T^n) ; \int_{T^n} vw \, dx = 0 \text{ for all } w \in \mathcal{G}^s(T^n) \text{ with } ^tPw = 0 \} \) there exists \( u \in \mathcal{E}'_s(T^n) \) such that \( Pu = f \) in \( T^n \).

This definition allows, in principle, an infinite-dimensional \( \text{Ker}^tP \) (see [18] or Remark 3.2).

Let us define

\[
\mathcal{E}(P) := \left\{ v \in \mathcal{G}^s(T^n) ; \int_{T^n} vw \, dx = 0 \text{ for all } w \in \mathcal{G}^s(T^n) \text{ with } ^tPw = 0 \right\}.
\]

Then \( \mathcal{E}(P) \) is clearly a closed subspace of \( \mathcal{G}^s(T^n) \) and hence it is also a (DFS)-space with respect to the locally convex topology \( \tau \) induced on it by \( \mathcal{G}^s(T^n) \) (cf. [16, Theorem 7′]); actually,

\[
(\mathcal{E}(P), \tau) = \text{ind lim } \eta \to 0 \mathcal{G}^s(T^n, \eta) \cap \mathcal{E}(P).
\]

(1.2)

We denote by \( \mathcal{E}(P, \eta) \) the Banach space \( (\mathcal{G}^s(T^n, \eta) \cap \mathcal{E}(P), |v; K_\pi, s, \eta|) \) and by \( |\cdot|'_\eta \) the canonical dual norm defined on \( (\mathcal{G}^s(T^n, \eta), |v; K_\pi, s, \eta|)' \). We point out that, for each \( \eta > \eta' > 0 \), \( ((\mathcal{G}^s(T^n, \eta'))', |\cdot|'_{\eta'}) \) is continuously embedded in \( ((\mathcal{G}^s(T^n, \eta)), |\cdot|_{\eta})' \).
Other notation is standard. We refer the reader for functional analysis to [13,24], and for the theory of linear partial differential operators to [11].

2. A necessary condition for the \( s \)-global solvability

In this section we will prove a necessary condition for the \( s \)-global solvability for linear partial differential operators with variable coefficients and acting on \( G^s(T^n) \). In order to give such a result we need the following lemma which should be compared with [6, Proposition 2.2].

**Lemma 2.1.** Let \( P \) be a linear differential operator of order \( m \) with coefficients in \( G^s(T^n) \), with \( s \geq 1 \). If \( P \) is \( s \)-globally solvable in \( T^n \), then for every \( \eta > 0 \) there exists \( C_\eta > 0 \) such that

\[
\forall f \in \mathcal{E}(P, \eta), \exists u \in (G^s(T^n, \eta))' \text{ such that } Pu = f \text{ in } T^n \\
\text{and } |u|'_\eta \leq C_\eta |f|_{K_\pi, s, \eta}.
\]  

**(2.3)**

**Proof.** Fixed \( \eta > 0 \), for each \( h \in \mathbb{N} \) let us define

\[ F_h := \{ f \in \mathcal{E}(P, \eta); \exists u \in (G^s(T^n, \eta))' \text{ such that } Pu = f \text{ in } T^n \text{ and } |u|'_\eta \leq h \}. \]

Since \( P \) is \( s \)-globally solvable in \( T^n \), it clearly holds that

\[
\bigcup_{h=1}^{\infty} F_h = \mathcal{E}(P, \eta);
\]

hence, by Baire’s theorem, there exists \( h_0 \in \mathbb{N} \) so that \( \hat{F}_{h_0} \neq \emptyset \), where \( F_{h_0} \) is an absolutely convex set. Thus, it follows that

\[ F_{h_0} \supseteq U = \{ f \in \mathcal{E}(P, \eta); |f|_{K_\pi, s, \eta} \leq \delta \} \]

for some \( \delta > 0 \). Consequently, given \( f \in \mathcal{E}(P, \eta) \), \( f \neq 0 \), it holds that \( \delta |f|_{K_\pi, s, \eta}^{-1} f \in U \) and then \( Pu = \delta |f|_{K_\pi, s, \eta}^{-1} f \) for some \( v \in (G^s(T^n, \eta))' \) with \( |v|'_\eta \leq h_0 \). If we set \( u = \delta^{-1} |f|_{K_\pi, s, \eta} v \), \( C_\eta = \delta^{-1} h_0 \), we obtain that

\[ |u|'_\eta \leq C_\eta |f|_{K_\pi, s, \eta}. \]

This completes the proof. \( \square \)

Now, we are able to state and show the main result of this section.

