Scattering of nucleons in the classical Skyrme model

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Abstract

Classically spinning $B = 1$ Skyrmions can be regarded as approximations to nucleons with quantised spin. Here, we investigate nucleon–nucleon scattering through numerical collisions of spinning Skyrmions. We identify the dineutron/diproton and dibaryon short-lived resonance states, and also the stable deuteron state. Our simulations lead to predictions for the polarisation states occurring in right angle scattering.
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1. Introduction

The Skyrme model [1] is a nonlinear theory of pion fields that has topological soliton solutions. Encouragingly, Witten identified the Skyrme model as a low energy effective model of QCD [2,3]. The conserved topological charge is interpreted as the baryon number B, and the minimal energy static solutions for each integer B are called Skyrmions. They can be treated as rigid bodies, free to rotate in space and also in isospace (the three-dimensional space of the pion fields). When the rotational motion is quantised, the Skyrmions are models for nucleons and nuclei. In particular the $B = 1$ Skyrmions, quantised with spin and isospin half, model protons and neutrons.

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A full quantum mechanical treatment of the interaction of two $B = 1$ Skyrmions is difficult. Each Skyrmion has three position coordinates and three orientational coordinates, so two-Skyrmion dynamics involves twelve coordinates [4]. A truncation to ten coordinates has been useful for modelling the deuteron bound state [5], but no proper discussion of two-nucleon scattering is possible with this truncation.

An alternative is a classical approach, using the idea that a classically spinning Skyrmion is a reasonable model of a nucleon [6,7]. The classical angular velocity of the Skyrmion is fixed to match the quantised spin and isospin of the nucleon. Scattering of non-spinning Skyrmions was first performed using an axially symmetric ansatz [8], the first full-field simulation was reported in [9], and multi-charge Skyrmion scattering was considered in [10]. The scattering of spinning Skyrmions representing nucleons was considered by Gisiger and Paranjape [6], and they obtained analytical formulae for the scattering angle for large impact parameters. There has also been some recent work on multi-charge, purely isospinning Skyrmions [12].

A systematic, numerical investigation of Skyrmion scattering without spin, at moderate and small impact parameters [11], confirmed that matter is exchanged between the Skyrmions when the Skyrmions are close, and also showed that a substantial rotation of the Skyrmion's orientation can occur. Here, we numerically investigate the scattering of two classically spinning $B = 1$ Skyrmions to model proton–proton, neutron–neutron and proton–neutron scattering at various impact parameters, including head-on collisions. We are particularly interested in the change in the polarisation states of the nucleons when they scatter, and also in finding evidence, using our classical approximation, for the deuteron and for the known two-nucleon resonance states.

The format of this paper is as follows. We first introduce the Skyrme model and its calibration. Then we discuss how to identify classically spinning Skyrmions as protons or neutrons. The next section presents the results of our numerical scattering of Skyrmions, and our identification of the dineutron/diproton resonances, the deuteron bound state, and the excited dibaryon state. The concluding section summarises and discusses our results.

2. The Skyrme model

The Skyrme model is defined by the Lagrangian density

$$\mathcal{L} = -\frac{F_\pi^2}{16} \operatorname{Tr}(R_\mu R^\mu) + \frac{1}{32e^2} \operatorname{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) + \frac{m_\pi^2 F_\pi^2}{8} \operatorname{Tr}(U - I_2), \tag{1}$$

where the Skyrme field $U(t, \boldsymbol{x})$ is an SU(2)-valued scalar and $R_\mu = \partial_\mu U U^\dagger$ is its su(2)-valued current. F_π, e and m_π are parameters, which are fixed by comparison with experimental data. Their values will be discussed later. It is convenient for us to work in dimensionless Skyrme units. One Skyrme length unit corresponds to $\frac{2}{eF_\pi}$ in inverse MeV, and one Skyrme energy unit corresponds to $\frac{F_\pi}{4e}$ MeV. Conversion of inverse MeV to fm is as usual through $\hbar = 197.3$ MeV fm. The dimensionless pion mass in Skyrme units is

$$m = \left(\frac{2}{eF_\pi}\right) m_\pi . \tag{2}$$

In Skyrme units, the energy of a static field is

$$E = \int \left(-\frac{1}{2} \operatorname{Tr}(R_i R_i) - \frac{1}{16} \operatorname{Tr}([R_i, R_j][R_i, R_j]) + m^2 \operatorname{Tr}(I_2 - U) \right) d^3x .$$

Side   Top

Fig. 1. $B = 1$ Hedgehog Skyrmion. (For a colour image, we refer the reader to the web version of the article.)

It is often convenient, especially in numerical simulations, to express U in terms of a triplet of pion fields $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ and an additional auxiliary field σ as

$$U(t, \boldsymbol{x}) = \sigma(t, \boldsymbol{x}) I_2 + i\boldsymbol{\pi}(t, \boldsymbol{x}) \cdot \boldsymbol{\tau}, \tag{3}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the three Pauli matrices, with the constraint $\sigma^2 + \boldsymbol{\pi} \cdot \boldsymbol{\pi} = 1$ so that $U \in \mathrm{SU}(2)$.

