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On oscillating polynomials

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Abstract

Extremal problems of Markov type are studied, concerning maximization of a local extremum of the derivative in the class of real polynomials of bounded uniform norm and with maximal number of zeros in $[-1, 1]$. We prove that if a symmetric polynomial f, with all its zeros in $[-1, 1]$, attains its maximal absolute value at the end-points, then $|f'|$ attains maximal value at the end-points too. As an application of the method developed here, we show that the classic Zolotarev polynomials have maximal derivative at one of the end-points.

Keywords: Markov's inequality; Maximal derivative; Zolotarev polynomials; Orthogonal polynomials

1. Introduction

An algebraic polynomial that has all its zeros on the real line $\mathbb{R}$ is termed *oscillating polynomial*. For example, the orthogonal polynomials constitute an important class of oscillating polynomials. Sometimes we shall restrict our study to the class of polynomials f of a fixed degree n which have n zeros in a given finite interval $[a, b]$ on $\mathbb{R}$. In such a case we shall say that f is oscillating on $[a, b]$.

In this paper we concentrate on certain general properties of polynomials with real zeros that are relevant to the following

Conjecture 1. *Assume that the polynomial*

$$P(x) = (x - x_1) \cdots (x - x_n), \quad -1 < x_1 \le x_2 \le \cdots \le x_n < 1,$$

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with symmetric zeros, attains its maximal absolute value at the end-points. Then $P'(x)$ attains its maximal absolute value at the end-points too.

The conjecture without the requirement of symmetry was formulated first by Aleksei Shadrin and then, independently, by Geno Nikolov. However, a counterexample constructed by Naidenov [15] showed that it cannot be true in that former form and the present corrected formulation of the problem was proposed by the second author. Our main result is a proof of the above conjecture. Then we give some applications of the result to orthogonal polynomials and the Zolotarev polynomials.

Our motivation to study the conjecture comes from Duffin and Schaeffer extension of the Markov inequality (see [7]). They proved that

$$|f^{(k)}(x+iy)| \le |T_n^{(k)}(1+iy)|, \quad x \in [-1,1], \ -\infty < y < \infty,$$

for every $k = 1, \ldots, n$ and every algebraic polynomial f of degree n such that

$$|f(\eta_j)| \le 1, \quad j = 0, \ldots, n,$$

where $\eta_j := \cos \frac{(n-j)\pi}{n}$, $j = 0, \ldots, n$, are the extremal points of the Tchebycheff polynomial

$$T_n(x) := \cos n \arccos x, \quad -1 \le x \le 1.$$

A crucial ingredient of their proof was the following property of the Tchebycheff polynomials:

$$|T_n(x+iy)| \le |T_n(1+iy)|, \quad x \in [-1,1], \ -\infty < y < \infty.$$

It was conjectured in [2] (see Question 5 and the comments after it) that the last property holds for every symmetric oscillating polynomial P on $[-1, 1]$ that attains its maximal absolute value at the end-points. Note that the last mentioned conjecture is stronger than Conjecture 1. This follows from a result of Bernstein [1] and de Bruijn [6], according to which the inequality

$$|P_n(x+iy)| \le |P_n(1+iy)|,$$

for every $y \in (-\infty, \infty)$ and a fixed $x \in [-1, 1]$, implies the same inequality for every derivative of P_n provided all zeros of P_n are situated in the half plane $x < 1$.

Hopefully, being verified, Conjecture 1 could be useful in the effort to extend further the remarkable result of Duffin and Schaeffer [7] to polynomials f that satisfy the assumption

$$|f(t_j)| \le |P_n(t_j)|, \quad j = 0, \ldots, n,$$

where $t_0 = -1$, $t_n = 1$ and $t_1 < \cdots < t_{n-1}$ are the zeros of $P'(x)$, P being an oscillating polynomial, described in Conjecture 1 (see [20,21,17,18] for research in this direction).

We prove even stronger proposition than that in Conjecture 1. Namely, for every oscillating polynomial P on $[-1, 1]$ such that $|P(-1)| = \|P\|$, we show that each local extremum of P' from the first half (i.e., with an index less than or equal to $\lfloor (n-1)/2 \rfloor$) is smaller in absolute value than $|P'(-1)|$. Our main tool is Lemma 1.

Here, and in what follows, $\|f\|$ denotes the uniform norm of f on the interval in consideration and π_n is the class of all real algebraic polynomials of degree less than or equal to n.

2. Preliminaries

Let $[a, b]$ be any given finite interval on the real line. Often we take $[a, b] = [-1, 1]$. With any oscillating polynomial P on $[a, b]$ we associate the set $\mathbf{h}(P)$ of its local maximal absolute values, including the values at the end-points. More precisely, for any oscillating polynomial P of degree n on $[a, b]$, we define

$$\mathbf{h}(P) := (h_0(P), h_1(P), \ldots, h_n(P)),$$

where

$$h_j(P) := |P(t_j)|, \quad j = 0, \ldots, n,$$

$t_0 := a$, $t_n := b$, and $t_1 \le t_2 \le \cdots \le t_{n-1}$ are the zeros of P'.

The following result from [3], inspired by a problem of Davis [5], shows that any oscillating polynomial is completely determined by its local extrema and the values at the end-points.

Theorem A. *For any given finite interval* $[a, b]$ *and a set of non-negative numbers*

$$\mathbf{h} := (h_0, h_1, \ldots, h_n),$$

there exists a unique polynomial $P(\mathbf{h}; x)$ *of degree n and a set of points*

$$a = t_0 \le t_1 \le \cdots \le t_{n-1} \le t_n = b$$

such that

$$P(\mathbf{h}; t_j) = (-1)^{n-j} h_j, \quad j = 0, \ldots, n,$$
$$P'(\mathbf{h}; t_j) = 0, \quad j = 1, \ldots, n-1.$$

Note that if

$$h_{k-1} > 0, \quad h_k = h_{k+1} = \cdots h_{k+\nu-1} = 0, \quad h_{k+\nu} > 0,$$

then t_k is considered as a $(\nu + 1)$-tuple zero of $P(\mathbf{h}; x)$. This theorem was proved first by Mycielski and Paszkowski [14] in the case where $h_j > 0$ for all j.

The derivative of any oscillating polynomial is also oscillating. There is a monotone dependence of the parameters $h_j(P')$ on the parameters $h_0, \ldots, h_n$ of P. This relation was discovered in [4] where the following was proved.

Theorem B. *For each* $j \in \{0, \ldots, n-1\}$, $h_j(P'(\mathbf{h}; \cdot))$ *is a non-decreasing function of* $h_0, \ldots, h_n$ *in the domain* $h_0 \ge 0, \ldots, h_n \ge 0$. *Moreover, it is strictly increasing, provided* $h_j(P'(\mathbf{h}; \cdot)) > 0$.

Let us introduce the class

$$H_0^n := \left\{(h_0, h_1, \ldots, h_n) \in \mathbb{R}^{n+1} : h_0 = 1, h_i \in [0, 1], i = 1, \ldots, n\right\}.$$

For any given $[a, b]$ and $i \in \{1, 2, \ldots, n-2\}$, we study the extremal problem

$$r_i(P) := \frac{h_i(P')}{h_0(P')} \quad \longrightarrow \quad \max, \tag{1}$$

where $\mathbf{h}(P) \in H_0^n$. More precisely, we want to find how large an interior extremum of the first derivative of an oscillating polynomial P can be with respect to the value $|P'(a)|$ at the end-point, under the condition that

$$|P(a)| = \|P\| := \max_{x \in [a,b]} |P(x)|.$$

We would like to characterize those oscillating polynomials P for which $r_i(P) \le 1$, $i = 1, \dots, n-2$, that is, for which the condition $\|P\| = |P(a)|$ implies $\|P'\| = \max\{|P'(a)|, |P'(b)|\}$.

3. Main result

Extremal problem (1) is a particular case of the more general problem of maximizing the ratio

$$r_{ij}(P) := \frac{h_i(P')}{h_j(P')}, \quad i, j \in \{0, 1, \dots, n-1\},$$

in a set of oscillating polynomials P for which $h_j(P')$ is bounded from below by a positive constant.

In what follows we shall reserve the notation

$$t_j = t_j(P) = t_j(\mathbf{h}), \quad j = 0, \dots, n,$$

for the ordered points at which $P(\mathbf{h}; x)$ attains its extremal values (including the values at the end-points) in $[a, b]$. The extremal points of $P'(\mathbf{h}; x)$ will be denoted by

$$a = \tau_0(\mathbf{h}) \le \tau_1(\mathbf{h}) \le \cdots \le \tau_{n-1}(\mathbf{h}) = b.$$

We first recall some simple observations, that will be used in the sequel.