**Theorem 2.1.** Let \( P \) be a linear differential operator of order \( m \) with coefficients in \( G^s(T^n) \), with \( s \geq 1 \). Let \( \{ p_i; i \in I \} \) be a fundamental system of continuous seminorms of \( G^s(T^n) \). If \( P \) is \( s \)-globally solvable in \( T^n \), then there exist \( i_0 \in I \) and \( C > 0 \) such that

\[
\left| \int_{T^n} fg \, dx \right| \leq C p_{i_0}(f) p_{i_0}(g) \]

\[ (2.4) \]

for every \( f \in \mathcal{E}(P) \) and \( g \in G^s(T^n) \).
Proof. First, we put $\tau = P(G^s(T^n))$ (the closure of $P(G^s(T^n))$ in $G^s(T^n)$). Since $\mathcal{R}$ is a closed subspace of $G^s(T^n)$, it is a (DFS)-space with respect to the locally convex topology $\tau'$ induced on it by $G^s(T^n)$; in particular, it holds that

$$(\mathcal{R}, \tau') = \text{ind lim}_{\eta \to 0} G^s(T^n, \eta) \cap \mathcal{R}.$$  

On the other hand, it is well known that

$$\ker P = \mathcal{R},$$

when $P$ acts from $\mathcal{E}'_f(T^n)$ into itself (here, we denote by “$^\circ$” the polar with respect to the duality $(G^s(T^n), \mathcal{E}'_f(T^n))$).

Since $P$ is $s$-globally solvable in $T^n$, we can define a form $\Phi$ on $\mathcal{E}(P) \times \mathcal{R}$ in the following way. For each $f \in \mathcal{E}(P)$ and $g \in \mathcal{R}$, let

$$\Phi(f, g) := (g, u),$$

where $u$ is any $s$-ultradistribution on $T^n$ satisfying $Pu = f$ in $T^n$. By (2.5), $\Phi$ is well defined. Indeed, if $f \in \mathcal{E}(P)$ and $u_1, u_2 \in \mathcal{E}'_f(T^n)$ such that $Pu_1 = f = Pu_2$, then $u_1 - u_2 \in \ker P$ and hence

$$\langle Pu_1 - Pu_2, u \rangle = 0$$

for all $g \in G^s(T^n)$, thereby implying that $(g, u_1 - u_2) = 0$ for all $g \in \mathcal{R}$ by (2.5). Moreover, $\Phi$ is a bilinear form as it is proved in the following.

Given any $f \in \mathcal{E}(P)$, it holds that, for every $g_1, g_2 \in \mathcal{R}$ and for every $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\Phi(\lambda_1 f_1 + \lambda_2 f_2, g) = (\lambda_1 f_1 + \lambda_2 f_2, u) = \lambda_1 \Phi(f_1, g) + \lambda_2 \Phi(f_2, g),$$

where $u \in \mathcal{E}'_f(T^n)$ satisfies $Pu = f$ in $T^n$.

On the other hand, given any $g \in \mathcal{R}$, for every $f_1, f_2 \in \mathcal{E}(P)$ we have that $\Phi(f_1 + f_2, g) = (g, u_1)$ and $\Phi(f_1, g) = (g, u_1)$, $\Phi(f_2, g) = (g, u_2)$, where $u, u_1, u_2 \in \mathcal{E}'_f(T^n)$ satisfy $Pu = f_1 + f_2, Pu_1 = f_1$ and $Pu_2 = f_2$ in $T^n$, respectively. Consequently, $Pu = f_1 + f_2 = Pu_1 + Pu_2 = Pu_1 + Pu_2$ in $T^n$ so that $u = (u_1 + u_2) \in \ker P$, thereby implying that $\Phi(f_1 + f_2, g) = \Phi(f_1, g) + \Phi(f_2, g)$. In the same way, one proves that, for every $f \in \mathcal{E}(P)$ and for every $\lambda \in \mathbb{C}$, $\Phi(\lambda f, g) = \lambda \Phi(f, g)$.

Moreover, we claim that $\Phi$ is separately continuous when $\mathcal{E}(P)$ and $\mathcal{R}$ are both endowed with the locally convex topology induced on them by $G^s(T^n)$.

First we prove that, fixed any $g \in \mathcal{R}$, the linear functional

$$f \in \mathcal{E}'(P) \mapsto \Phi(f, g)$$

is $\tau$-continuous on $\mathcal{E}(P)$.

Suppose that $g \in G^s(T^n, \eta_0)$ for some $\eta_0 > 0$. Let $\eta > \eta_0$ and let $(f_h)_h \subset \mathcal{E}(P, \eta)$ be a sequence which converges to 0 in $\mathcal{E}(P, \eta)$. By (2.3), for each $h \in \mathbb{N}$ there exists $u_h \in (G^s(T^n, \eta)^\prime)$ for which $Pu_h = f_h$ in $T^n$ and

$$|u_h|_\eta \leq C_\eta |f_h|_{K_\eta, s, \eta}.$$
This implies that \( u_h \overset{h}{\rightharpoonup} 0 \) in \( (G^s(T^n, \eta))' \). Since the inclusion map
\[
(G^s(T^n, \eta))', \quad \|\cdot\|_0 \hookrightarrow \left( (G^s(T^n, \eta_0))', \quad \|\cdot\|_{m_0} \right)
\]
is continuous, it follows that \( u_h \overset{h}{\rightharpoonup} 0 \) in \( (G^s(T^n, \eta_0))' \) too; hence \( \Phi_g(f_h) = \langle g, u_h \rangle \overset{h}{\rightharpoonup} 0 \) because \( g \in G^s(T^n, \eta_0) \).