At fixed time, $U(t, \boldsymbol{x})$ is a map $U : \mathbb{R}^3 \to \mathrm{SU}(2)$, where $U \to I_2$ at spatial infinity. This boundary condition compactifies $\mathbb{R}^3 \cup \{\infty\}$ to S^3. The group manifold of $\mathrm{SU}(2)$ is S^3, so a finite energy configuration U extends to a map $U : S^3 \to S^3$, and then belongs to a class of $\pi_3(S^3) = \mathbb{Z}$ indexed by an integer $B \in \mathbb{Z}$, called the baryon number. B is also the degree of the map U which can be explicitly calculated as

$$B \equiv \int \mathcal{B}(\boldsymbol{x}) \, d^3 x = -\frac{1}{24\pi^2} \int \varepsilon_{ijk} \, \mathrm{Tr}(R_i R_j R_k) \, d^3 x, \tag{4}$$

where $\mathcal{B}(\boldsymbol{x})$ is the baryon density. Skyrmions are the static field configurations of minimal energy for each value of B. In the figures we plot level-sets of baryon density $\mathcal{B}(\boldsymbol{x})$.

It is well known that the $B = 1$ Skyrmion can be found using the so-called Hedgehog ansatz [1]

$$U_H(\boldsymbol{x}) = \cos f(r) \, I_2 + i \sin f(r) \, \hat{\boldsymbol{x}} \cdot \boldsymbol{\tau}, \tag{5}$$

where $r = |\boldsymbol{x}|$ and $\hat{\boldsymbol{x}}$ is the radial unit vector. $f(r)$ is a real radial profile function satisfying $f(0) = \pi$, $f(\infty) = 0$. This produces a solution with rotationally symmetric energy and baryon density. Fig. 1 shows a $B = 1$ Hedgehog Skyrmion coloured as in [7]. The centres of the white and black regions are where $\pi_1^2 + \pi_2^2 = 0$ and $\pi_3 > 0$, $\pi_3 < 0$ respectively. The centres of the red, green and blue regions are where $\pi_3 = 0$ and $\tan^{-1}\left(\frac{\pi_2}{\pi_1}\right) = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ respectively. This is the colouring scheme used throughout this paper.

An important feature of the Hedgehog ansatz is that an isorotation, $U(\boldsymbol{x}) \to AU(\boldsymbol{x})A^\dagger$, which acts as a rotation among the pion fields, is equivalent to a rotation in space, $U(\boldsymbol{x}) \to U(D(A)\boldsymbol{x})$, where $D(A)_{ij} = \frac{1}{2}\mathrm{Tr}(\tau_i A \tau_j A^\dagger)$ is the standard $\mathrm{SO}(3)$ rotation matrix corresponding to the $\mathrm{SU}(2)$ matrix A. This can be understood in terms of the coloured Skyrmions, where a spatial rotation can be 'undone' by a reordering of colours, i.e. an isorotation of the pion fields. A further use of the colouring is that it shows when two Hedgehog Skyrmions are in the attractive channel [13]. This is when the colours on the nearest, facing sides of two Skyrmions match.

The $B = 2$ Skyrmion plays an important role in the dynamics of two $B = 1$ Skyrmions. This Skyrmion is toroidal, and is shown in Fig. 2.
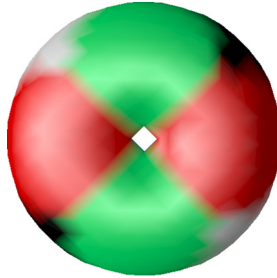
Fig. 2. $B = 2$ Skyrmion.

To relate the Skyrme model to nuclear physics, Skyrmions should be quantised. As the Skyrmions are energy minima they can be treated as rigid bodies rotating in space and isospace. For general Skyrmions, semi-classical quantisation is the quantisation of the time-dependent space and isospace rotations A_2 and A_1 in the rigid-body ansatz

$$U(t, \boldsymbol{x}) = A_1(t) U_0(D(A_2(t)) \boldsymbol{x}) A_1(t)^\dagger, \tag{6}$$

where $U_0(\boldsymbol{x})$ is a static solution. $A_1(t)$ and $A_2(t)$ are SU(2) matrices, and $D(A_2(t))$ is, as before, the SO(3) matrix corresponding to $A_2(t)$. The spherical symmetry of the Hedgehog ansatz (5) implies that the rotational motion of the $B = 1$ Skyrmion only depends on the combined SU(2) matrix $A(t) = A_1(t) A_2(t)$.

Under rigid rotation the $B = 1$ Skyrmion has body-fixed angular momentum $L_i = \lambda(-a_i + b_i)$ and body-fixed isospin angular momentum $K_i = \lambda(a_i - b_i)$, where $a_j = -i \operatorname{Tr}(\tau_j A_1^\dagger \dot{A}_1)$ is the angular velocity in isospace, $b_j = i \operatorname{Tr}(\tau_j \dot{A}_2 A_2^\dagger)$ is the angular velocity in space and λ is the moment of inertia [14]. From these one obtains the space-fixed angular momentum $J_i = -D(A_2)_{ij}^T L_j$ and the "space-fixed" isospin angular momentum $I_i = -D(A_1)_{ij} K_j$. The space-fixed angular momenta correspond to physical spin and isospin, and these can be expressed purely in terms of $A(t)$ and its time derivative. Nucleons have both spin and isospin half. A proton is an isospin-up state, and a neutron is an isospin-down state.

Due to the spherical symmetry of the Hedgehog ansatz, $L_i + K_i = 0$ ($i = 1, 2, 3$). Something similar is true for the $B = 2$ toroidal Skyrmion where $L_3 + 2K_3 = 0$ [15]. We shall make use of these properties later. For an in-depth discussion of Skyrmion quantisation we point the reader to [16,15,17,14].