Set

$$\omega(x) := (x - t_0) \cdots (x - t_n), \quad \omega_k(x) := \frac{\omega(x)}{x - t_k}.$$

Lemma A. *Assume that $t_0 < t_1 < \cdots < t_n$. Let*

$$\ell_{nk}(x) := \frac{\omega_k(x)}{\omega_k(t_k)}, \quad k = 0, \dots, n,$$

be the Lagrange basic polynomials for the nodes $t_0, \dots, t_n$. Then, for any fixed x,

$$\frac{\partial P^{(j)}(\mathbf{h}; x)}{\partial h_k} = (-1)^{n-k} \ell_{nk}^{(j)}(x),$$

in the domain $h_0 > 0, \dots, h_n > 0$.

Lemma B. *For any $k \in \{0, 1, \dots, n\}$ and $s \in \{0, 1, \dots, n-1\}$,*

$$\frac{\partial h_s(P')}{\partial h_k} = (-1)^{k+s+1} \ell'_{nk}(\tau_s).$$

These are known relations (see formula (4.1) in [2] or the proof of Lemma 2 in [4]).

Next we give a lemma which is basic for the proof of our results.

Lemma 1. *Let $i, j \in \{0, \dots, n-1\}$, $k \in \{1, \dots, n-1\}$. For every oscillating polynomial P of degree n on $[a, b]$ with $h_k(P), h_i(P'), h_j(P') \neq 0$, holds the relation*

$$\operatorname{sign} \frac{\partial r_{ij}(P(\mathbf{h}; \cdot))}{\partial h_k} = \operatorname{sign} \left\{ (i - j) \left(t_k - \frac{\tau_i + \tau_j}{2} \right) \right\}.$$

Proof. The lemma says that the ratio $r_{ij}(\mathbf{h})$ increases as h_k increases if t_k is closer to τ_i than to τ_j, and it decreases if t_k is closer to τ_j.

Making use of Lemma B we first find a closed form expression for the derivative of r_{ij}. We have

$$\begin{aligned}\frac{\partial r_{ij}}{\partial h_k} &= \frac{1}{h_j^2(P')}\left(\frac{\partial h_i(P')}{\partial h_k}h_j(P') - \frac{\partial h_j(P')}{\partial h_k}h_i(P')\right)\\ &= \frac{1}{h_j^2(P')}\left((-1)^{i+k+1}\ell'_{nk}(\tau_i)h_j(P') - (-1)^{j+k+1}\ell'_{nk}(\tau_j)h_i(P')\right).\end{aligned}$$

Let us write the polynomial $\omega(x)$ in the form

$$\omega(x) = (x-t_0)(x-t_1)\cdots(x-t_n) =: (x-a)(x-b)g(x)$$

and represent the derivative above in terms of g. Note that if $P(x) = Kx^n + \cdots$, then $P'(x) = nKg(x)$. Taking this observation into account, we modify further the expression for the derivative and obtain

$$\begin{aligned}\frac{\partial r_{ij}}{\partial h_k} &= \frac{1}{h_j^2(P')}\left((-1)^{i+k+1}\frac{\omega_k'(\tau_i)}{\omega_k(t_k)}h_j(P') - (-1)^{j+k+1}\frac{\omega_k'(\tau_j)}{\omega_k(t_k)}h_i(P')\right)\\ &= \frac{(-1)^{n+1}}{h_j^2(P')|\omega_k(t_k)|}\left((-1)^i\omega_k'(\tau_i)|Kng(\tau_j)| - (-1)^j\omega_k'(\tau_j)|Kng(\tau_i)|\right).\end{aligned}$$

Since $k \neq 0, n$,

$$\begin{aligned}\omega_k'(\tau_s) &= \left(\frac{(x-a)(x-b)g(x)}{x-t_k}\right)'\bigg|_{x=\tau_s} = \left(\frac{(x-a)(x-b)}{x-t_k}\right)'\bigg|_{x=\tau_s}\cdot g(\tau_s)\\ &= \frac{\tau_s^2 - 2\tau_s t_k + (a+b)t_k - ab}{(\tau_s-t_k)^2}\,g(\tau_s)\\ &= \left[1 - \frac{(t_k-a)(t_k-b)}{(\tau_s-t_k)^2}\right]g(\tau_s),\end{aligned}$$

for all $s = 0, 1, \ldots, n-1$, where for $s \neq 0, n-1$ we have used the equality $g'(\tau_s) = 0$, while for $s = 0$ and $s = n-1$ we used that $(x-a)(x-b)|_{x=\tau_s} = 0$. Substituting this expression for the values of ω_k' and taking into account that

$$\operatorname{sign} g(\tau_s) = (-1)^{n-s-1}$$

we obtain

$$\begin{aligned}\frac{\partial r_{ij}}{\partial h_k} &= \frac{|K|n(-1)^{n+1}}{h_j^2(P')|\omega_k(t_k)|}\left((-1)^i\left[1 - \frac{(t_k-a)(t_k-b)}{(\tau_i-t_k)^2}\right]g(\tau_i)|g(\tau_j)|\right.\\ &\quad \left. - (-1)^j\left[1 - \frac{(t_k-a)(t_k-b)}{(\tau_j-t_k)^2}\right]g(\tau_j)|g(\tau_i)|\right)\\ &= \frac{|Kng(\tau_i)g(\tau_j)|}{h_j^2(P')|\omega_k(t_k)|}\left(\left[1 - \frac{(t_k-a)(t_k-b)}{(\tau_i-t_k)^2}\right] - \left[1 - \frac{(t_k-a)(t_k-b)}{(\tau_j-t_k)^2}\right]\right)\\ &= \frac{|g(\tau_i)|}{h_j(P')|\omega(t_k)|}(t_k-a)(b-t_k)\left(\frac{1}{(\tau_i-t_k)^2} - \frac{1}{(\tau_j-t_k)^2}\right)\end{aligned}$$

$$= \frac{2|g(\tau_i)|(t_k - a)(b - t_k)}{h_j(P')|\omega(t_k)|(\tau_i - t_k)^2(\tau_j - t_k)^2}(\tau_j - \tau_i)\left(\frac{\tau_i + \tau_j}{2} - t_k\right)$$
$$= C\,(\tau_i - \tau_j)\left(t_k - \frac{\tau_i + \tau_j}{2}\right), \quad C = C(\mathbf{h}, i, j, k) > 0.$$

The last equality proves the lemma. □

Remark 1. The increase of h_n is equivalent to a compression of the polynomial $P(\mathbf{h}; x)$, that is,

$$P(\mathbf{h}; x) \quad \to \quad P(\mathbf{h}; a + \lambda(x - a)), \quad \lambda > 1.$$

But it is easily seen that the ratio $r_i(P)$ remains the same after such a change of the variable for $i = 1, \ldots, n-2$, and thus, $r_i(P)$ does not depend on h_n. Moreover, the quantity $r_i(P)$ is invariant with respect to any linear change $x = \lambda t + \mu$, $\lambda > 0$, of the variable, provided a is chosen as the smallest solution of the equation $|P(x)| = h_0$.

We next prove a lemma which will be used in our study. In order to formulate it we need a notation. With any pair of natural numbers $n > m$, a given interval $[a, b]$, and a set of parameters

$$\mathbf{h} := (h_0, \ldots, h_m, \ldots, h_n)$$

we associate the oscillating polynomial $P_{n,m}(\mathbf{h}; x)$ of degree n defined uniquely (as described in Theorem A) on $[a, d]$, by the parameters $\mathbf{h}$ where $d > b$ is chosen so that the mth zero x_m coincides with b.

Lemma 2. *Given a finite interval $[a, b]$ and natural numbers $n > m$, let $P_m(h_0, \ldots, h_{m-1}, 0; x)$ be the oscillating polynomial of degree m defined on $[a, b]$ by the parameters $(h_0, \ldots, h_{m-1}, 0)$ with $h_0 > 0$ and normalized by the condition $P_m(h_0, \ldots, h_{m-1}, 0; a) = (-1)^n h_0$. Then, the polynomial $P_{n,m}(\mathbf{h}; x)$ associated with*

$$\mathbf{h} := (h_0, \ldots, h_{m-1}, h, h_{m+1}, \ldots, h_n)$$

tends uniformly to $P_m(h_0, \ldots, h_{m-1}, 0; x)$ on $[a, b]$ as $h \to \infty$.