Since \((E(P), \tau)\) is an (LB)-space given by (1.2), it follows that \( \Phi_g \) is \( \tau \)-continuous.

Next, fix \( f \in E(P) \) and \( u \in E'_s(T^n) \) for which \( Pu = f \) in \( T^n \). Then the linear functional
\[
g \in \mathcal{R} \overset{\Phi}{\mapsto} \langle g, u \rangle
\]
is \( \tau' \)-continuous on \( \mathcal{R} \). Indeed, if \((g_\alpha)_{\alpha} \subset \mathcal{R} \) is \( \tau' \)-convergent to \( 0 \) (or equivalently converges to \( 0 \) in \( G^s(T^n) \)), then \( \Phi_g(g_\alpha) = \langle g_\alpha, u \rangle \overset{\alpha}{\rightharpoonup} 0 \) because \( u \in E'_s(T^n) \).

Finally, since a bilinear form in the product of two strong duals of reflexive Fréchet spaces is continuous if it is separately continuous (cf. [24, Theorem 41.1] or [13, Chapter 4, §7]), we obtain that \( \Phi \) is continuous on \( (E(P), \tau) \times (\mathcal{R}, \tau') \), being \((E(P), \tau)\) and \((\mathcal{R}, \tau')\) (DFS)-spaces. Now, the locally convex topologies \( \tau \) and \( \tau' \) are both generated by the same system \((p_i)_{i \in I}\) of continuous seminorms; hence, there exist \( i_0 \in I \) and \( C > 0 \) for which
\[
\|\Phi(f, g)\| \leq C p_{i_0}(f) p_{i_0}(g)
\]
for all \( f \in E(P) \) and \( g \in \mathcal{R} \).

At this point, we observe that, for every \( f \in E(P) \) and \( g \in G^s(T^n) \),
\[
\Phi(f, \xi Pg) = \langle \xi Pg, u \rangle = \langle g, Pu \rangle = \langle g, f \rangle = \int_{T^n} f g \, dx,
\]
where \( u \in E'_s(T^n) \) satisfies \( Pu = f \) in \( T^n \); by (2.6) it follows that, for every \( f \in E(P) \) and \( g \in G^s(T^n) \),
\[
\left| \int_{T^n} f g \, dx \right| \leq C p_{i_0}(f) p_{i_0}(\xi Pg).
\]
This completes the proof. \( \square \)

We point out that Theorem 2.1 continues to hold on large classes of compact manifolds too.

**Remark 2.1.** Using Fourier-expansion, in [25, Lemma 7.1] (see also [25, Lemma 8.1]) it was given a sequence space representation for the space of all \( 2\pi \)-periodic functions in the \( \omega \)-ultradifferentiable function space \( \mathcal{E}_\omega([0])\) of Roumieu type, with \( \omega \) nonquasianalytic weight-function (for their definition let us refer to [5, §4]). Since for \( \omega(t) = t^{1/s}, t \geq 0 \) and \( s > 1 \), the class \( \mathcal{E}_\omega([0]) \) coincides with the Gevrey class \( G^s \), from such a result it follows that
\[
G^s(T^n) \simeq \underset{m \to \infty}{\text{ind lim}} \left\{ f = \sum_{\nu \in \mathbb{Z}^n} a_\nu e^{i\nu x}; \quad |f|_m := \sum_{\nu \in \mathbb{Z}^n} |a_\nu| e^{m|\nu|} < +\infty \right\},
\]
where the topological isomorphism onto is given by
\[
f \in G^s(T^n) \leftrightarrow (a_\nu)_{\nu \in \mathbb{Z}^n}.
with \( a_v := (2\pi)^{-n} \int_{\mathbb{T}^n} f(x) e^{-ivx} \, dx \), \( v \in \mathbb{Z}^n \) (observe that \( \{ f = \sum_{\nu \in \mathbb{Z}^n} a_\nu e^{ivx}; |f|_m := \sum_{\nu \in \mathbb{Z}^n} |a_\nu| e^{1/m|\nu|^{1/s}} < +\infty \} \)

is a Banach space with respect to the norm \( |·|_m \). Consequently, a fundamental system of continuous seminorms for \( G^s(T^n) \) can be defined as follows.

For each \( m \in \mathbb{N} \) and \( \nu \in \mathbb{Z}^n \) let \( v_m(\nu) := e^{1/m|\nu|^{1/s}} \). Clearly, \( v_m(v) \geq v_{m+1} \) on \( \mathbb{Z}^n \) for all \( m \in \mathbb{N} \).