2.1. Skyrmion calibration

For the model to be relevant to nuclear physics it requires calibration. Adkins and Nappi [16] calibrated the quantised $B = 1$ Skyrmion against the masses of the spin $\frac{1}{2}$ nucleons, the spin $\frac{3}{2}$ delta resonances, and the pions, finding $F_\pi = 108$ MeV, $e = 4.84$ and $m_\pi = 138$ MeV (so $m = 0.526$). This calibration assumes that rigid rotation is a good approximation, but the surface fields of the delta are rotating close to the speed of light and pion radiation is very strong. Therefore, for the delta, the approximation is not reliable and this calibration is not valid for higher charge Skyrmions. Recently, Manton and Lau calibrated the Skyrme model against the states of Carbon-12 [18] and found $m = 0.7$ to be optimal. For the $B = 1$ Skyrmion this leads to an energy (Skyrmion mass) $M = 159.89$, $\lambda = 55.89$ and an rms matter radius $\langle r^2 \rangle^{1/2} = 1.006$ in Skyrme units. The calibration requires $F_\pi = 117.5$ MeV and $e = 3.93$. Planck's constant in

Skyrme units is always $\hbar = 2e^2$ so for this calibration $\hbar = 30.8$. These are the values we use for our simulations and throughout this paper: $m = 0.7$ and $\hbar = 30.8$.

The resulting nucleon mass is the static $B = 1$ Skyrmion energy M plus the spin-energy contribution, $M + \frac{3}{8\lambda}\hbar^2$, which in physical units is

$$M_{\mathrm{N}} = M\frac{F_\pi}{4e} + \frac{3}{8\lambda}e^3 F_\pi . \tag{7}$$

With our calibration $M_{\mathrm{N}} = 1243$ MeV, about 30% larger than the physical nucleon mass $M_{\mathrm{N}} = 939$ MeV. Also the rms matter radius is $\langle r^2 \rangle^{1/2} = 0.87$ fm.

The $B = 2$ toroidal Skyrmion can also be quantised as a rigid body. The two lowest-energy states are a spin 1, isospin 0 state representing the deuteron, and a spin 0, isospin 1 state representing the low-energy resonance known as the diproton/dineutron. With our calibration, and following the quantisation procedure in [17], we find the mass for the deuteron to be 2324 MeV, which is similarly larger than the experimental value of 1876 MeV. We also find the rms matter radius to be 1.08 fm, which is less than half the experimentally found rms radius of the deuteron ~ 2.14 fm, showing that rigid-body quantisation of the static $B = 2$ Skyrmion gives a too tightly bound representation of the loosely bound deuteron. A better representation of the deuteron in the Skyrme model was obtained in [5]. Using a Yang–Mills instanton ansatz for the Skyrme field [19], an additional, radial degree of freedom could be included, allowing the two $B = 1$ Skyrmions to separate. In the deuteron state, this radial degree of freedom oscillates with the zero point motion of a vibrational ground state. We will see later that modest oscillations of the radial degree of freedom are generated when spinning Skyrmions collide classically.

Although this calibration is not optimal for the proton, neutron or deuteron, we have found that the processes discussed later are not sensitive to small changes of calibration.

3. Spinning Skyrmions as protons or neutrons

It was discovered by Adkins, Nappi and Witten [20] that the properly normalised wavefunctions for the proton and neutron in spin-up and spin-down states, along the z-axis, are

$$p^\uparrow = \frac{1}{\pi}(a_1 + i a_2), \quad p^\downarrow = -\frac{i}{\pi}(a_0 - i a_3), \tag{8}$$

$$n^\uparrow = \frac{i}{\pi}(a_0 + i a_3), \quad n^\downarrow = -\frac{1}{\pi}(a_1 - i a_2), \tag{9}$$

where $A = a_0 I_2 + i a_i \tau_i$ is the SU(2) matrix that controls the $B = 1$ Skyrmion's orientation and $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$. It has been proposed that these wavefunctions can be used to identify protons and neutrons as classically spinning Skyrmions [6,7]. This is because each wavefunction has maximal magnitude on a great circle in SU(2) and motion along the great circle is equivalent to a 4π rotation of the Skyrmion in space. The phase of the wavefunction changes by 2π around the circle, so the spin is $\frac{1}{2}$.

The wavefunctions of $p^\downarrow$ and $n^\uparrow$ are maximal when the Hedgehog Skyrmion (initially in its standard orientation) is simply rotated about the z-axis, whereas the wavefunctions of $p^\uparrow$ and $n^\downarrow$ are maximal when the Skyrmion is flipped over, and then rotated about the z-axis.

A Hedgehog Skyrmion spinning with angular frequency ω about the z-axis is obtained by acting with the isorotation matrix $A(t) = \exp(\frac{i\omega t}{2}\tau_3) = \cos(\frac{\omega t}{2}) + i\sin(\frac{\omega t}{2})\tau_3$. This isospinning Skyrmion is simultaneously spinning about its white–black axis in space. It has spin and isospin
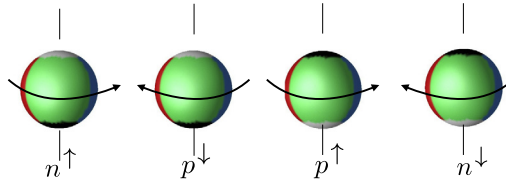
Fig. 3. Identifying Skyrmions as neutrons and protons.

projections $J_3 = \lambda\omega$, $I_3 = -\lambda\omega$, which are opposite. An anticlockwise spin, $\omega > 0$, corresponds to an $n^\uparrow$ state and a clockwise spin, $\omega < 0$, corresponds to a $p^\downarrow$ state.

