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Existence results for dynamic inclusions on time scales with nonlocal initial conditions☆

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Abstract

This paper is mainly concerned with the existence of solutions for first order dynamic inclusions on time scales with nonlocal initial conditions. By using Bohnenblust–Karlin's fixed point theorem and Leray–Schauder nonlinear alternative for multivalued maps, some sufficient conditions are established. An example is also included to illustrate our results.

Keywords: Dynamic inclusions; Time scales; Nonlocal initial conditions; Fixed point; Nonlinear alternative

1. Introduction

Recently, much attention has been paid to existence results for dynamic inclusions on time scales, for example, Atici and Biles [1], Belarbi, Benchohra and Ouahab [2], Bohner and Tisdell [3]. In this paper, we are interested in the existence of solutions for the following first order dynamic inclusions on time scales with nonlocal initial conditions

$$y^{\Delta}(t) + p(t)\, y^{\sigma}(t) \in F(t, y(t)) \quad \text{a.e. } t \in [0, b], \tag{1.1}$$

$$y(0) + \sum_{k=1}^{m} c_k y(t_k) = y_0, \tag{1.2}$$

where $0, b \in \mathbb{T}$, $[0, b] = \{t \in \mathbb{T} : 0 \le t \le b\}$ and $\mathbb{T}$ is a time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$, p is regressive and right-dense continuous, $F : [0, b] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued function, $y_0 \in \mathbb{R}$, $t_1 < \cdots < t_m$, and $c_k \neq 0$, $t_k \in [0, b]$ for all $k = 1, 2, \ldots, m$. σ is a function that will be defined later and $y^{\sigma}(t) = y(\sigma(t))$.

Nonlocal Cauchy problems for ordinary differential equations (inclusions) have been studied by several authors, see, for example, Boucherif [4,5], Byszewski [6] and the references therein. As pointed out by Byszewski [6], the

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nonlocal conditions can be applied in physics with better effects than the classical initial conditions. We note that (1.1) and (1.2) have been studied by Boucherif [4] with $\mathbb{T} = \mathbb{R}, b = 1$ and $p(t) \equiv 0$. Since dynamic equations provide a unifying structure for the study of differential equations and finite difference equations, it is natural to consider existence results for dynamic inclusions with nonlocal initial conditions. Based upon Bohnenblust–Karlin's fixed point theorem [7] and Leray–Schauder nonlinear alternative for multivalued maps [8], we shall prove some existence results for the problem (1.1) and (1.2). Our results generalize those of [4], and an example is also given to illustrate the main results.

For other more recent results about dynamic equations on time scales, we refer to [9–25] and references therein.

2. Preliminaries

In this section, we shall recall some basic definitions and lemmas which are used throughout this paper.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined, respectively, by

$$\sigma(t) = \inf\{s > t : s \in \mathbb{T}\} \quad \text{and} \quad \rho(t) = \sup\{s < t : s \in \mathbb{T}\}.$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the "delta derivative" of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}$ of t such that

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$$

for all $s \in \mathcal{N}$.

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t.

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T},$$

where $\mu(t) = \sigma(t) - t$, which is called the graininess function.

If p is a regressive function, then generalized exponential function e_p is defined by

$$\mathrm{e}_p(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\}, \quad \text{for } s, t \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1 + hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q = p + q + \mu pq, \qquad \ominus p := -\frac{p}{1 + \mu p}, \qquad p \ominus q := p \oplus (\ominus p).$$

Then the generalized function e_p has the following properties.

Lemma 2.1 (*[17]*). *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then*

(i) $\mathrm{e}_0(t, s) \equiv 1$ *and* $\mathrm{e}_p(t, t) \equiv 1$;
(ii) $\mathrm{e}_p(\sigma(t), s) = (1 + \mu(t)p(t))\mathrm{e}_p(t, s)$;

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)\, e_p(s,r) = e_p(t,r)$;
(vi) $e_p(t,s)\, e_q(t,s) = e_{p\oplus q}(t,s)$;
(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

When we say "a.e." in this paper, the measure we consider is Δ-measure on $\mathbb{T}$. The construction of Δ-measure on $\mathbb{T}$ and the following lemmas can be found in [19].