Then the space \( K^s(T^n) := \{ f = \sum_{\nu \in \mathbb{Z}^n} a_\nu e^{ivx}; \forall \tilde{v} \in \tilde{V}, \max_{\nu \in \mathbb{Z}^n} |a_\nu| \tilde{v}(\nu) < +\infty \} \)

is clearly a complete locally convex space with respect to the topology generated by the system \( (\tilde{p}_\tilde{v})_{\tilde{v} \in \tilde{V}} \) of seminorms. Moreover, by (2.7) and by proceeding exactly as in the proofs of [4, Lemmas 2.1 and 2.2, and Theorem 2.3], one proves that \( G^s(T^n) \simeq K^s(T^n) \), where the topological isomorphism onto is again given by \( f \in G^s(T^n) \leftrightarrow (a_\nu)_{\nu \in \mathbb{Z}^n} \); hence \( (\tilde{p}_\tilde{v})_{\tilde{v} \in \tilde{V}} \) is a fundamental system of continuous seminorms for \( G^s(T^n) \).

Finally, we point out that, given any increasing sequence \( (J_m)_m \) of finite sets of \( \mathbb{Z}^n \) with \( \bigcup_{m \in \mathbb{N}} J_m = \mathbb{Z}^n \), for every decreasing sequence \( (d_m)_m \) of positive numbers the positive weight function \( \tilde{w}(v) \), defined by

\[
\tilde{w}(v) := d_m v_m(v) \quad \text{if} \quad v \in J_m \setminus J_{m-1} \quad \text{for some} \quad m \in \mathbb{N},
\]

belongs to \( \tilde{V} \). Indeed, for a fixed \( m \in \mathbb{N} \) and an arbitrary \( k \geq m \), if \( v \in J_k \setminus J_{k-1} \),

\[
\frac{\tilde{w}(v)}{v_m(v)} \leq \frac{\tilde{w}(v)}{v_k(v)} = d_k \leq d_m,
\]

hence

\[
\sup_{v \in \mathbb{Z}^n} \frac{\tilde{w}(v)}{v_m(v)} = \sup_{k \in \mathbb{N}} \sup_{v \in J_k \setminus J_{k-1}} \frac{\tilde{w}(v)}{v_m(v)} = \max \left\{ \frac{\tilde{w}(v)}{v_m(v)} : \sup_{k \geq m} \right\} \leq \max_{v \in J_m} \frac{\tilde{w}(v)}{v_m(v)} < +\infty.
\]

Vice versa, for each \( \tilde{w} \in \tilde{V} \) there exists a weight function \( \tilde{w} \) defined as in (2.8), where \( (d_m)_m \) is a suitable decreasing sequence of positive numbers, such that \( \tilde{w} \leq \tilde{w} \). Indeed, it suffices to take

\[
d_m := \sup_{v \in \mathbb{Z}^n} \frac{\tilde{w}(v)}{v_m(v)} \quad \text{for all} \quad m \in \mathbb{N}.
\]
3. Application

In the following we will give a complete characterization of the $s$-global solvability for the following class of self-adjoint operators on $T^4$:

$$P = -\partial_t^2 - (\partial_x + a_1(t)\partial_{x_1} + a_2(t)\partial_{x_2})^2,$$

where $a_1, a_2 \in G^s(T^1) (s > 1)$ and are real valued (we will use the variables $(t, y, x_1, x_2) \in T^4$). We recall that such a class of operators was studied in [10,18] and In [9] in the frame of $C^\infty$-class and of $G^s$-classes, respectively. In [9,10] the authors investigated conditions which are necessary and sufficient for the global hypoellipticity; while, in [18] the global solvability in $D'(T^4)$ was studied.

In order to do this, we need to give further definitions and results.

**Definition 3.1** (see [9]). Let $s \geq 1$. A collection of real numbers $\alpha_1, \ldots, \alpha_n$ is said to be not simultaneously $s$-exponentially approximable in $\mathbb{Q}$ if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for any $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{Z}^n$ and $\xi \in \mathbb{N}$ we have

$$|\alpha_j \xi - \eta_j| \geq C_\epsilon e^{-\epsilon \xi^{1/s}} \text{ for some } j = 1, \ldots, n.$$

It follows that $\alpha_1, \ldots, \alpha_n$ are simultaneously $s$-exponentially approximable in $\mathbb{Q}$ if there exists $\epsilon_0 > 0$ for which there exist two sequences $(\eta_k)_k \subset \mathbb{Z}^n$ and $(\xi_k)_k \subset \mathbb{N}$ such that

$$|\alpha_j \xi_k - \eta_{k,j}| \leq C \epsilon_0 \xi_k^{1/s} \quad \text{for all } j = 1, \ldots, n. \quad (3.10)$$