The isorotation matrix $A(t)(i\tau_2)$ flips the Skyrmion by π about the y-axis, and rotates it with angular frequency ω about the z-axis. Such an isospinning Skyrmion has $J_3 = -\lambda\omega$ and $I_3 = -\lambda\omega$, so the spin and isospin projections are now equal. An anticlockwise spin, $\omega > 0$, corresponds to an $n^\downarrow$ state and a clockwise spin, $\omega < 0$, corresponds to a $p^\uparrow$ state.

The replacement $A(t) \to A(t)(i\tau_2)$ produces the spin flips $n^\uparrow \to n^\downarrow$ and $p^\downarrow \to p^\uparrow$ because $D_{33}(A) \to -D_{33}(A)$. This leads to the identification that a Skyrmion spinning anticlockwise about its white–black axis is always a neutron and a Skyrmion spinning clockwise about the same axis is always a proton, as shown in Fig. 3. This is regardless of the orientation of the axis in space.

To classically model a nucleon whose projected spin has magnitude $\frac{1}{2}$ we require $\lambda|\omega| = \frac{1}{2}\hbar$, so

$$|\omega| = \frac{\hbar}{2\lambda} = 0.28\,,$$

as $\hbar = 30.8$ and $\lambda = 55.89$. We use $|\omega| = 0.28$ throughout this paper. With this angular velocity, the surface fields of the Skyrmion are not rotating close to the speed of light.

There are classical models of the two lowest-energy quantum states of the $B = 2$ toroidal Skyrmion too. In the deuteron state, the torus spins around an axis orthogonal to the symmetry axis of the torus, with spin 1. The colour orientation is arbitrary and unchanging in time, as the deuteron has isospin zero. In the dineutron/diproton state the torus is coloured as in Fig. 2, and the colours circulate steadily, so that the isospin is 1. The black and white regions on the circular edge of the torus are fixed. The torus is not spatially rotating, as the dineutron/diproton has spin zero. The torus does not rotate about its symmetry axis in either state. If it did, the constraint $L_3 + 2K_3 = 0$ would imply that the state had non-zero spin and non-zero isospin.

4. Skyrmion scattering and its interpretation

4.1. Spinning Skyrmion collisions

We are interested in classically replicating nucleon scattering, namely proton–proton, neutron–neutron and proton–neutron scattering. To do this we identify protons and neutrons as spinning Skyrmions, as discussed above. Instead of colliding non-spinning Skyrmions as in some previous work [11], we numerically collide spinning Skyrmions. We initiate a collision by forming two configurations U_1 and U_2 out of spinning and translated Hedgehog Skyrmions,

$$U_1(t, \boldsymbol{x}) = U_H\big(D_1(t)(\boldsymbol{x} - \boldsymbol{X})\big),$$
$$U_2(t, \boldsymbol{x}) = U_H\big(D_2(t)(\boldsymbol{x} + \boldsymbol{X})\big), \tag{10}$$

where $D_1(t)$ and $D_2(t)$ are two SO(3) time-dependent rotation matrices and $\pm \boldsymbol{X}$ are the locations of the Skyrmions. We choose $\boldsymbol{X} = (b, Y, 0)$, where Y is large compared with the Skyrmion radius. Then, using the product ansatz, we create an initial configuration of two Skyrmions boosted towards each other, at velocities $\pm \mathrm{v}$, by

$$U(t, \boldsymbol{x}) = U_1\big(x, \gamma(y + \mathrm{v}t), z\big) U_2\big(x, \gamma(y - \mathrm{v}t), z\big), \tag{11}$$

where $\gamma = 1/\sqrt{1 - \mathrm{v}^2}$ is the usual Lorentz boost factor.

Each Skyrmion is initially moving parallel to the y-axis. b is the impact parameter chosen to give the desired orbital angular momentum $l\hbar = 2M\mathrm{v}b$, where M is the Skyrmion mass. We define transverse polarisation to be where the Skyrmion's white–black axis is aligned parallel to the z-axis and linear polarisation to be where the Skyrmion's white–black axis is aligned parallel to the y-axis.

We numerically evolve these initial configurations using the field equations obtained from the Skyrme Lagrangian density (1). We use a finite difference leap-frog numerical algorithm on a suitably large lattice. Leap-frog was chosen because it is a symplectic integrator.

All of the simulations are presented as videos at http://www.bristoltheory.org/people/david.foster/skyrmevideos/; we urge the reader to view these videos.

4.2. The dineutron/diproton

There are no physical bound states in the 2-proton or 2-neutron systems. However, a 2-proton low energy resonance is observed in proton–proton collisions and is called the diproton. It has spin $\boldsymbol{J} = 0$ and isospin $\boldsymbol{I} = 1$. The dineutron is the corresponding 2-neutron resonance. Neutron–neutron scattering is less easy to achieve than proton–proton scattering, but the dineutron resonance is observed as a decay product of the neutron-rich nucleus beryllium-16, ^{16}Be [21], where it is energetically favourable for ^{16}Be to decay to ^{14}Be by simultaneously ejecting two neutrons. The trajectories of the two detected neutrons trace back to the same point, where they form a state of lifetime $\sim 10^{-22}$ s. The ^{16}Be and ^{14}Be ground states are both $\boldsymbol{J} = 0$, hence the dineutron is a $\boldsymbol{J} = 0$ state. The relative energy of the two neutrons is found to be ≈ 0.25 MeV. So, assuming no other radiation, the neutrons separate with initial speed $\mathrm{v} = 0.02$.