Lemma 2.2. *For each $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, the single-point set t_0 is Δ-measurable, and its Δ-measure is given by*

$$\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0 = \mu(t_0).$$

Lemma 2.3. *If $a, b \in \mathbb{T}$ and $a \le b$, then*

$$\mu_\Delta([a,b)) = b - a \quad \text{and} \quad \mu_\Delta((a,b)) = b - \sigma(a).$$

If $a, b \in \mathbb{T} \setminus \{\max \mathbb{T}\}$ and $a \le b$, then

$$\mu_\Delta((a,b]) = \sigma(b) - \sigma(a) \quad \text{and} \quad \mu_\Delta([a,b]) = \sigma(b) - a.$$

The Lebesgue integrals associated with the measure μ_Δ on $\mathbb{T}$ is called the Lebesgue Δ-integral on $\mathbb{T}$. For a measurable set $E \subset \mathbb{T}$ and a function $f : E \to \mathbb{R}$ the corresponding integrals of f on E is denoted by

$$\int_E f(t)\Delta t.$$

For further results about Lebesgue Δ-integral on $\mathbb{T}$, we refer to a recent Ref. [20].

Let $C([0,\sigma(b)], \mathbb{R})$ be the Banach space of all continuous functions from $[0,\sigma(b)]$ into $\mathbb{R}$ with the norm

$$\|y\| = \sup\{|y(t)| : t \in [0,\sigma(b)]\}.$$

$L^1([0,\sigma(b)], \mathbb{R})$ denotes the space of functions from $[0,\sigma(b)]$ into $\mathbb{R}$ which are Lebesgue integrable in the time scales sense (see [1]) normed by

$$\|y\|_{L^1} = \int_0^{\sigma(b)} |y(t)|\Delta t \quad \text{for each } y \in L^1([0,\sigma(b)], \mathbb{R}).$$

$AC((0,\sigma(b)), \mathbb{R})$ is the space of all continuous functions on $(0,\sigma(b))$ such that they are a.e. Δ-differentiable on $(0,\sigma(b))$ with their first delta derivative y^Δ belonging to $L^1([0,\sigma(b)], \mathbb{R})$ (see [20]).

Let $(X, |\cdot|)$ denote a Banach space. Then a multivalued map $G : X \to \mathcal{P}(X)$ is convex (compact) valued if $G(x)$ is convex (compact) for all $x \in X$. G is bounded on bounded sets if $G(\Omega) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set Ω of X (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set Ω of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq \Omega$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_\star$, $y_n \to y_\star$, $y_n \in G(x_n)$ imply $y_\star \in G(x_\star)$).

Let $CC(X)$ be the set of all nonempty compact and convex subsets of X.

For each $y \in C([0,\sigma(b)], \mathbb{R})$, let $S_{F,y}$ be the set of selections of F defined by

$$S_{F,y} = \left\{ f \in L^1([0,\sigma(b)], \mathbb{R}) : f(t) \in F(t, y(t)) \text{ a.e. } t \in [0,\sigma(b)] \right\}.$$

Definition 2.1. A function $y \in AC([0,\sigma(b)], \mathbb{R})$ is said to be a solution of the problem (1.1) and (1.2), if there exists a function $f \in L^1([0,\sigma(b)], \mathbb{R})$ such that

$$f(t) \in F(t, y(t)), \qquad y^\Delta(t) + p(t)\, y^\sigma(t) = f(t) \quad \text{a.e. } t \in [0,\sigma(b)]$$

and the condition (1.2) holds.

The following lemmas are of great importance in the proof of our main results.

Lemma 2.4 (*Bohnenblust–Karlin [7]*). *Let X be a Banach space, D a nonempty subset of X, which is bounded, closed, and convex. Suppose $G : D \to 2^X \setminus \{0\}$ is u.s.c. with closed, convex values, and such that $G(D) \subset D$ and $\overline{G(D)}$ compact. Then G has a fixed point.*

Lemma 2.5 (*Leray–Schauder Nonlinear Alternative [8]*). *Let X be a Banach space with $C \subset X$ convex. Assume V is a relatively open subset of C with $0 \in V$ and $G : \overline{V} \to 2^C$ is a compact multivalued map, u.s.c. with convex closed values. Then either*

(I) *G has a fixed point in $\overline{V}$; or*
(II) *there exists a point $v \in \partial V$ such that $v \in \lambda G(v)$ for some $\lambda \in (0, 1)$.*

In [1], the authors extend the following lemma from the case $[a, d]$ any compact real interval [26] to the case $[a, d]$ a compact interval on time scales $\mathbb{T}$.