**Definition 3.2** (see [7,9]). A vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \setminus \mathbb{Q}^n$ is said to be not an $s$-Liouville vector if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\alpha \xi - \eta| \geq C_\epsilon e^{-\epsilon |\xi|^{1/s}} \quad \text{for all } (\eta, \xi) \in \mathbb{Z}^{1+n}. \quad (3.11)$$

Therefore a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \setminus \mathbb{Q}^n$ is an $s$-Liouville vector if there exists $\epsilon_0 > 0$ such that for every $C > 0$ there exists a sequence $((\eta_k, \xi_k))_k \subset \mathbb{Z}^{1+n}$, $\xi_k = (\xi_k, 1, \ldots, \xi_k, n) \neq 0$, for which

$$|\alpha \xi_k - \eta_k| \leq C e^{-\epsilon_0 |\xi_k|^{1/s}} \quad \text{for all } k \in \mathbb{N}.$$

We refer, e.g., to [18, Definition 1.5] (or see [10]) for the proper analogue definitions in the setting of $C^\infty$-class.

**Remark 3.1.** Let $n = 1$. Using the arguments of [17], in [7] it was proved that there exist real numbers $\alpha$ satisfying (3.10) or (3.11). Moreover, such numbers $\alpha$ could be found in the set of irrationals and they have the density of the continuum (see [7, Appendix, Lemma A]). On the other hand, a pair of real numbers $\alpha$ and $\beta$ are simultaneously $s$-exponentially approximable in $\mathbb{Q}$ if both are rational numbers, or if one is rational and the other is an $s$-Liouville number. If one of the numbers is not $s$-Liouville, then the numbers are not simultaneously $s$-exponentially approximable in $\mathbb{Q}$.

From the proof of [9, Theorem 1] it follows that for any dimension it holds:
**Lemma 3.1.** Let $s \geq 1$. Let $T^{1+n}$ be the $(1 + n)$-dimensional torus with variables $(t, x_1, \ldots, x_n)$ and $Q$ be given by

$$Q = -\partial_t^2 - \left( \sum_{j=1}^{n} a_j(t) \partial_{x_j} \right)^2,$$

where the functions $a_j \in G^s(T^1)$ and are real valued. Let

$$w(t, \xi) = \sum_{j=1}^{n} a_j(t) \xi_j \text{ for all } \xi \in \mathbb{Z}^n.$$

If there exist constants $\alpha > 0$ and $\delta > 0$ such that for every $\epsilon > 0$ and $\xi \in \mathbb{Z}^n \setminus \{0\}$ there exists an open interval $I_{\xi} \subset T^1$ for which

$$w^2(t, \xi) \geq \alpha C_{\epsilon} e^{-\epsilon |\xi|^{1/s}} \forall t \in I_{\xi}, \ |I_{\xi}| > \delta,$$

(3.12)

where $C_{\epsilon} > 0$ depends only on $\epsilon$, then $Q$ is $s$-globally hypoelliptic on $T^{1+n}$.

Moreover, in [18, Lemma 2.1] the following lemma was proved.

**Lemma 3.2.** Let $P$ be the self-adjoint operator on $T^4$ given in (3.9). If at least one of the $a_1$ and $a_2$ is not a constant function and one of the $a_1$ and $a_2$ can be written as a linear combination of the other and 1 with coefficients that at least one of them is not a rational number, then $\text{Ker} P = \text{Ker} tP = \mathbb{C}$ (as $P = tP$).

It is worth also recalling that in [1, Theorem 2.1] (or see [2] where such a Theorem 2.1 was announced) the $s$-global hypoellipticity of an operator $P$ and the $s$-global solvability of its transposed operator $tP$ were connected, thus proving the $s$-global solvability of large classes of partial differential operators on the torus, that is:

**Theorem 3.1.** Let $P$ be a linear differential operator of order $m$ with coefficients in $G^s(T^n)$, with $s \geq 1$. If $P$ is $s$-globally hypoelliptic in $T^n$, then $tP$ is $s$-globally solvable in $T^n$.

We are now able to state and to show that in the setting of Gevrey classes the following holds.

**Theorem 3.2.** Let $P$ be the self-adjoint operator on $T^4$ given in (3.9). If $a_1$ and $a_2$ are both constants then $P$ is $s$-globally solvable on $T^4$ if and only if either $(a_1, a_2)$ is not an $s$-Liouville vector or $(a_1, a_2) \in \mathbb{Q}^2$. If at least one of the $a_1$ and $a_2$ is not a constant function, then $P$ is $s$-globally solvable on $T^4$ if and only if none of the $a_1$ and $a_2$ can be written as a linear combination of the other and 1 with coefficients that are simultaneously $s$-exponentially approximable and at least one of them is not a rational number.