There is no significant electrostatic repulsion between two neutrons, and since we have not included Coulomb effects in our Skyrme model, our simulations of Skyrmion scattering are likely to better model the dineutron rather than the diproton. So we collide two Skyrmions which are spinning as anti-aligned neutrons (illustrated in Fig. 3), giving a $\boldsymbol{J} = 0$, $\boldsymbol{I} = 1$, $I_3 = -1$ state, with initial speeds $\mathrm{v} = 0.02$. The Skyrmions are either transversely polarised (parallel to the z-axis) or linearly polarised. Both types of scattering are shown in Fig. 4.

It is seen that no bound state is produced, but the $B = 2$ torus configuration is present for a short time. This torus forms in the (y, z)-plane and it does not spin, but it has isospin because the colouring continues to circulate at all times. It has the quantum numbers $\boldsymbol{J} = 0$ and $\boldsymbol{I} = 1$, $I_3 = -1$ which is consistent with the short-lived dineutron state observed in the decay of ^{16}Be. For both polarisations, the black and white regions alternate around the circular edge of the torus.

The scattering shown in Fig. 4 is for a head-on collision of two neutrons. For both polarisations, the outgoing Skyrmions are moving at right angles to the incoming Skyrmions, and they
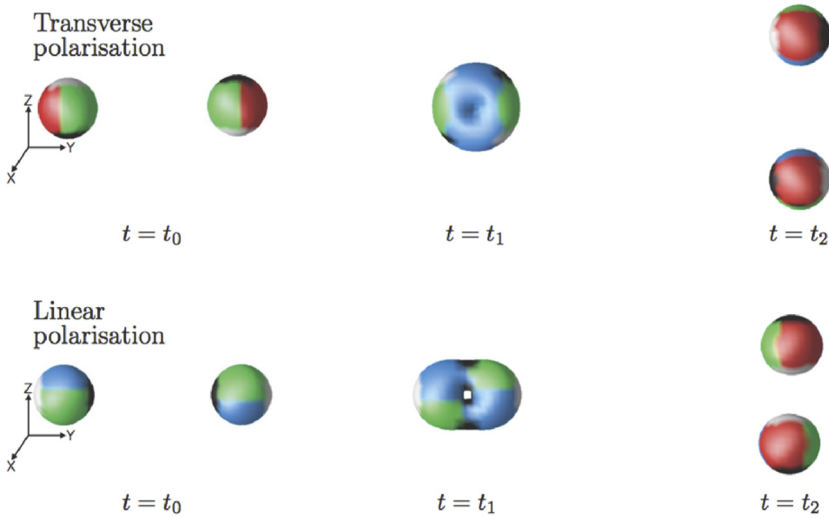
Fig. 4. Level-sets of baryon density $\mathcal{B}(\mathbf{x})$ for the $J = 0$ scattering of two neutrons. Three time-snapshots are shown here. The transverse polarisation scattering and the linear polarisation scattering videos are shown online.

are spinning as neutrons too, because of isospin conservation. Similar right angle scattering was observed for non-spinning Skyrmions in [11].

In the case of transverse polarisation the outgoing direction of motion is precisely the direction of the initial polarisation axis. There is another interesting phenomenon: the polarisation axis, the white–black axis, jumps by 90 degrees during the scattering, from being parallel to the z-axis to being parallel to the y-axis. This is possible because of the way the Skyrmions merge and break up, such that each outgoing Skyrmion is a combination of half of each incoming Skyrmion.

From our numerical simulations we can observe the sense of the polarisation, before and after the collision. For example, the neutron incoming from the negative y-direction (the left) has its polarisation in the positive z-direction, and the neutron that is outgoing in the positive z-direction has its polarisation in the negative y-direction. The result can be stated in a more invariant way, both for 2-neutron and 2-proton right angle scattering. Each particle, incoming or outgoing, has a momentum $\mathbf{p}$ and classical spin $\mathbf{s}$. The two incoming particles and the two outgoing particles all have the same value for the vector $\mathbf{p} \times \mathbf{s}$, so the scattering is accompanied by spin rotation.

In the case of linear polarisation the outgoing direction of motion can be any direction at right angles to the y-axis, and is determined by the precise initial colour orientations. The scattering shown in Fig. 4 has been selected to be from the y-direction to the z-direction. Again there is a 90 degree jump in the polarisation axis, so that after scattering the neutrons are still linearly polarised. There is however a flip in the sense of the polarisation. In our example, for both incoming neutrons the polarisation is outwards, but for the outgoing neutrons it is inwards. Helicity, $\mathbf{p} \cdot \mathbf{s}$, is therefore the same for incoming and outgoing particles, and this result is more general, holding in the diproton case too.

Although the scattering here is classical, the results suggest that quantised 2-neutron and 2-proton scattering depends strongly on polarisations. For Skyrmions transversely polarised in the z-direction, the outgoing motion breaks the axial symmetry around the y-axis, because the outgoing particle motion is concentrated in the z-direction. Also, the outgoing polarisations are completely determined. It would be interesting to compare these classical simulations with phys-
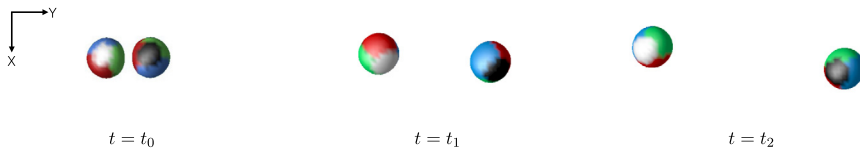
Fig. 5. The repulsion of a Skyrmion proton and a Skyrmion neutron with zero initial speed. The dynamics is shown online.

ical proton–proton scattering in which the polarisations of both incoming particles and both outgoing particles were measured.