Lemma 2.6 (*[1]*). *Let X be a Banach space. Let $F : [a, d] \times X \to CC(X); (t, y) \mapsto F(t, y)$ measurable with respect to t for any $y \in X$ and u.s.c. with respect to y for a.e. $t \in [a, d]$ and $S_{F,y} \neq \emptyset$ for any $y \in C([a, d], X)$ and let Γ be a linear continuous mapping from $L^1([a, d], X)$ to $C([a, d], X)$, then the operator*

$$\Gamma \circ S_F : C([a, d], X) \to CC(C([a, d], X))$$
$$y \mapsto (\Gamma \circ S_F)(y) := \Gamma\left(S_{F,y}\right)$$

is a closed graph operator in $C([a, d], X) \times C([a, d], X)$ (i.e. the graph of $\Gamma \circ S_F$ is a closed subset of $C([a, d], X) \times C([a, d], X)$).

3. Existence results

In this section, we shall present and prove our main results.

We first consider the following 'linear' problem

$$y^{\Delta}(t) + p(t) y^{\sigma}(t) = f(t), \quad y(t_0) = \eta. \tag{3.1}$$

For this 'linear' problem (3.1), we have the following lemma.

Lemma 3.1 (*[18]*). *Let $p : \mathbb{T} \to \mathbb{R}$ be right-dense continuous and regressive. Suppose $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous, $t_0 \in \mathbb{T}$ and $\eta \in \mathbb{R}$. Then y is the unique solution of the initial value problem (3.1) if and only if*

$$y(t) = \mathrm{e}_{\ominus p}(t, t_0)\eta + \int_{t_0}^{t} \mathrm{e}_{\ominus p}(t, s) f(s) \Delta s.$$

Let us list the following hypothesis:

(H1) $c_k \neq 0$ for each $k = 1, 2, \ldots, m$ and $1 + \sum_{k=1}^{m} c_k \mathrm{e}_{\ominus p}(t_k, 0) \neq 0$. Let

$$c = \left(1 + \sum_{k=1}^{m} c_k \mathrm{e}_{\ominus p}(t_k, 0)\right)^{-1}, \qquad \gamma = \sup_{t \in [0, \sigma(b)]} \mathrm{e}_{\ominus p}(t, 0)|c||y_0|$$

and

$$\delta = \left(1 + \sup_{t \in [0, \sigma(b)]} \mathrm{e}_{\ominus p}(t, 0)|c| \sum_{k=1}^{m} |c_k|\right).$$

(H2) $t \mapsto F(t, y)$ is measurable for each $y \in \mathbb{R}$.
(H3) $y \mapsto F(t, y)$ is u.s.c. for a.e. $t \in [0, \sigma(b)]$.
(H4) $F : [0, \sigma(b)] \times \mathbb{R} \to CC(\mathbb{R})$.

(H5) For each $r > 0$, there exists a function $\varphi_r \in L^1([0, \sigma(b)], \mathbb{R}_+)$ such that

$$\|F(t, y)\| = \sup\{|f| : f \in F(t, y)\} \le \varphi_r(t)$$

for each $(t, y) \in [0, \sigma(b)] \times \mathbb{R}$ with $|y| \le r$, and

$$\liminf_{r \to +\infty} \frac{1}{r} \int_0^{\sigma(b)} \varphi_r(t) \Delta t = \beta.$$

(H6) There exist a continuous nondecreasing function $\phi : [0, \infty) \to (0, \infty)$, a function $q \in L^1([0, \sigma(b)], \mathbb{R}_+)$ and a positive constant M such that

$$\|F(t, y)\| \le q(t)\phi(|y|)$$

for each $(t, y) \in [0, \sigma(b)] \times \mathbb{R}$, and

$$\frac{M}{\gamma + \delta \sup_{(t,s) \in [0,\sigma(b)] \times [0,\sigma(b)]} e_{\ominus p}(t, s)\phi(M) \int_0^{\sigma(b)} q(s)\Delta s} > 1.$$

Remark 3.1. By [1,27], it follows that for each $y \in C([0, \sigma(b)], \mathbb{R})$, the set $S_{F,y}$ is nonempty.