Proof. For the sake of simplicity, we may suppose without loss of generality that $[a, b]$ is $[0, 1]$. For every $h > 0$ we have

$$P_{n,m}(\mathbf{h}; x) = C(h) \prod_{j=1}^{n} (x - x_j(h))$$
$$= C(h) x_1(h) \cdots x_n(h)(x - 1) \prod_{j=1, j \neq m}^{n} \left(\frac{x}{x_j(h)} - 1\right),$$

remembering that $x_m(h) = 1$. Since $P_{n,m}(\mathbf{h}; 0) = (-1)^n h_0$ by construction, we observe that

$$C(h) x_1(h) \cdots x_n(h) = h_0,$$

and passing to limit as h tends to ∞, we obtain

$$\lim_{h \to \infty} P_{n,m}(\mathbf{h}; x) = (-1)^n h_0 (1 - x) \prod_{j=1}^{m-1} (1 - \alpha_j x) \prod_{j=m+1}^{n} (1 - \alpha_j x)$$

where

$$\alpha_j = \lim_{h \to \infty} \frac{1}{x_j(h)}, \quad j \neq m.$$

We have to show that these limits exist. Indeed, if we assume that $x_1(h) \to 0$, then $P'_{n,m}(\mathbf{h}; x)$ would become very large for some $x \in (0, x_1(h))$. But this contradicts the Markov inequality since $\|P_{n,m}(\mathbf{h}; \cdot)\|$ remains bounded on [0, 1] for every h. Thus, every of the sequences $1/x_j(h)$, for $1 \le j \le m-1$, is bounded and hence it contains a convergent subsequence. As we shall see at the end of this proof, there is only one accumulation point of the sequence $1/x_j(h)$ and thus, it converges to a certain number α_j. Further, for $m < j \le n$, clearly $x_j(h) \to \infty$ as $h \to \infty$. This follows from the Tchebycheff inequality, i.e., from a known property of the Tchebycheff polynomials $\tilde{T}_n$ (associated with the interval [0, 1]), namely,

$$|P_{n,m}(\mathbf{h}; x)| \le M\, \tilde{T}_n(x)$$

for every $x > 1$, where M is the uniform norm of $P_{n,m}(\mathbf{h}; \cdot)$ on [0, 1]. Now taking x to be the point $t_m(h)$ of the mth extremum of $P_{n,m}(\mathbf{h}; x)$, we conclude that $t_m(h)$ should satisfy $M\, \tilde{T}_n(t_m(h)) \ge h$ which implies that $t_m(h) \to \infty$. Because of the obvious inequalities $t_m(h) < x_{m+1}(h) \le \cdots \le x_n(h)$ we conclude that $x_j(h) \to \infty$ for $j > m$. Therefore, $\alpha_j \to 0$ for $j > m$ and hence $P_{n,m}(\mathbf{h}; x)$ tends uniformly on [0, 1] to the polynomial

$$P(x) = (-1)^n h_0 (1 - x) \prod_{j=1}^{m-1} (1 - \alpha_j x)$$

of degree m. Since for every h the polynomials $P_{n,m}(\mathbf{h}; x)$ have on [0, 1] local extrema $h_0, \ldots, h_{m-1}$ and 0, the limit polynomial P will have the same property. Moreover, $P(0) = (-1)^n h_0$. Then, by the uniqueness of the oscillating polynomial of degree m defined on [0, 1] by $(h_0, \ldots, h_{m-1}, 0)$, it follows that

$$P \equiv P_m(h_0, \ldots, h_{m-1}, 0; \cdot).$$

The proof is complete. □

Lemma 2 allows us to consider any oscillating polynomial $P(\mathbf{h}; x)$ on $[a, b]$ of degree m as an oscillating polynomial on $[a, \infty)$ of degree $n > m$ with parameters $(h_0, \ldots, h_{m-1}, \infty, \ldots, \infty)$ where $h_0, \ldots, h_{m-1}$ are the first m components of $\mathbf{h}$. Making use of such an interpretation, we derive from Lemma 2 the following consequence.

Lemma 3. *Let $P = P(\mathbf{h}; x)$ be an oscillating polynomial of degree n on the finite interval $[a, b]$ with $h_0(P) = 1$. Let Q be the oscillating polynomial on $[a, b]$ of degree $n - 1$, defined by the parameters*

$$h_k(Q) = h_k(P), \quad k = 0, \ldots, n-2, \quad h_{n-1}(Q) = 0.$$

Then

$$r_j(P) < r_j(Q), \quad \textit{for } j = 1, \ldots, n-3.$$

Proof. By Lemma 2, for sufficiently large h, the corresponding polynomial $P_{n,n-1}(\mathbf{h}; x)$ with $\mathbf{h} = (h_0, \ldots, h_{n-2}, h, h_n)$ will be uniformly close to Q on $[a, b]$ (up to multiplication by -1). Then, by Lemma 1 and Remark 1, for every $h > h_{n-1}(P)$,

$$r_j(P) < r_j(P_{n,n-1}),$$

and, increasing, $r_j(P_{n,n-1})$ tends to $r_j(Q)$ as $h \to \infty$. Thus $r_j(P) < r_j(Q)$. □

Now we are prepared to prove our central result.

Theorem 1. *Let P be an oscillating polynomial on $[a,b]$ of degree $n \geq 3$. Assume that*

$$\|P\| = |P(a)| = 1.$$

Then

$$|P'(\tau_j)| < |P'(a)|, \quad \text{for } j = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$$

where $\tau_1 \leq \cdots \leq \tau_{n-2}$ are the zeros of $P''(x)$ and $\lfloor t \rfloor$ denotes the integer part of t.

Proof. The theorem says that any local maximum of $|P'(x)|$ from the first half of the set of ordered maxima (including the central one) is less than the value of $|P'(x)|$ at the first end-point a. The assertion is easily verified for $n = 3$. Indeed, if $n = 3$, then we consider $P(x)$ over the interval $[a_1, b_1]$ of those x for which

$$-h_1(P) \leq P(x) \leq h_2(P) \quad (\text{provided } P(a) > 0).$$

Since $P(a) = 1 \geq \max\{h_1(P), h_2(P)\}$, we have $a \leq a_1$. Besides, $P(x)$, considered on $[a_1, b_1]$, coincides with the corresponding Tchebycheff polynomial of degree 3 for $[a_1, b_1]$, up to addition of a constant. Then by a well-known property of the Tchebycheff polynomials, the maximal value of the first derivative of P is attained only at the end-points a_1 and b_1. Thus,

$$|P'(a)| \geq |P'(a_1)| > |P'(x)|, \quad \text{for every } x \in (a_1, b_1),$$

and in particular for $x = \tau$, the zero of P''.

Now the proof goes by induction on n. For the sake of convenience, let us assume that $[a, b]$ is $[-1, 1]$ and $h_n(P) = 1$ (see Remark 1). Assume that the theorem is true for all oscillating polynomials on $[-1, 1]$ of degree $n - 1$. Let P be any oscillating polynomial on $[-1, 1]$ of degree n. Then we consider the oscillating polynomial Q of degree $n - 1$ defined on $[-1, 1]$ by the parameters

$$h_0(Q) = 1, \quad h_k(Q) = h_k(P), \ k = 1, \ldots, n-2, \quad h_{n-1}(Q) = 0.$$

According to Lemma 3,

$$r_j(P) = \frac{h_j(P')}{h_0(P')} < \frac{h_j(Q')}{h_0(Q')} = r_j(Q). \tag{2}$$

Assume first that $j < \left\lfloor \frac{n-1}{2} \right\rfloor$. Then, for any integer n we have

$$j \leq \left\lfloor \frac{(n-1)-1}{2} \right\rfloor. \tag{3}$$

Therefore, Q is an oscillating polynomial on $[-1, 1]$ of degree $n - 1$, satisfying the conditions in the theorem and j is from the range stated there. Then, by the induction hypothesis, we conclude that $r_j(Q) < 1$. This, combined with (2), implies $r_j(P) < 1$, which was to be shown.

Let now $j = \left\lfloor \frac{n-1}{2} \right\rfloor$. In the case of even n, (3) still holds and we can apply the induction hypothesis. If n is odd, then the jth local maximum of $|P'(x)|$ is its central maximum. Recall that, for odd n, the central maximum of $|T_n'|$ is attained at 0, i.e., $h_j(T_n') = |T_n'(0)|$. Then, by the monotonicity theorem (Theorem B),

$$h_j(P') \leq h_j(T_n') = |T_n'(0)| = n.$$

Notice that the equality sign holds here only if $P \equiv T_n$.

In order to find a lower bound for $h_0(P')$, we consider the polynomial $\varphi(x) := x^n$ on $[-1, 1]$. Again by the monotonicity theorem,

$$h_0(P') \geq h_0(\varphi') = n,$$

where the equality occurs only if $P \equiv \varphi$. Since P cannot coincide simultaneously with T_n and φ, at least one of the above inequalities is strict. Therefore, in case $j = \left\lfloor \frac{n-1}{2} \right\rfloor$ we have

$$r_j(P) < \frac{h_j(T_n')}{h_0(\varphi')} = 1,$$

which finishes the proof of the theorem. □

Next immediate consequence from the theorem verifies Conjecture 1.