In the case $a_1$ and $a_2$ are both constants, the result follows from the following remark.
Remark 3.2. Let us consider a linear partial differential operator $P$ of order $m$ with constant coefficients on $\mathbb{T}^n$, i.e., $P = P(D)$. If $f = \sum_{v \in \mathbb{Z}^n} f_v e^{i\nu x}$ and $u = \sum_{v \in \mathbb{Z}^n} u_v e^{i\nu x}$, then

$$P(D)u = f \iff \sum_{v \in \mathbb{Z}^n} P(v)u_v e^{i\nu x} = \sum_{v \in \mathbb{Z}^n} f_v e^{i\nu x}$$

$$\iff P(v)u_v = f_v \quad \forall v \in \mathbb{Z}^n.$$  

Define $A' = \{v \in \mathbb{Z}^n; P(v) = 0\}$ and $A'' = \{v \in \mathbb{Z}^n; P(v) \neq 0\};$ clearly, $A' \cap A'' = \emptyset$ and $A' \cup A'' = \mathbb{Z}^n$. Therefore

$$P(D)u = f \iff \begin{cases} f_v = 0 & \forall v \in A', \\ u_v = \frac{f_v}{P(v)} & \forall v \in A'' \end{cases}.$$  

Moreover, for each $f \in \mathcal{E}'_s(\mathbb{T}^n)$ the corresponding solution

$$u = \sum_{v \in A''} \frac{f_v}{P(v)} e^{i\nu x}$$

belongs to $\mathcal{E}'_s(\mathbb{T}^n)$ if and only if for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $|P(v)| \geq C_\epsilon e^{-\epsilon|v|^{1/2}}$ for all $v \in A''$. In this case, $\dim \ker P = \#\{v \in \mathbb{Z}^n; v \in A'\}$ and it can be infinite-dimensional.

If the operator $P$ is the following type:

$$P = -\partial_1^2 - (\partial_2 + a_1 \partial_3 + a_2 \partial_4)^2,$$

with $a_1, a_2 \in \mathbb{R}$, then $P = D_1^2 + (D_2 + a_1 D_3 + a_2 D_4)^2$ so that

$$A' = \{ (\tau, \eta, \xi_1, \xi_2) \in \mathbb{Z}^4; \tau^2 + (\eta + a_1 \xi_1 + a_2 \xi_2)^2 = 0 \}$$

$$A'' = \mathbb{Z}^4 \setminus A'.$$

and $A'' = \mathbb{Z}^4 \setminus A'$. Thus, $P$ is $s$-globally solvable if and only if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $\tau^2 + (\eta + a_1 \xi_1 + a_2 \xi_2)^2 \geq C_\epsilon e^{-\epsilon(|\tau|^{1/2} + |\eta, \xi_1, \xi_2|^{1/2})}$ for all $(\tau, \eta, \xi_1, \xi_2) \in A'$. Consequently, by (3.11) $P$ is $s$-globally solvable if and only if either $(a_1, a_2)$ is not an $s$-Liouville vector or $(a_1, a_2) \in \mathbb{Q}^2$ (in the case $(a_1, a_2) \in \mathbb{Q}^2$ it clearly holds that $\inf (\tau, \eta, \xi_1, \xi_2) \in A'' | P(\tau, \eta, \xi_1, \xi_2) = \text{const} > 0$).

For the proof of Theorem 3.2 it will be also useful to split the action of the operator $P$ into certain subspaces as follows (see, e.g., [3,18]). For any $A \subset \mathbb{Z}^3$ let us define

$$\mathcal{E}'_{s,A}(\mathbb{T}^4) = \left\{ u \in \mathcal{E}'(\mathbb{T}^4); u = \sum_{\xi \in A} \hat{u}(t, \xi)e^{i\nu x}, \text{where } \hat{u}(t, \xi) \text{ is the partial Fourier coefficient of } u. \right\}$$

The space $G_{s,A}^2(\mathbb{T}^4)$ is defined in the analogous way. Also, we denote by $P_A$ the operator $P$ acting from $\mathcal{E}'_{s,A}(\mathbb{T}^4)$ into $\mathcal{E}'_{s,A}(\mathbb{T}^4)$ and put $\mathcal{E}(P_A) = \mathcal{E}(P) \cap G_{s,A}^2(\mathbb{T}^4)$.

Definition 3.3. Let $A \subset \mathbb{Z}^3$. The operator $P_A$ is said to be $s$-globally solvable on $\mathbb{T}^4$ if for every $f \in \mathcal{E}(P_A)$ there exists $u \in \mathcal{E}'_{s,A}(\mathbb{T}^4)$ such that $P_Au = f$. The operator $P_A$ is said to
be $s$-globally hypoelliptic on $T^4$ if the conditions $u \in \mathcal{E}_{s,A}^\prime(T^4)$ and $P_A u \in G^s_A(T^4)$ imply $u \in G^s_A(T^4)$.