Theorem 3.1. *Assume that* (H1)–(H5) *are satisfied. Then the problem* (1.1) *and* (1.2) *has at least one solution on* $[0, \sigma(b)]$, *provided that*

$$\sup_{(t,s) \in [0,\sigma(b)] \times [0,\sigma(b)]} e_{\ominus p}(t, s)\beta\delta < 1. \tag{3.2}$$

Proof. We transform the problem (1.1) and (1.2) into a fixed point problem. Consider the operator $N : C([0, \sigma(b)], \mathbb{R}) \to 2^{C([0,\sigma(b)],\mathbb{R})}$ defined by

$$N(y) = \left\{ h \in C([0, \sigma(b)], \mathbb{R}) : h(t) = e_{\ominus p}(t, 0)c\left(y_0 - \sum_{k=1}^{m} c_k \int_0^{t_k} e_{\ominus p}(t_k, s) f(s)\Delta s\right) \right.$$
$$\left. + \int_0^t e_{\ominus p}(t, s) f(s)\Delta s,\ f \in S_{F,y} \right\}. \tag{3.3}$$

Clearly, the fixed points of N are solutions of the problem (1.1) and (1.2). We shall show that N satisfies all the assumptions of Lemma 2.4. For the sake of convenience, we break the proof into four steps.

Step 1. $N(y)$ is convex for each $y \in C([0, \sigma(b)], \mathbb{R})$.

In fact, if $h_1, h_2 \in N(y)$, then there exist $f_1, f_2 \in S_{F,y}$ such that for each $t \in [0, \sigma(b)]$ we have

$$h_i(t) = e_{\ominus p}(t, 0)c\left(y_0 - \sum_{k=1}^{m} c_k \int_0^{t_k} e_{\ominus p}(t_k, s) f_i(s)\Delta s\right) + \int_0^t e_{\ominus p}(t, s) f_i(s)\Delta s, \quad i = 1, 2.$$

Let $0 \le \epsilon \le 1$. Then for each $t \in [0, \sigma(b)]$ we have

$$(\epsilon h_1 + (1 - \epsilon)h_2)(t) = e_{\ominus p}(t, 0)c\left(y_0 - \sum_{k=1}^{m} c_k \int_0^{t_k} e_{\ominus p}(t_k, s)[\epsilon f_1 + (1 - \epsilon) f_2](s)\Delta s\right)$$
$$+ \int_0^t e_{\ominus p}(t, s)[\epsilon f_1 + (1 - \epsilon) f_2](s)\Delta s.$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$\epsilon h_1 + (1 - \epsilon)h_2 \in N(y).$$

Step 2. For each constant $r > 0$, let $B_r = \{y \in C([0, \sigma(b)], \mathbb{R}) : \|y\| \le r\}$. Then B_r is a bounded closed convex set in $C([0, \sigma(b)], \mathbb{R})$. We claim that there exists a positive number r such that for each $y \in B_r$, $N(y) \subseteq B_r$. If it is not true, then for each positive number r, there exists a function $y_r \in B_r$ such that $h_r \in N(y_r)$ but $\|N(y_r)\| > r$ and

$$h_r(t) = e_{\ominus p}(t,0)\, c\left(y_0 - \sum_{k=1}^{m} c_k \int_0^{t_k} e_{\ominus p}(t_k,s)\, f_r(s)\,\Delta s\right) + \int_0^t e_{\ominus p}(t,s)\, f_r(s)\,\Delta s$$

for some $f_r \in S_{F,y_r}$. However, on the other hand, we have from (H5)