Corollary 1. *Let P be a symmetric oscillating polynomial on $[-1, 1]$ of degree n. Assume that*

$$\|P\| = |P(1)| = 1.$$

Then, for $k = 1, \ldots, n$, we have

$$\|P^{(k)}\| = |P^{(k)}(1)|.$$

Proof. Clearly, $P'(x)$ is symmetric too. Then, in view of the theorem,

$$\|P'\| = \max_{x \in [-1,0]} |P'(x)| = \max_{0 \leq j \leq \lfloor \frac{n-1}{2} \rfloor} |P'(\tau_j)| = |P'(-1)|.$$

Since $P'(x)$ is symmetric oscillating polynomial, we derive the same conclusion for the second derivative and so on. □

4. Applications

4.1. Orthogonal polynomials

As an immediate consequence of Theorem 1 we obtain the following property of a wide class of orthogonal polynomials. Assume that $w(x)$ is an even weight function on $[-1, 1]$ and let $\{P_n(x)\}_0^\infty$ be the sequence of the corresponding orthogonal polynomials. It follows from the uniqueness of $P_n(x)$ (normalized by $P_n(1) = 1$) that $P_n(x)$ is symmetric (i.e., it is an even or odd function). The verified conjecture then implies that if $|P_n(x)|$ attains its maximal value in $[-1, 1]$ at the end-points, then all derivatives of P_n attain maximal absolute value at the end-points too. This property was known for some classical orthogonal polynomials. For example, the ultraspherical polynomials $P_n^{(\lambda)}(x)$ (that are orthogonal on $[-1, 1]$ with respect to the weight $(1-x^2)^{\lambda-1/2}$) possess this property for $\lambda > -\frac{1}{2}$. Recall that the famous Tchebycheff polynomials $T_n(x)$ correspond to $\lambda = 0$ while the case $\lambda = 1/2$ gives the Legendre polynomials. The property follows from the representation (see [22, p. 93])

$$P_n^{(\lambda)}(x) = \sum_{k=0}^{n} \alpha_{k,n}^{(\lambda)} T_k(x) \quad \text{with } \alpha_{k,n}^{(\lambda)} \geq 0, \ \forall k$$

and the fact that $\max_{-1 \leq x \leq 1} |T_k'(x)| = T_k'(1)$ for all k.

4.2. Zolotarev polynomials

In his fundamental paper V.A. Markov [13] proved the inequality

$$\|f^{(k)}\| \le T_n^{(k)}(1)\|f\|, \quad k = 1, \ldots, n,$$

for every algebraic polynomial f of degree n on $[-1, 1]$. To show this, he considered first the problem: Given a fixed point x in $[-1, 1]$, characterize the polynomial P_* in π_n, with $\|P_*\| = 1$, for which

$$|P_*^{(k)}(x)| = \max\left\{|f^{(k)}(x)| : f \in \pi_n, \|f\| \le 1\right\} =: M_k(x).$$

A standard now variational argument yields that the extremal polynomial P_* should alternate n times in $[-1, 1]$, i.e., it should possess the property: There exist n points $s_1 < \cdots < s_n$ in $[-1, 1]$ such that

$$P_*(s_j) = (-1)^{n-j}\|P_*\|, \quad j = 1, \ldots, n.$$

Polynomials of this kind have been studied first by Zolotarev [23]. The class of all such polynomials can be described by a single parameter A that traverses the whole real line outside the interval (η_1, η_{n-1}), where

$$-1 = \eta_0 < \eta_1 < \cdots < \eta_n = 1$$

are the extremal points of the Tchebycheff polynomial $T_n(x)$ in $[-1, 1]$. The most difficult part in Markov's proof was to show that $\max M_k(x) = M_k(1)$. Then, by Lagrange interpolation at the nodes $\{\eta_j\}$ (and with basic polynomials denoted $\{\ell_{nj}(x)\}$) one easily see that for every $f \in \pi_n$ with $\|f\| \le 1$, we have

$$|f^{(k)}(1)| = \left|\sum_{j=0}^{n} f(\eta_j)\ell_{nj}^{(k)}(1)\right| \le \left|\sum_{j=0}^{n}(-1)^j \ell_{nj}^{(k)}(1)\right| = T^{(k)}(1)$$

which implies Markov's inequality. Having in mind that the original proof of V.A. Markov [13] goes on for 110 pages, it was always of a great interest to show directly that the Zolotarev polynomials Z_n have the property that for every $1 \le k \le n$, the maximal value $\max_{x\in[-1,1]} |Z_n^{(k)}(x)|$ is attained at one of the end-points (1 or -1). This would imply immediately Markov's inequality. The question is still open. We shall show here, based on Theorem 1, that this is true for $k = 1$.

Theorem 2. *For every natural n, every Zolotarev polynomial of degree n attains the maximal absolute value in* $[-1, 1]$ *of its first derivative at one of the end-points.*

Proof. An interesting analysis of the Zolotarev polynomials have been done by Erdős and Szegő [9]. Following the presentation there, with every point $A \ge \eta_{n-1}$ we associate the Zolotarev polynomial $Z_n(A; x)$ of degree n defined as follows.

If $\eta_{n-1} \le A \le 1$, then $Z_n(A; x)$ is simply a stretch of T_n,

$$Z_n(A; x) := -T_n\left(\frac{(1 + \eta_{n-1})x + \eta_{n-1} - A}{1 + A}\right).$$

The point A is the point of the last local extremum of $Z_n(A; x)$. In case $A = \eta_{n-1}$ the Zolotarev polynomial coincides with the Tchebycheff polynomial $-T_n$. Therefore, for $\eta_{n-1} \le A \le 1$ the theorem is evidently true for every k, since it is a well-known fact that $|T_n^{(k)}(x)|$ attains its maximal value at the end-points.

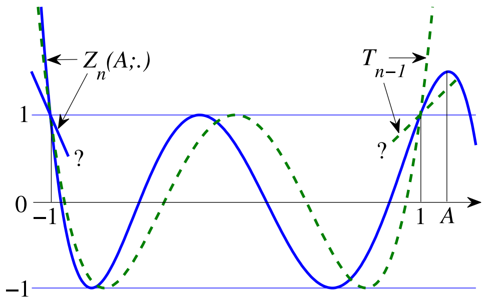

Fig. 1.

We concentrate next on the case $1 < A$. Then $Z_n(A; x)$ is defined as the unique polynomial of degree n such that

$$Z_n(-1) = (-1)^{n-1}, \quad Z_n(1) = 1, \quad Z_n'(A) = 0,$$

and having $n-2$ local extrema in $(-1, 1)$, all of them equal to 1 in absolute value. The fact that these conditions define $Z_n(A; x)$ uniquely is proved in [9]. The uniqueness can be derived also from Theorem A here. Besides, given $A > 1$, there exist unique constants B and C, such that $A < B < C$,

$$Z_n(A; B) = 1, \quad Z_n(A; C) = -1,$$

and the polynomial $y = Z_n(A; x)$ satisfies the differential equation

$$n^2(1 - y^2) = (1 - x^2)y'^2 \frac{(B - x)(C - x)}{(A - x)^2}.$$

Another result from [9] (see Lemma 1 there) is that $-|Z_n'(A; -1)|$ and $|Z_n'(A; 1)|$ are increasing functions of A in $[1, \infty)$. Adding the fact that $Z_n(A; \cdot)$ tends uniformly on $[-1, 1]$ to T_{n-1} when $A \to \infty$, we arrive at the inequality

$$|Z_n'(A; -1)| > |T_{n-1}'(-1)| > |Z_n'(A; 1)| \tag{4}$$

for every $A > 1$.

Note that (4) follows directly by zero counting argument with respect to $Z_n(A; x) - T_{n-1}(x)$. Indeed (see Fig. 1), for $x < -1$ the graph of T_{n-1} should lie below the graph of $Z_n(A; x)$ since, otherwise, the graph of $Z_n(A; x)$ (being a polynomial of degree n, and hence, increasing faster than $T_{n-1}(x)$ as $x \to -\infty$) will cross that of T_{n-1} at some point $x_0 < -1$. But $Z_n(A; \cdot) - T_{n-1}$ must have another n zeros in $[-1, 1]$, thus, to be identically zero, a contradiction. Similar argument is applied for $x > 1$.