4.3. Deuteron formation

The deuteron is a stable, two-baryon bound state comprised of a proton and a neutron. It has been understood as the quantised ground state of the $B = 2$ Skyrmion [15,5,17] with spin $\boldsymbol{J} = 1$ and isospin $\boldsymbol{I} = 0$.

Here we are interested in classically modelling deuteron formation, so we numerically collide two spinning Skyrmions, representing a proton and neutron with transverse polarisation parallel to the z-axis. The deuteron total angular momentum projected along the z-direction has three contributions, the individual spins J_p and J_n of the proton and neutron, which are $\pm\frac{1}{2}$, and the orbital angular momentum, an integer l, combining to give $J_3 = J_p + J_n + l$ [22]. Quantum models predict that the $l = 0$ state dominates, and the deuteron spin is accounted for by the proton and neutron spins being aligned. We can replicate the deuteron state by colliding two Skyrmions head-on, where one Skyrmion is spinning as a neutron with $J_n = \frac{1}{2}$ and the other Skyrmion has the opposite orientation of the white–black axis and is spinning as a proton with $J_p = \frac{1}{2}$. The impact parameter is zero, so $l = 0$.

The deuteron has slightly lower energy than a proton plus a neutron, so we cannot use a simple energetic argument to choose an initial collision speed. Whatever the initial speed, some energy needs to be dissipated in order to truly produce a deuteron. It is therefore a considerable surprise that our collisions do produce a configuration similar to what is expected for a deuteron in a classical Skyrmion model.

There is, however, a problem to be overcome. If we minimise the energy by choosing initial speed $v = 0$, and the initial relative orientations arbitrary, then averaged over time the Skyrmions repel. This is because the Skyrmions are spinning, and most of the time the colours do not match, producing a repulsion. Only briefly do the colours match, and the Skyrmions attract. Similar behaviour was observed for two spinning baby Skyrmions in 2-dimensions [23]. The repulsion of spinning Skyrmions initially at rest, which is not a Coulomb effect, is shown in Fig. 5.

Therefore, to model deuteron formation we need to collide the Skyrmions at a positive speed, and we choose $v = 0.2$. There is now enough kinetic energy to overcome the repulsive barrier. The dynamic $B = 2$ Skyrmion that forms is stable even with this extra energy. The collision along the y-axis of two Skyrmions spinning about the z-direction is shown in Fig. 6, from two viewpoints.

As expected [24], the $B = 2$ torus forms in the (y, z)-plane and spins about the z-axis, which is an axis orthogonal to the symmetry axis of the torus. Therefore there is no projection of spin or isospin along the symmetry axis, and the constraint $L_3 + 2K_3 = 0$ is satisfied.
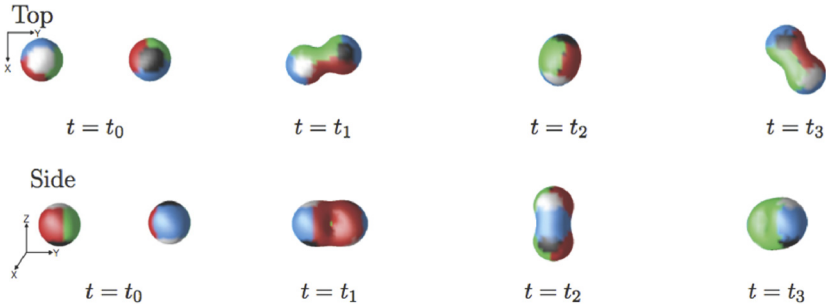
Fig. 6. Formation of a deuteron bound state in a spinning Skyrmion collision with $J = 1$, $I = 0$. This is shown dynamically online.
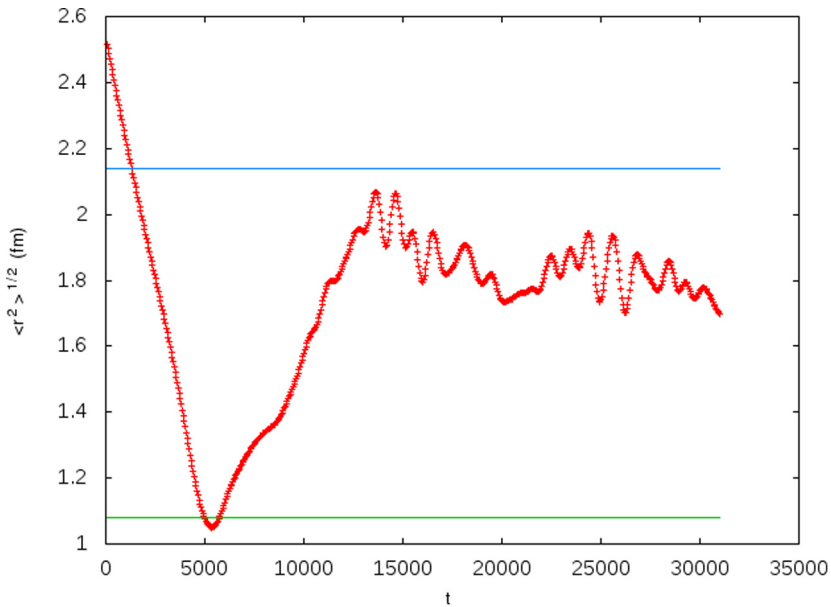

Fig. 7. Skyrmion matter radius in the classical simulation of a neutron–proton collision, as a function of time t. The upper line is the experimental deuteron matter radius and the lower line is the matter radius of the static $B = 2$ toroidal Skyrmion.