When $A = \mathbb{Z}^3$ the previous notions of $s$-global solvability and hypoellipticity are recovered. In the case that $B = \mathbb{Z}^3 \setminus A$, it clearly holds that $P$ is $s$-globally solvable on $T^4$ if and only if $P_A$ and $P_B$ are both $s$-globally solvable on $T^4$.

We can now give the proof of Theorem 3.2.

**Proof of Theorem 3.2.** By Remark 3.2 we can suppose that at least one of the $a_1$ and $a_2$ is not a constant function.

**Sufficiency.** In the case that $a_2(t) = \lambda a_1(t) + \mu$ with $\lambda, \mu$ real numbers not simultaneously $s$-exponentially approximable in $\mathbb{Q}$ or that $a_2(t) \neq \lambda a_1(t) + \mu$ for all $\lambda, \mu \in \mathbb{R}$, $P$ is $s$-globally hypoelliptic on $T^4$ by [9, Theorem 1]. By Theorem 3.1 (see [1, Theorem 2.1]) $P$ is then $s$-globally solvable on $T^4$.

We now assume that $a_2(t) = \lambda a_1(t) + \mu$ with $\lambda, \mu \in \mathbb{Q}$. We can suppose that $\lambda = s/r$ and $\mu = m/r$, where $s, m \in \mathbb{Z}$ and $r \in \mathbb{N}$.

We first prove that $P_A$ and $P_B$ are $s$-globally solvable on $T^4$, where

$$A = \{(\eta, \xi_1, \xi_2) \in \mathbb{Z}^3; \eta = -lm, \xi_1 = -ls, \xi_2 = lr, l \in \mathbb{Z}\}$$

and

$$B = \mathbb{Z}^3 \setminus A.$$ 

Since $P_A = -\partial^2_\eta$, it is clearly $s$-globally solvable on $T^4$.

We claim that $P_B$ is $s$-globally hypoelliptic on $T^4$. By Lemma 3.1 it is enough to show that there exist constants $\alpha > 0$ and $\delta > 0$ such that for every $\epsilon > 0$ and $\xi \in \mathbb{Z}^n \setminus \{0\}$ there exists an open interval $I_\xi \subset T$, with $|I_\xi| > \delta$, for which

$$w^2(t, \eta, \xi_1, \xi_2) = \left[\left(\eta + \frac{m}{r} \xi_2\right) + a_1(t)\left(\xi_1 + \frac{s}{r} \xi_2\right)\right]^2 \geq \alpha C_\epsilon e^{\|\xi\|^2/\epsilon} \quad \forall i \in I_\xi,$$

where $C_\epsilon > 0$ depends only on $\epsilon$.

Proceeding exactly as in the proof of [18, Theorem 1.10], one proves that such a condition is satisfied. Thus $P_B$ is $s$-globally hypoelliptic on $T^4$; thereby $P_B$ is $s$-globally hypoelliptic on $T^4$ by Theorem 3.1 (see [1, Theorem 2.1]).

**Necessity.** Without loss of generality, we can assume that

$$a_2(t) = \lambda a_1(t) + \mu,$$

where $a_1 \neq \text{const}$, $\lambda$ and $\mu$ are simultaneously $s$-exponentially approximable in $\mathbb{Q}$ and $\lambda$ is an irrational number. We claim that such conditions imply that $P$ is not $s$-globally solvable in $T^4$ by violating the inequality (2.4) of Theorem 2.1.

Since $\lambda, \mu$ are simultaneously $s$-exponentially approximable in $\mathbb{Q}$ there is $\epsilon_0 > 0$ for which there exist two sequences $(p_j, \tilde{p}_j) \subset \mathbb{Z}^2$ and $(q_j) \subset \mathbb{N}$ for all $j \in \mathbb{N}$ such that

$$\left|\frac{p_j}{q_j}\right| < q_j^{-1} e^{-\epsilon_0 q_j^{1/3}} \quad \text{and} \quad \left|\frac{\tilde{p}_j}{q_j}\right| < q_j^{-1} e^{-\epsilon_0 q_j^{1/3}} \quad \forall j \in \mathbb{N},$$

where $q_j \to +\infty$ as $j \to +\infty$. 