Let $\sigma_1 < \cdots < \sigma_{n-2}$ be the zeros of $Z_n''(A; x)$ and $\sigma_0 := -1$, $\sigma_{n-1} := 1$. Clearly, the extremal points of $Z_n'(A; x)$ in $[-1, 1]$ are $\{\sigma_i\}_{i=0}^{n-1}$ if $\sigma_{n-2} < 1$ and without σ_{n-2}, otherwise. By Lemma 3,

$$r_j(Z_n) < r_j(T_{n-1}) < 1, \quad j = 1, \ldots, n-3.$$

In the case $\sigma_{n-2} < 1$, increasing A, and consequently $h_{n-1}(Z_n(A; .))$, we arrive at a situation when $\sigma_{n-2}(A) = 1$ for a certain $A = A_1$. The monotone dependence of $h_{n-1}(Z_n(A; .))$ on A is implied by the uniqueness part of Theorem A; The existence of A_1 follows by the continuity and the property $\sigma_{n-2}(A) \to \infty$ as $A \to \infty$. Then, according to Lemma 1 and (4),

$$r_{n-2}(Z_n(A; .)) < r_{n-2}(Z_n(A_1; .)) = \frac{|Z_n'(A_1; 1)|}{|Z_n'(A_1; -1)|} < 1.$$

Therefore, for all local extrema σ of $Z_n'(A;x)$ in $(-1,1)$, as well as for $\sigma=1$, we have $|Z_n'(A;\sigma)|<|Z_n'(A;-1)|$ and consequently, the maximum of $|Z_n'(A;x)|$ is attained at -1.

The case $A\le\eta_1$ follows by symmetry. The proof is complete. □

Note that Markov's inequality was proved first by A.A. Markov [12] for $k=1$. Then his younger brother Vladimir Markov extended it in [13] for any k. Thus, the property of Zolotarev polynomials, proved here, supplies another proof of A.A. Markov's result.

4.3. Polynomials with real critical points

We showed in Theorem 1 that each of the local extrema, from the first half, of the derivative P' is majorized by the value of $|P'(x)|$ at a, provided $P(x)$ is oscillating on $[a,b]$ and attains its maximum on $[a,b]$ at a. Can we relax these conditions if we consider only the first local extremum of P'? We answer here this question in affirmative combining our main result with an interesting old theorem about oscillating polynomials. It is due to Grünwald [10] (extended later by Kuhn [11] to polynomials with real critical points) and gives an important property of the oscillating polynomials. Historical notes and the proof can be found in [19].

Theorem C (*Grünwald–Kuhn Theorem*)**.** *Let f be a polynomial of degree $n\ge 2$ with real coefficients and only real critical points, the smallest, denoted by ξ, being simple. Suppose that $\mu<\xi<\lambda$, $f(\mu)=f(\lambda)=0$, and $f'(x)>0$ for $x\in(\xi,\lambda)$. Then*

$$\int_{\mu-(\xi-\mu)}^{\lambda} f(x)\mathrm{d}x\ge 0.$$

Equality holds if and only if f' is a positive constant multiple of $(x-\xi)(x-\lambda)^{n-2}$.

Theorem 3. *Let P be a real polynomial of degree $n\ge 3$ and $t_1<t_2\le\cdots\le t_{n-1}$ be the zeros of $P'(x)$, all of them lying in (a,b). Assume that*

$$|P(t_1)-P(a)|\ge|P(t_1)-P(t_2)|.$$

Then

$$|P'(\tau)|<|P'(a)|,$$

where τ is the point of the first local extremum of $P'(x)$ in (a,b).

Proof. Assume that P satisfies the conditions in the theorem. For the sake of definiteness, we may suppose further that $P'(a)>0$.

Consider now the polynomial $f(x):=P'(x)$ on the interval $[a,t_2]$ (see Fig. 2). The condition $|P(t_1)-P(a)|\ge|P(t_1)-P(t_2)|$ implies

$$\left|\int_a^{t_1} f(x)\mathrm{d}x\right|\ge\left|\int_{t_1}^{t_2} f(x)\mathrm{d}x\right|.$$

Then there is point t, $a\le t<t_1$, such that

$$\left|\int_t^{t_1} f(x)\mathrm{d}x\right|=\left|\int_{t_1}^{t_2} f(x)\mathrm{d}x\right|. \tag{5}$$

On the other hand, by Grünwald–Kuhn theorem,

$$\left|\int_t^{t_1} f(x)\mathrm{d}x\right|>\left|\int_{t_1}^{t_2} f(x)\mathrm{d}x\right|$$

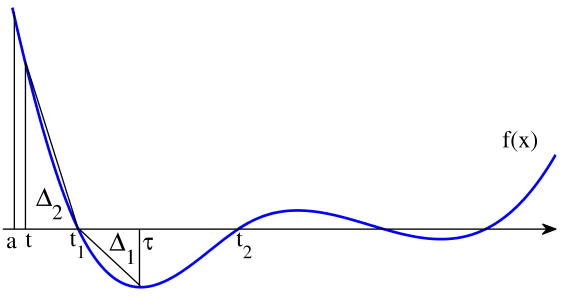

Fig. 2.

provided $\tau - t_1 < t_1 - t$. Therefore

$$\tau - t_1 \geq t_1 - t.$$

Assuming, in addition, that $|f(\tau)| \geq |f(t)|$ we will conclude that the area A_1 of the right triangle Δ_1 with base on $[t_1, \tau]$ and height $|f(\tau)|$ is greater than or equal to the area A_2 of the right triangle Δ_2 with base on $[t, t_1]$ and height $|f(t)|$ (see Fig. 2). But the function $f(x)$ is convex on $[t, \tau]$ and thus

$$\left| \int_{t_1}^{\tau} f(x) \mathrm{d}x \right| > A_1, \quad A_2 > \left| \int_{t}^{t_1} f(x) \mathrm{d}x \right|.$$

Therefore

$$\left| \int_{t_1}^{\tau} f(x) \mathrm{d}x \right| > \left| \int_{t}^{t_1} f(x) \mathrm{d}x \right|,$$

a contradiction with (5). Therefore $|f(\tau)| < |f(t)|$ and consequently $|f(\tau)| < |f(a)|$ since evidently $|f(t)| \leq |f(a)|$. The proof is complete. □

As an immediate consequence we obtain a corollary for oscillating polynomials, which also can be derived by Lemma 1.

Corollary 2. *Let P be an oscillating polynomial on $[a, b]$ of degree $n \geq 2$. Assume that*

$$h_0(P) \geq h_i(P), \quad i = 1, 2.$$

Then, $h_0(P') > h_1(P')$, unless $h_0(P) = 0$.

5. Characterization of the extremal polynomial

In this section we return to the original extremal problem

$$r_i^{(n)} := \max \left\{ r_i(P) : P = P(\mathbf{h}; x), \mathbf{h} \in H_0^n \right\} \tag{6}$$

for a fixed $i \in \{1, \ldots, n-2\}$, where

$$r_i(P) := \frac{h_i(P')}{h_0(P')}.$$

More precisely, we shall study the question of characterizing the corresponding oscillating polynomial P of degree n which, for a given i, supplies the maximum of the ratio r_i. We first derive from Lemma 1 the following partial characterization.

Theorem 4. *For any fixed* $i \in \{1, \dots, n-2\}$, *each solution of the extremal problem* (6) *is a polynomial* $P(\mathbf{h}; x)$ *defined by parameters* $\mathbf{h}$ *such that*

$$h_0 = 1, \quad h_1 = \cdots = h_{k-1} = 0, \quad h_k \in [0, 1], \quad h_{k+1} = \cdots = h_{n-1} = 1,$$

with some $k \in \{1, \dots, i\}$.

Proof. Assume that P is an extremal polynomial in the set of oscillating polynomials of degree n on a certain interval $[a, b]$. If

$$t_j < \frac{a+\tau_i}{2} \quad \text{and} \quad h_j > 0,$$

then, according to Lemma 1, the ratio $r_i(P(\mathbf{h}; \cdot))$ is a decreasing function of h_j and thus, if we decrease slightly h_j the quantity $r_i(P(\mathbf{h}; \cdot))$ will become bigger, a contradiction with the extremality of P. Therefore, for $t_j < (a+\tau_i)/2$, $j \geq 1$, it is necessary that $h_j = 0$. If

$$t_j > \frac{a+\tau_i}{2}, \quad j < n,$$

then again by Lemma 1, $r_i(P(\mathbf{h}; \cdot))$ is an increasing function of h_j for $h_j > 0$ but, because of the continuity of r_i, this holds also for $h_j = 0$. Therefore, if $h_j < 1$, then the ratio $r_i(P(\mathbf{h}; \cdot))$ can be increased by a small perturbation of the parameter h_j, a contradiction. In particular, since $t_i < \tau_i < t_{i+1}$, the above inequality holds for $i < j < n$ and hence $h_j(P(\mathbf{h}; \cdot)) = 1$, $j = i+1, \dots, n-1$.