The important feature of this collision is that the $B = 1$ Skyrmions do not scatter, but merge into a long-lived spinning $B = 2$ Skyrmion, with superimposed oscillations. The colours are not independently spinning. When the oscillation is at its maximal amplitude, two $B = 1$ Skyrmions can be identified. They represent two nucleons with parallel spins. This classical bound state models the physical, quantised deuteron [15], with the correct spin, and zero isospin. The oscillation is similar to that of the $B = 2$ Skyrmion in its quantised ground state obtained using the instanton ansatz [5], but the amplitude of oscillation is somewhat greater, and the energy somewhat larger. The classical oscillation produces an oscillating matter radius, which is shown in Fig. 7 as a function of time. The time-averaged matter radius is comparable with the experimental value for the deuteron.
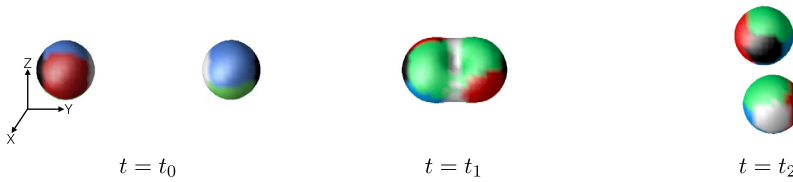
Fig. 8. Head-on collision of linearly polarised proton and neutron with $J = 1$, $I = 0$. Note: at $t = t_1$ the torus is spinning about its black–black axis. The dynamics is shown online.

We have also attempted to simulate deuteron formation by colliding Skyrmions spinning as a neutron and proton, both with linear polarisation. This is shown in Fig. 8. The process does not produce a classical bound state, even though the spin and isospin are correct for a deuteron. The initial state can be understood as Skyrmions approaching in the attractive channel with two white faces being closest. The Skyrmions scatter at right angles and the initial spin is converted to orbital angular momentum, the Skyrmions separating with some non-zero impact parameter. They also spin slowly about a coloured axis as they separate – this is inferred from a scattering of Skyrmions with larger initial spin. Skyrmions spinning like this should be interpreted as a superposition of a neutron and a proton.

Although the torus forms briefly in the linearly polarised case, it breaks apart. This is because Skyrmions in the attractive channel accelerate towards each other and the excess energy quickly breaks the torus up. In the transversely polarised case the excess energy is less and goes into the vibrational mode. This shows that polarisation has implications for proton–neutron fusion.

4.4. Excited dibaryon

Recently an excited proton–neutron state has been discovered [25], called the dibaryon. It can be viewed as an excited spin 3 state of the spin 1 deuteron, with mass $M_{\mathrm{Dib}} \approx 2380$ MeV and quantum numbers $J = 3$, $I = 0$.

To create the Skyrmion analogue requires colliding two Skyrmions each with kinetic energy $T = \frac{1}{2}(M_{\mathrm{Dib}} - 2M_{\mathrm{N}})$; this is achieved with initial, relativistic speeds of $\mathrm{v} = 0.6$. The $J = 3$, $I = 0$ state is replicated by colliding a Skyrmion spinning as a neutron and an oppositely orientated Skyrmion spinning as a proton, with the spins parallel and with orbital angular momentum $l = 2$, such that $J_3 = \frac{1}{2} + \frac{1}{2} + 2$. The orbital angular momentum is achieved with an impact parameter $b = 0.32$ at this speed. This impact parameter is the same order of magnitude as half the spatial width of the $B = 2$ toroidal Skyrmion. The scattering is shown in Fig. 9, with the initial motion parallel to the y-axis, and the proton and neutron polarisations parallel to the z-axis. The dibaryon appears briefly in the form of the $B = 2$ toroidal Skyrmion spinning with $J = 3$ about the z-axis. The spin axis is at right angles to the torus's symmetry axis, as for the deuteron.

There is a small amount of excess energy and as a result the $B = 2$ Skyrmion almost breaks up into two rapidly spinning $B = 1$ Skyrmions moving back-to-back along the z-axis. These have close to spin $\frac{3}{2}$ each, but rapidly lose energy, presumably by pion radiation, although this is hard to see in the simulation. The spinning Skyrmions stop and return to form a spinning $B = 2$ Skyrmion again. This replicates the experimental result that two pions are emitted, and a deuteron remains.
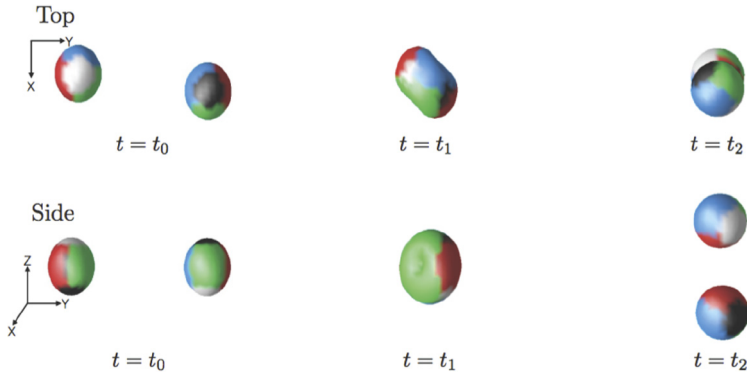
Fig. 9. Skyrmion scattering modelling proton–neutron scattering with $J = 3$, $I = 0$ at the energy of the dibaryon resonance. The orbital angular momentum is $l = 2$, requiring an impact parameter $b = 0.32$. Note: at $t = t_2$ the Skyrmions are spinning about the red–cyan axis. The dynamical scattering is shown online.
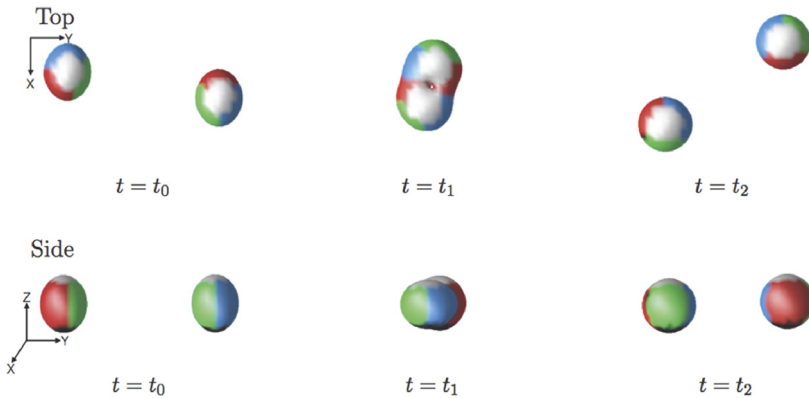