(3.13)
Denoting by $z = (y, x_1, x_2) \in T^3$ and by $\xi_j = (-\tilde{p}_j, -p_j, q_j) \in \mathbb{Z}^3$ for all $j \in \mathbb{N}$, we set

$$f_j(t, z) = e^{-iz\xi_j} \quad \text{and} \quad g_j(t, z) = e^{iz\xi_j}, \quad j \in \mathbb{N}.$$ 

Since $\lambda$ is an irrational number, by Lemma 3.2 we have that $\dim \ker^{t}P = 1$, therefore easily implying that $f_j \in \mathcal{E}(P)$ for all $j \in \mathbb{N}$. In particular, for each $j \in \mathbb{N},$

$$\int_{T^4} f_j g_j \, dt \, dy \, dx_1 \, dx_2 = \int_{T^4} dt \, dy \, dx_1 \, dx_2 = (2\pi)^4. \quad (3.14)$$

Moreover, as it is easy to compute, we have that, for every $j \in \mathbb{N},$

$$(^tP g_j = P g_j = g_j \Delta_j(t, \xi_j),$$

where

$$\Delta_j(t, \xi_j) = \left[ (\tilde{p}_j - \mu q_j) + a_1(t)(p_j - \lambda q_j) \right]^2$$

$$= (\tilde{p}_j - \mu q_j)^2 + 2(\tilde{p}_j - \mu q_j)(p_j - \lambda q_j)a_1(t) + (p_j - \lambda q_j)^2 a_1^2(t),$$

with $a_1, a_1^2 \in G^*(T^1)$; hence $a_1(t) = \sum_{a \in \mathbb{Z}} e^{ia\alpha t}, \quad a_1^2(t) = \sum_{a \in \mathbb{Z}} d_a e^{ia\alpha t}$ with $|c_a| \leq c e^{-\epsilon_0 |a|^{1/s}}$ and $|d_a| \leq c e^{-d_0 |a|^{1/s}}$ for all $\alpha \in \mathbb{Z}$ and for some positive constants $c, c_0$ and $d_0$.

Fix any $\tilde{\nu} \in \mathcal{V}$. Then, for each $j \in \mathbb{N}$, $p_{\tilde{\nu}}(f_j) = \tilde{\nu}(0, \xi_j)$ and

$$p_{\tilde{\nu}}(P g_j) = p_{\tilde{\nu}}(g_j \Delta_j)$$

$$\leq (\tilde{p}_j - \mu q_j)^2 \tilde{\nu}(0, \xi_j) + 2(\tilde{p}_j - \mu q_j)(p_j - \lambda q_j) \sum_{a \in \mathbb{Z}} |c_a| \tilde{\nu}(\alpha, \xi_j)$$

$$+ (p_j - \lambda q_j)^2 \sum_{a \in \mathbb{Z}} |d_a| \tilde{\nu}(\alpha, \xi_j)$$

as $e^{iz\xi_j} \cdot e^{ia\alpha t} = e^{i(\alpha t + z\xi_j)}$.

Since for each $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that $\tilde{\nu}(v) \leq C(\epsilon) e^{\epsilon |v|^{1/s}}$ for all $v \in \mathbb{Z}^4$ (according to the definition of $\tilde{\nu}$, see Remark 2.1) and by (3.13) there exist two positive constants $k_0, k_1$ such that $k_0 q_j \leq |\xi_j| \leq k_1 q_j$ for all $j \in \mathbb{N}$, we have that

$$p_{\tilde{\nu}}(f_j) \leq C(\epsilon) e^{\epsilon |\xi_j|^{1/s}} \leq C(\epsilon) e^{k_0 q_j^{1/s}} \quad (3.15)$$

and

$$p_{\tilde{\nu}}(P g_j) \leq C(\epsilon) e^{(k_0 - 2\epsilon_0)q_j^{1/s}} + c C(\epsilon) e^{(k_0 - 2\epsilon_0)q_j^{1/s}} \sum_{a \in \mathbb{Z}} e^{(\epsilon - c_0) |a|^{1/s}}$$

$$+ c C(\epsilon) e^{(k_0 - 2\epsilon_0)q_j^{1/s}} \sum_{a \in \mathbb{Z}} e^{(\epsilon - d_0) |a|^{1/s}}, \quad (3.16)$$

where the positive constant $k$ depends only on $k_1$. Since $\epsilon$ is arbitrary, we can choose $0 < \epsilon < \min\{1, c_0, d_0, \epsilon_0 / k\}$. Then

$$\sum_{a \in \mathbb{Z}} e^{(\epsilon - c_0) |a|^{1/s}} < +\infty \quad \text{and} \quad \sum_{a \in \mathbb{Z}} e^{(\epsilon - d_0) |a|^{1/s}} < +\infty;$$
hence, by (3.16) we deduce that

\[ p_\bar{v}(P_{g_j}) \leq C''(\epsilon) e^{(k\epsilon - 2\epsilon_0)q_j^{1/s}} \tag{3.17} \]

for all \( j \in \mathbb{N} \). Thus, by (3.15) and (3.17), we obtain that

\[ p_\bar{v}(f_j) p_\bar{v}(P_{g_j}) \leq C'(\epsilon) C''(\epsilon) e^{2(k\epsilon - \epsilon_0)q_j^{1/s}} \to 0 \quad \text{as} \quad j \to +\infty. \tag{3.18} \]

It follows from (3.14) and (3.18) that the inequality (2.4) of Theorem 2.1 is violated and this leads to the non-s-global solvability of \( P \). \( \square \)

References