In conclusion, it is necessary that those h_k for which $t_k < (a+\tau_i)/2$ be zero, while those with $t_k > (a+\tau_i)/2$ should be equal to 1. The only exceptions are: $h_0 = 1$, which follows from the assumption that $\mathbf{h} \in H_0^n$; h_n can be arbitrary (because in this case r_i does not depend on h_n, as it was noticed already in Remark 1); that special h_k for which $t_k = (a+\tau_i)/2$, if such an h_k exists at all. □

Proposition 1 (*Refinement of* Theorem 4). *For* $n \geq 4$ *and* $i = 1, \dots, n-2$, *each solution* P *of the extremal problem* (6) *satisfies the condition* $h_1(P) = 0$.

Proof. Assume that $h_1(P) > 0$. Multiplying if necessary by -1, we get an extremal polynomial P such that

$$P(a) = 1, \quad P(t_1) = -h_1, \quad P(t_2) = 1.$$

Then, with $f = P'$, we have

$$-\int_a^{t_1} f(x)\mathrm{d}x = \int_{t_1}^{t_2} f(x)\mathrm{d}x.$$

But by Grünwald–Kuhn theorem this is impossible unless $\tau_1 - t_1 > t_1 - a$. This means that t_1 is closer to the end-point a than to τ_1 (and consequently to any other τ_i). Thus, by Lemma 1, the ratio r_i can be increased by a small decrease of h_1, a contradiction. □

Remark 2. In the case $n = 3$, for all oscillating polynomials P with $h_0 = h_2 = 1$, we have $r_1(P) = r_1^{(3)} = 1/3$ independently of the values of h_1 and h_3 in [0, 1].

The parameter h_k of the extremal polynomial remains still unspecified in [0, 1]. Notice that if there exists at all such a special h_k for which $t_k = (a+\tau_i)/2$, then (see the proof of Lemma 1) $\partial r_i / \partial h_k = 0$ and in order to describe the behavior of r_i in a neighborhood of h_k one has to perform some additional technical considerations what we do in the sequel. In order to investigate

$\partial^2 r_i/\partial h_k^2$ we arrive at the following quantity, which plays also an important role in other related studies (see e.g. [8, Lemma 2]).

For every oscillating polynomial $P \in \pi_n$ with $h_k(P) > 0$, $1 \le k \le n-1$, consider the ratio

$$R_k(P) := \frac{P'''(t_k)}{2P''(t_k)},$$

where t_k is the kth zero of P'.

Lemma 4. *For every integer $n \ge 3$ and $k \in \{1, \ldots, n-1\}$, the quantity $R_k(P)$ has the properties:*

1) *With $[a, b] = [-1, 1]$ and $P = P(\mathbf{h}; x)$, we have*

$$\frac{\partial t_k}{\partial h_k} = \frac{1}{|P''(t_k)|}\left(R_k(P) + \frac{1}{t_k+1} + \frac{1}{t_k-1}\right),$$

provided $h_k > 0$;

2) *the functional R_k possess the following affine invariance properties:*

$$R_k(L.P(x) + M) = R_k(P), \quad L \ne 0, \quad \textit{and}$$
$$\operatorname{sign}(R_k(P(\lambda x + \mu))) = \operatorname{sign}(R_k(P)), \quad \lambda > 0;$$

3) *let P and Q be oscillating polynomials of degree n such that*

$$h_i(Q) \le h_i(P) \quad \textit{for } i = 1, \ldots, k-1,$$
$$h_k(Q) = h_k(P) > 0,$$
$$h_i(Q) \ge h_i(P) \quad \textit{for } i = k+1, \ldots, n-1.$$

Then $R_k(P) > 0$ implies the inequality $R_k(Q) > 0$;

4)

$$R_k(P) = \sum_{i=1, i\ne k}^{n-1} \frac{1}{t_k - t_i}.$$

Proof. 1) Differentiating the equality $P'(\mathbf{h}; t_i) = 0$, $i \in \{1, \ldots, n-1\}$, with respect to h_j and making use of Lemma A, we obtain

$$(-1)^{n-j} l'_{nj}(t_i) + \frac{\partial t_i}{\partial h_j}.P''(t_i) = 0,$$

and consequently, for $P''(t_i) \ne 0$,

$$\frac{\partial t_i}{\partial h_j} = (-1)^{n-j+1} \frac{l'_{nj}(t_i)}{P''(t_i)}. \tag{7}$$

For $j \ne 0, n$ we have

$$l_{nj}(x) = \frac{q(x)P'(x)}{(1-t_j^2)P''(t_j)} = \frac{(-1)^{n-j+1}}{C_j}(qP')(x), \tag{8}$$

where

$$q(x) = \frac{1-x^2}{x-t_j} = \frac{1-t_j^2}{x-t_j} - (x+t_j)$$

and $C_j := (1-t_j^2)|P''(t_j)| > 0$. In particular, for $i = j = k$,

$$\frac{\partial t_k}{\partial h_k} = \frac{(qP')'(t_k)}{C_k P''(t_k)} = \frac{1}{C_k P''(t_k)} \lim_{t \to t_k} \left(q'(t)P'(t) + q(t)P''(t)\right).$$

In order to compute the limit of the last expression we use Taylor's expansion around the point t_k and obtain

$$\begin{aligned} q'(t)&P'(t) + q(t)P''(t) \\ &= \left(-\frac{1-t_k^2}{(t-t_k)^2} - 1\right) \sum_{m=0}^{n-1} \frac{P^{(m+1)}(t_k)}{m!}(t-t_k)^m + \frac{1-t^2}{t-t_k} \sum_{m=1}^{n-1} \frac{P^{(m+1)}(t_k)}{(m-1)!}(t-t_k)^{m-1} \\ &= -\frac{(1-t_k^2)P''(t_k)}{t-t_k} - \frac{(1-t_k^2)P'''(t_k)}{2} + \frac{1-t^2}{t-t_k}P''(t_k) \\ &\quad + (1-t^2)P'''(t_k) + \mathcal{O}(|t-t_k|). \end{aligned}$$

Thus

$$\lim_{t \to t_k} \left(q'(t)P'(t) + q(t)P''(t)\right) = \frac{1-t_k^2}{2} P'''(t_k) - 2t_k P''(t_k).$$

Now using this expression, we find

$$\frac{\partial t_k}{\partial h_k} = \frac{1}{|P''(t_k)|} \left[R_k(P) - \frac{2t_k}{1-t_k^2} \right].$$

2) is easily verified.

3) In view of 2) we can consider P and Q as oscillating on $[-1, 1]$ polynomials with positive h_0, h_n. Then, for $j \neq k$, applying Lemma A, (7), (8) and $P'(t_k) = 0$ we calculate

$$\begin{aligned} \frac{\partial R_k}{\partial h_j} &= \frac{1}{2P''^2(t_k)} \left((-1)^{n-j} l'''_{nj}(t_k) + \frac{\partial t_k}{\partial h_j} P^{(4)}(t_k) \right) P''(t_k) \\ &\quad - \frac{1}{2P''^2(t_k)} P'''(t_k) \left((-1)^{n-j} l''_{nj}(t_k) + \frac{\partial t_k}{\partial h_j} P'''(t_k) \right) \\ &= \frac{(-1)^{n-j}}{2P''^2} \left(\left(l'''_{nj} - \frac{l'_{nj}}{P''}.P^{(4)} \right) P'' - P''' \left(l''_{nj} - \frac{l'_{nj}}{P''}.P''' \right) \right) \Bigg|_{x=t_k} \\ &= \frac{(-1)}{2C_j P''^2} \left((qP')''' P'' - (qP')' P^{(4)} - P'''(qP')'' + \frac{P'''^2}{P''}(qP')' \right) \Bigg|_{x=t_k} \\ &= \frac{(-1)}{2C_j P''^2} \Bigg(\left(q'''P' + 3q''P'' + 3q'P''' + qP^{(4)} \right) P'' \\ &\quad + (q'P' + qP'') \left(\frac{P'''^2}{P''} - P^{(4)} \right) - P''' \left(q''P' + 2q'P'' + qP''' \right) \Bigg) \Bigg|_{x=t_k} \\ &= \frac{(-1)}{2C_j P''^2} \left(3q''P''^2 + q'P'''P'' \right) \Big|_{x=t_k} = -\frac{1}{C_j} \left(\frac{3}{2} q''(t_k) + q'(t_k) R_k(P) \right) \\ &= -\frac{1}{C_j} \left(3 \frac{1-t_j^2}{(t_k-t_j)^3} - \left(\frac{1-t_j^2}{(t_k-t_j)^2} + 1 \right) R_k(P) \right) \\ &= \frac{1}{|P''(t_j)|} \left(\frac{3}{(t_j-t_k)^3} + \left(\frac{1}{(t_k-t_j)^2} + \frac{1}{1-t_j^2} \right) R_k(P) \right). \end{aligned}$$

Assume now that $R_k(P) > 0$. Then, for $k < j < n$, the expression in the brackets is positive. Thus $\partial R_k/\partial h_j > 0$, and hence, if we increase h_j, the quantity R_k remains positive. Next, let us decrease an $h_j > 0$ with $j : 0 < j < k$. Assume that R_k changes its sign, and let h_j^* be "the first" value of the parameter for which R_k vanishes. In other words,

$$h_j^* := \max\{h_j \in [0, h_j(P)) : R_k(h_j) = 0\}.$$

Then, $\partial R_k/\partial h_j \geq 0$ at h_j^* since R_k decreases from a positive value to 0 when h_j decreases. On the other hand, by the above formula for the derivative and the condition $R_k(h_j^*) = 0$, we obtain $\partial R_k/\partial h_j < 0$ for $h_j = h_j^*$, a contradiction.