$$\begin{aligned}
r &< \|N(y_r)\| \\
&\le \sup_{t\in[0,\sigma(b)]} e_{\ominus p}(t,0)\,|c|\left(|y_0| + \sum_{k=1}^{m}|c_k| \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s)\int_0^{t_k}|f_r(s)|\Delta s\right) \\
&\quad + \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s)\int_0^t |f_r(s)|\Delta s \\
&\le \sup_{t\in[0,\sigma(b)]} e_{\ominus p}(t,0)\,|c|\left(|y_0| + \sum_{k=1}^{m}|c_k| \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s)\int_0^{t_k}\varphi_r(s)\,\Delta s\right) \\
&\quad + \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s)\int_0^{\sigma(b)} \varphi_r(s)\,\Delta s \\
&\le \left(1 + \sup_{t\in[0,\sigma(b)]} e_{\ominus p}(t,0)\,|c|\sum_{k=1}^{m}|c_k|\right) \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s)\int_0^{\sigma(b)}\varphi_r(s)\,\Delta s \\
&\quad + \sup_{t\in[0,\sigma(b)]} e_{\ominus p}(t,0)\,|c||y_0| \\
&= \delta \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s)\int_0^{\sigma(b)}\varphi_r(s)\,\Delta s + \gamma.
\end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \to \infty$, we conclude that

$$\sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s)\,\beta\delta \ge 1,$$

which contradicts (3.2). Hence there exists a positive number r' such that for each $y \in B_{r'}$, $N(y) \subseteq B_{r'}$.

Step 3. $N(B_{r'})$ is equicontinuous.

Let $t', t'' \in [0,\sigma(b)]$, $t' < t''$ and $y \in B_{r'}$. For each $h \in N(y)$ we have

$$\begin{aligned}
&|h(t'') - h(t')| \\
&\quad\le |e_{\ominus p}(t'',0) - e_{\ominus p}(t',0)|\,|c|\left(|y_0| + \sum_{k=1}^{m}|c_k| \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s)\int_0^{t_k}\varphi_{r'}(s)\,\Delta s\right) \\
&\qquad + \int_0^{t'} |e_{\ominus p}(t'',s) - e_{\ominus p}(t',s)|\,\varphi_{r'}(s)\,\Delta s + \int_{t'}^{t''} e_{\ominus p}(t'',s)\,\varphi_{r'}(s)\,\Delta s.
\end{aligned}$$

The right hand side of the above inequality tends to zero independently of $y \in B_{r'}$ as $t'' \to t'$.

As a consequence of Step 1 to Step 3 together with the Ascoli–Arzela theorem, we can conclude that N is a compact valued map.

Step 4. N has closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$ as $n \to \infty$. We need to show that $h_* \in N(y_*)$. The relation $h_n \in N(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that for each $t \in [0,\sigma(b)]$,

$$h_n(t) = e_{\ominus p}(t,0)\, c\left(y_0 - \sum_{k=1}^{m} c_k \int_0^{t_k} e_{\ominus p}(t_k,s)\, f_n(s)\,\Delta s\right) + \int_0^t e_{\ominus p}(t,s)\, f_n(s)\,\Delta s.$$

We must show that there exists $f_* \in S_{F,y_*}$ such that for each $t \in [0,\sigma(b)]$,

$$h_*(t) = e_{\ominus p}(t,0)\, c\left(y_0 - \sum_{k=1}^{m} c_k \int_0^{t_k} e_{\ominus p}(t_k,s)\, f_*(s)\,\Delta s\right) + \int_0^t e_{\ominus p}(t,s)\, f_*(s)\,\Delta s.$$

Consider the continuous linear operator

$$\Gamma : L^1([0,\sigma(b)],\mathbb{R}) \to C([0,\sigma(b)],\mathbb{R}),$$
$$f \mapsto \Gamma(f)(t) = \int_0^t e_{\ominus p}(t,s) f(s)\Delta s - e_{\ominus p}(t,0)\, c \sum_{k=1}^m c_k \int_0^{t_k} e_{\ominus p}(t_k,s) f(s)\Delta s.$$

Clearly,

$$\| \left(h_n(t) - e_{\ominus p}(t,0) y_0\right) - \left(h_*(t) - e_{\ominus p}(t,0) y_0\right) \| \to 0, \quad \text{as } n \to \infty.$$

From Lemma 2.6 it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t) - e_{\ominus p}(t,0) y_0 \in \Gamma\left(S_{F,y_n}\right).$$

Since $y_n \to y_*$ as $n \to \infty$. Lemma 2.6 implies that

$$h_*(t) - e_{\ominus p}(t,0) y_0 = \int_0^t e_{\ominus p}(t,s) f_*(s)\Delta s - e_{\ominus p}(t,0)\, c \sum_{k=1}^m c_k \int_0^{t_k} e_{\ominus p}(t_k,s) f_*(s)\Delta s$$

for some $f_* \in S_{F,y_*}$.