Fig. 10. Modelling dibaryon production in proton–neutron scattering, with orbital angular momentum $l = 3$, and impact parameter $b = 0.48$. This is shown dynamically online.

The transient production of spin $\frac{3}{2}$ particles matches other theoretical models of the dibaryon resonance involving two deltas with spin $\frac{3}{2}$ [26]. Our simulation shows that (in the proton–neutron centre-of-mass frame) the deltas are emitted transversely to the line of collision, but parallel to the proton and neutron polarisation axes. Each delta is linearly polarised, and because its spin axis is an axis perpendicular to the white–black axis, it is in a superposition of states with different charges. This superposition is not surprising, because the average charge emerging along the positive or negative z-axis is half the proton charge.

We have also attempted to produce a $J = 3$, $I = 0$ dibaryon state by colliding a proton Skyrmion and a neutron Skyrmion with the spin polarisations transverse but opposite, so that the spin angular momenta cancel. The orbital angular momentum is $l = 3$, which requires the same initial velocities, but an impact parameter $b = 0.48$. This scattering is shown in Fig. 10.

Here, the toroidal Skyrmion also appears but is not an excited deuteron, because it forms in the plane normal to the initial orbital angular momentum vector. In this case, one might anticipate the torus to be spinning about its symmetry axis, but it would then have non-zero isospin, which is not allowed. On closer inspection one observes that the $B = 2$ torus is not spinning, but each

$B = 1$ Skyrmion can be interpreted as a wave propagating along the outer edge, such that the colours do not rotate. This curious dynamical configuration is not a dibaryon.

In summary, dibaryon formation can be observed in classical Skyrmion scattering, but only if the initial polarisations of the proton and neutron are transverse and parallel.

5. Conclusion

In this work we have classically scattered spinning Skyrmions to model nucleon–nucleon scattering. The initial spin angular momenta are fixed to be $\frac{1}{2}\hbar$. Interestingly, we have found short-lived states modelling the dineutron/diproton and the dibaryon. We have also found that in a collision with the quantum numbers of the deuteron, the Skyrmions do not scatter, but merge into a spinning and oscillating form of the $B = 2$ toroidal Skyrmion. So, even without quantisation, the Skyrme model usefully captures important physical features of two-nucleon dynamics. This is gratifying but a little surprising, especially when it is recalled that our long-lived deuteron state was made from two Skyrmions colliding with a larger kinetic energy than that needed to produce the short-lived dineutron/diproton.

Our most valuable observations concern the polarisations. The classically spinning Skyrmions must be polarised along some axis, so we inevitably have complete polarisation information for the nucleons before and after scattering. Such complete information is hard to obtain experimentally, and also difficult to determine in more traditional nucleon–nucleon potential models. Classically we also fix the impact parameter. Several of our simulations lead to right angle scattering of the Skyrmions. Experimentally one cannot fix the impact parameter, but one can focus on scattering events where there is right angle scattering (in the centre-of-mass frame), and can hope that these are modelled by our collisions. For such events, our Skyrmion simulations make strong predictions for the polarisations. For example, in our simulations of a proton–neutron collision leading to the dibaryon, with total spin 3, we have seen the transient production of two linearly polarised deltas each with spin close to $\frac{3}{2}$.

We have observed that in a 2-proton or 2-neutron head-on collision, with total angular momentum zero, if the particles are transversely polarised initially then they are transversely polarised finally. If they are linearly polarised initially they are linearly polarised finally. In both cases the polarisation axes rotate through a right angle. Moreover, there is a fixed plane containing the initial and final momenta and also the polarisations. The sense of the polarisations is determined by conservation of $\mathbf{p} \times \mathbf{s}$ for each particle in the transverse polarisation case, and conservation of $\mathbf{p} \cdot \mathbf{s}$ in the linear polarisation case. It would be very interesting if something similar was observed experimentally. Such an advanced experiment would require the development of new experimental techniques, but it would help understand the structure of nuclei.

We intend to extend our simulations to scattering of polarised Skyrmions with a range of impact parameters, and from the results deduce a differential cross section. This cross section would be easier to compare with existing data. We also plan to extend our numerical simulations to model nucleon–deuteron and deuteron–deuteron collisions, possibly forming Helium-3, Tritium or Helium-4 bound states. The physical spins must be correctly modelled in the initial data, and the results will differ from those obtained with non-spinning Skyrmions.

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