We have to consider also the exceptional case when $P''(t_j) = 0$, for $h_j = h_j^* = 0$. This is the only possibility for non-existence of $\partial R_k/\partial h_j$, since the other denominators in the formula for $\partial R_k/\partial h_j$ do not vanish because of the assumption $h_i(P) > 0$, $i = 0, k, n$. In this case a contradiction is obtained when R_k attains "firstly" a sufficiently small δ. This contradiction shows that R_k remains positive with decreasing of h_j, $j = 1, \dots, k-1$. In the end, by 2), a change of h_0 or h_n preserves the positivity of R_k. The property 3) is proved.

4) follows from the equality

$$\sum_{i=1, i\neq k}^{n-1} \frac{1}{x - t_i} = \frac{P''(x)}{P'(x)} - \frac{1}{x - t_k}. \quad \square$$

Proposition 2 (*Further refinement of* Theorem 4). *Let $n \geq 4$ and $i \in \{1, \dots, n-2\}$. Assume that P_* is a solution of the extremal problem* (6). *If there is a local extremum $h_k(P_*) \in (0, 1)$ for some $k < n$, then $k = 2$.*

Proof. Without loss of generality we may suppose that $[a, b] = [-1, 1]$. Let P_* be an extremal polynomial in (6) with some intermediate h_k, that is, with $h_k \in (0, 1)$. In view of Proposition 1, we have $k \neq 1$. Assume that $k \geq 3$. Then, $\frac{\partial r_i}{\partial h_k}(P_*) = 0$ and consequently $t_k^* = \frac{\tau_i^* + a}{2}$ for this polynomial. We are going to prove next that under these conditions

$$\frac{\partial^2 r_i}{\partial h_k^2}(P_*) > 0$$

which implies that $r_i(P)$, as a function of h_k, has a local minimum at the specified point, and this would be a contradiction with the extremality of P_*.

In the proof of Lemma 1 (the last equality) we obtained the closed form expression

$$\frac{\partial r_i}{\partial h_k} = C(\tau_i - a)\left(t_k - \frac{\tau_i + a}{2}\right) = C_1\left(t_k - \frac{\tau_i + a}{2}\right),$$

where $C_1 = C_1(\mathbf{h}, i, k) > 0$. Let us differentiate it and put $\mathbf{h} = \mathbf{h}(P_*)$. We obtain

$$\frac{\partial^2 r_i}{\partial h_k^2}(P_*) = \frac{\partial C_1}{\partial h_k} \cdot 0 + C_1\left(\frac{\partial t_k}{\partial h_k} - \frac{1}{2}\frac{\partial \tau_i}{\partial h_k}\right)(P_*).$$

The first summand in the brackets was calculated in Lemma 4 and the second one can be found by differentiation of the identity $P''(\tau_i) = 0$. Namely, with $q(x)$ and C_k as in (8), we obtain

$$\frac{\partial \tau_i}{\partial h_k} = (-1)^{n-k+1}\frac{\ell''_{nk}}{P'''}(\tau_i) = \frac{(qP')''}{C_k P'''}(\tau_i)$$

$$= \frac{1}{C_k}\left(2\frac{1-t_k^2}{(\tau_i-t_k)^3}\cdot\frac{P'}{P'''}(\tau_i)+0+\frac{1-\tau_i^2}{\tau_i-t_k}\right)$$
$$= \frac{1}{|P_k''(t_k)|}\left(\frac{2}{(\tau_i-t_k)^3}\cdot\frac{P'}{P'''}(\tau_i)+\frac{1-\tau_i^2}{(1-t_k^2)(\tau_i-t_k)}\right).$$

We substitute these expressions in the above formula and taking into account that $t_k^* = \frac{-1+\tau_i^*}{2}$, we obtain

$$\frac{\partial^2 r_i}{\partial h_k^2}(P_*) = \frac{C_1}{|P''(t_k)|}\left(R_k(P)-\frac{2t_k}{1-t_k^2}-\frac{1}{(\tau_i-t_k)^3}\cdot\frac{P'}{P'''}(\tau_i)-\frac{1}{2}\frac{1-\tau_i^2}{(1-t_k^2)(\tau_i-t_k)}\right)\Bigg|_{P=P_*}$$
$$= \frac{C_1}{|P''(t_k)|}\left(R_k(P)-\frac{1}{(\tau_i-t_k)^3}\cdot\frac{P'}{P'''}(\tau_i)\right)\Bigg|_{P=P_*}.$$

Since P_*' is an oscillating polynomial with local extrema at $\tau_1,\ldots,\tau_{n-2}$ it is easily seen that $P_*'''(x)$ also alternates in sign at these points and thus $P_*'(\tau_i)/P_*'''(\tau_i) < 0$. Noticing that $\tau_i^* > t_k^*$ we conclude that the desired inequality $\frac{\partial^2 r_i}{\partial h_k^2} > 0$ would follow from $R_k(P_*) \ge 0$. In order to prove the later it suffices to find an oscillating polynomial $P_N(x)$ of degree $N \ge n$ for which $R_k(P_N) > 0$ and

$$h_k(P_N) = h_k(P_*);\quad h_i(P_N) \le h_i(P^*)\quad \text{for } i = k+1,\ldots,n-1. \tag{9}$$

Indeed, if $N = n$ and we add to (9) the conditions

$$h_i(P_N) \ge 0 = h_i(P_*)\quad \text{for } i = 1,\ldots,k-1,$$

then the conditions in Lemma 4 would be fulfilled and we could conclude that $R_k(P^*) > 0$.

If $N > n$, then, starting from P_N we decrease $\{h_i\}_{i=1}^{k-1}$ to $h_i(P_*) = 0$ and increase $\{h_i\}_{i=k}^{n-1}$ to $h_i(P_*) = 1$ in order to obtain $\tilde{P}_N$ so that the inequality $R_k(\tilde{P}_N) > 0$ remains valid. Next, set $h_n = h$ to be a parameter that tends to ∞ and with appropriate linear change of the argument, let P_N^h be the polynomials with parameters $\mathbf{h}(\tilde{P}_N)$, but with h in place of $h_n(\tilde{P}_N)$, such that $x_1(P_N^h) = x_1(P_*)$ and $x_n(P_N^h) = x_n(P_*)$. Using Lemma 2, we conclude that P_N^h tends uniformly to P_* as $h \to \infty$ on $[x_1(P_*), x_n(P_*)]$ and, therefore on every finite interval. In view of Lemma 4 and the construction of P_N^h from $\tilde{P}_N$ we have $R_k(P_N^h) > 0$, for sufficiently large h. Then, after the limit pass $h \to \infty$ we get $R_k(P_*) \ge 0$, provided we have found a polynomial P_N with the required properties.

Assume further that $k \ge 4$. In order to construct such a polynomial we consider the function

$$f_\alpha(x) = x^4(x^2-\alpha x+3\alpha-8)\mathrm{e}^{-x}$$

where the numerical parameter α is varying in a neighborhood of 10 so that the polynomial in the brackets has always 2 real zeros. Then $f_\alpha(x)$ is a uniform limit on every finite interval of the oscillating polynomials

$$Q_N(x) := x^4(x^2-\alpha x+3\alpha-8)\left(1-\frac{x}{N}\right)^{N-6},\quad N \ge 7.$$

We claim that for sufficiently large N, $R_4(Q_N) > 0$ and $h_4(Q_N) > h_i(Q_N)$, $i = 5,\ldots,N-1$. Actually the last inequality for $i \ge 7$ is clear from $h_i(Q_N) = 0$. The derivative

$$f_\alpha'(x) = -x^3(x-2)(x^2-(\alpha+4)x+6\alpha-16)\mathrm{e}^{-x}$$

has 6 real zeros and we denote them in increasing order by

$$t_1(f_\alpha) = t_2(f_\alpha) = t_3(f_\alpha) = 0, \quad t_4(f_\alpha) = 2, \quad t_{5,6}(f_\alpha).$$

In particular, for $\alpha = 10$, we have

$$t_4 = 2, \quad t_5 = 7 - \sqrt{5}, \quad t_6 = 7 + \sqrt{5},$$

and direct computations show that

$$h_4(f_\alpha) = 12.99\ldots, \quad h_5(f_\alpha) = 12.93\ldots, \quad h_6(f_\alpha) = 10.59\ldots.$$

Then, from the uniform convergence of $Q_N(x)$ to $f_\alpha(x)$ and by Lemma 4, statement 4), we obtain

$$\begin{aligned}
&\lim_{N\to\infty} R_4(Q_N)\\
&\quad = \lim_{N\to\infty}\left(\frac{3}{t_4(Q_N)-0} + \frac{1}{t_4(Q_N)-t_5(Q_N)} + \frac{1}{t_4(Q_N)-t_6(Q_N)} + \frac{N-7}{t_4(Q_N)-N}\right)\\
&\quad = \frac{3}{2} + \frac{1}{2-t_5(f_\alpha)} + \frac{1}{2-t_6(f_\alpha)} - 1 = \frac{1}{2} + \frac{4-(\alpha+4)}{4-2(\alpha+4)+6\alpha-16} = \frac{\alpha-10}{4(\alpha-5)}.
\end{aligned}$$

Now we choose α sufficiently close to 10 but greater than 10 in order to preserve the relations $h_4(f_\alpha) > h_{5,6}(f_\alpha)$ (like it is in the case $\alpha = 10$). Next we choose a sufficiently large N so that (by virtue of the above limit and the uniform convergence)

$$R_4(Q_{N+4-k}) > 0 \quad \text{and} \quad h_4(Q_{N+4-k}) > h_i(Q_{N+4-k}) \quad \text{for } i = 5, 6.$$

Then we construct, on the basis of Lemma 2, a polynomial $\tilde{Q}_N$ of degree N which is sufficiently close to Q_{N+4-k} on $[0, N]$ adding $k-4$ sufficiently large local extrema in $(-\infty, -1)$ (i.e., situated before the point corresponding to $h_0(Q_{N+4-k})$). We make the approximation good enough to preserve the property

$$R_k(\tilde{Q}_N) > 0 \quad \text{and} \quad h_k(\tilde{Q}_N) > h_i(\tilde{Q}_N), \quad i = k+1, \ldots N-1.$$

Notice here that adding local extrema before h_0 changes the numbering of the extremum points. For example $t_k(\tilde{Q}_N)$ corresponds to $t_4(Q_{N+4-k})$. Finally the wanted polynomial P_N is defined by

$$P_N(x) := \frac{h_k(P_*)}{h_k(\tilde{Q}_N)}\tilde{Q}_N(x).$$

Clearly, it satisfies the (9) since for $k < i < n$ we have

$$h_i(P_N) = h_k(P_*)\frac{h_i(\tilde{Q}_N)}{h_k(\tilde{Q}_N)} < h_k(P_*) < 1 = h_i(P_*).$$

This finishes the proof of the proposition for $k \geq 4$.

The remaining case $k = 3$ can be verified by similar reasoning using the exponential polynomial

$$f(x) = x^3 P_m(x) \mathrm{e}^{-x},$$

defined by

$$h_3(f) = \cdots = h_{m+3}(f) = 1$$

with some $m \geq 23$. Constructing numerically $f(x)$ we computed that $R_3(f) > 0$. This, as in the case $k = 4$, leads to the conclusion that the extremal problem (6) has not a solution P_* for which $h_3(P_*) \in (0, 1)$. We omit the details. □

Remark 3. Considering the polynomial $f(x) = x^2 P_l(x)\mathrm{e}^{-x}$ with $l \geq 7$ such that

$$h_2(f) = \frac{1}{3}, \quad h_3(f) = \cdots = h_{l+2}(f) = 1,$$

we compute that $R_2(f) > 0$. This shows in the same fashion as in the proof of the previous proposition that $h_2(P_*)$ cannot lie in $(0, \frac{1}{3}]$. Further computer experiments show that $h_2(P_*)$ does not lie also in $(\frac{1}{3}, 1)$. Moreover, we think that in the last case the equality $t_2^* = \frac{\tau_i^* + a}{2}$ cannot take place at all. These experiments make us suggest that all extremal polynomials P_* of problem (6) for $n \geq 4$ are of the form $P_* = \pm T_{n,k}(\lambda x + \mu)$. Let us recall that $T_{n,k}$ are the oscillating polynomials defined by the parameters $\mathbf{h}(T_{n,k}) = (1, 0, \ldots, 0, 1, \ldots, 1)$ where the zero sequence has a length k.

Remark 4. We should note however that there are oscillating polynomials for which $\frac{\partial r_i}{\partial h_k} = 0$ and $\frac{\partial^2 r_i}{\partial h_k^2} < 0$, but they are not in the class described in Theorem 4.

As we mentioned in the introduction, Conjecture 1 is not true for every n without the assumption of symmetry. But elementary considerations show that it is true for some small n. What is the highest degree D so that it holds for all polynomials of degree less than or equal to D? This is the question we are going to clarify below.

We have to check the inequalities

$$r_i^{(n)} \leq 1, \quad i = 1, \ldots, n-2,$$

for small consecutive n. For $n = 3$ the inequality follows from Theorem 1. For $n \geq 4$ the next proposition, which is of independent interest, shows that it suffices to check only the case $i = n - 2$.

Proposition 3. *For $n \geq 4$ we have*

$$r_{n-2}^{(n)} > r_i^{(n)}, \quad i = 1, \ldots, n-3.$$

Proof. Let P_* be an extremal polynomial in (6) for $i \in \{1, \ldots, n-3\}$. By Theorem 4, $h_{i+1}(P_*) = \cdots = h_{n-1}(P_*) = 1$. Without loss of generality we may assume also that $h_n(P_*) = 1$. We shall show that $h_{n-2}(P_*') > h_i(P_*')$. Indeed, comparing P_* with the Tchebycheff polynomial T_n we observe that $h_j(P_*) = h_j(T_n)$ for $j = i+1, \ldots, n$, whereas

$$h_j(P_*) \leq h_j(T_n) \quad \text{for } j = 0, \ldots, i,$$

with strict inequality for $j = 1$, according to Proposition 1. Moreover, $t_1 \leq \cdots \leq t_i \leq \tau_i \leq \tau_{n-2}$ for every oscillating polynomial. Then, by Lemma 1,

$$r_{i,n-2}(P_*) < r_{i,n-2}(T_n) \leq 1,$$

and consequently

$$r_{i,n-2}(P_*) = \frac{h_i(P_*')}{h_{n-2}(P_*')} < 1.$$

Therefore

$$r_i^{(n)} = r_i(P_*) = \frac{h_i(P'_*)}{h_0(P'_*)} < \frac{h_{n-2}(P'_*)}{h_0(P'_*)} = r_{n-2}(P_*) \le r_{n-2}^{(n)}$$

and the proof is complete. □

For a fixed n, the verification of the inequality $r_{n-2}^{(n)} < 1$ is reduced to numerical construction of the special polynomials $T_{n,k}$ for $k = 1, \ldots, n-2$, and comparing the extremal values of their derivatives. More precisely, according to the characterization of the extremal polynomial P_* (with $[a, b] = [-1, 1]$, $h_n = 1$) given above, we distinguish the following three cases:

(i) $P_* = T_{n,k}$, $3 \le k \le n-2$, $n \ge 5$. We verify numerically the inequality

$$h_0(T'_{n,k}) \ge h_{n-2}(T'_{n,k})$$

for these k;

(ii) P_* is defined by $h_1 = 0$, $h_2 \in [0, 1]$, $h_3 = \cdots = h_n = 1$. Then, from Theorem B we have

$$r_{n-2}(P_*) = \frac{h_{n-2}(P'_*)}{h_0(P'_*)} < \frac{h_{n-2}(T'_{n,1})}{h_0(T'_{n,2})}$$

and it suffices to verify that $h_{n-2}(T'_{n,1}) < h_0(T'_{n,2})$;

(iii) P_* is defined by $h_1 = h_2 = 0$, $h_3 \in [0, 1]$, $h_4 = \cdots = h_n = 1$, $n \ge 5$. Like in (ii),

$$r_{n-2}(P_*) < \frac{h_{n-2}(T'_{n,2})}{h_0(T'_{n,3})},$$

and it suffices to verify that $h_{n-2}(T'_{n,2}) < h_0(T'_{n,3})$.

After we did all these verifications by computer modifying slightly the algorithm used in [16], we came to the conclusion that Conjecture 1, without the requirement for symmetry, is true for all $n \le 11$, while for $n = 12$ it is no more true since $r_{10}(T_{12,k}) > 1$ for $k = 6, 7, 8$.

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