Therefore, N is a compact multivalued map, u.s.c. with convex closed values. As a consequence of Lemma 2.4, we deduce that N has a fixed point y which is a solution of the problem (1.1) and (1.2). □

As an immediate result of Theorem 3.1, we can obtain the following corollary.

Corollary 3.1. *Suppose that* (H1)–(H4) *and the following condition hold:*

(H5′) *There exist* $a(t), d(t) \in L^1([0,\sigma(b)],\mathbb{R}_+)$, $\theta \in [0,1)$ *such that*

$$\|F(t,y)\| \le a(t) + d(t)|y|^\theta \quad \textit{for each } (t,y) \in [0,\sigma(b)] \times \mathbb{R}.$$

Then the problem (1.1) *and* (1.2) *has at least one solution on* $[0,\sigma(b)]$.

Theorem 3.2. *Assume that* (H1)–(H4) *and* (H6) *are satisfied. Then the problem* (1.1) *and* (1.2) *admits at least one solution on* $[0,\sigma(b)]$.

Proof. Define the operator $N : C([0,\sigma(b)],\mathbb{R}) \to 2^{C([0,\sigma(b)],\mathbb{R})}$ as (3.3). In order to apply Lemma 2.5, we first give a priori bound.

Let y be such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then, there exists a function $f \in S_{F,y}$ such that for each $t \in [0,\sigma(b)]$, we have

$$y(t) = \lambda e_{\ominus p}(t,0)\, c \left(y_0 - \sum_{k=1}^m c_k \int_0^{t_k} e_{\ominus p}(t_k,s) f(s)\Delta s \right) + \lambda \int_0^t e_{\ominus p}(t,s) f(s)\Delta s.$$

This implies by (H6) that, for each $t \in [0,\sigma(b)]$,

$$\begin{aligned}
|y(t)| &\le \sup_{t\in[0,\sigma(b)]} e_{\ominus p}(t,0)|c| \left(|y_0| + \sum_{k=1}^m |c_k| \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s) \int_0^{t_k} |f(s)|\Delta s \right) \\
&\quad + \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s) \int_0^t |f(s)|\Delta s \\
&\le \left(1 + \sup_{t\in[0,\sigma(b)]} e_{\ominus p}(t,0)|c| \sum_{k=1}^m |c_k| \right) \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s) \int_0^{\sigma(b)} |f(s)|\Delta s \\
&\quad + \sup_{t\in[0,\sigma(b)]} e_{\ominus p}(t,0)|c||y_0| \\
&\le \gamma + \delta \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} e_{\ominus p}(t,s)\, \phi(\|y\|) \int_0^{\sigma(b)} q(s)\Delta s.
\end{aligned}$$

Therefore,

$$\frac{\|y\|}{\gamma+\delta \sup_{(t,s)\in[0,\sigma(b)]\times[0,\sigma(b)]} \mathrm{e}_{\ominus p}(t,s)\,\phi(\|y\|)\int_0^{\sigma(b)} q(s)\,\Delta s} \leq 1.$$

Then by (H6), there exists M such that $\|y\| \neq M$. Define

$$V = \{y \in C([0,\sigma(b)],\mathbb{R}) : \|y\| < M\}.$$

Just as in the proof of Theorem 3.1, we can show that the operator $N : \overline{V} \to 2^{C([0,\sigma(b)],\mathbb{R})}$ is a compact multivalued map, u.s.c. with convex closed values. From the choice of V, there is no $y \in \partial V$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of Lemma 2.5, we deduce that N has a fixed point y which is a solution of the problem (1.1) and (1.2). □

Remark 3.2. Theorem 3.2 is even new when used to differential inclusions which has been studied in [4] with $\mathbb{T} = \mathbb{R}$, $b = 1$ and $p(t) \equiv 0$.

4. An example

Suppose $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$ and p is a regressive function, where $\mathbb{N}_0$ is the set of nonnegative integers and $0, b \in \mathbb{T}$. We consider the following dynamic inclusions

$$y^{\Delta}(t) + p(t)\,y^{\sigma}(t) \in F(t, y(t)), \quad t \in [0,b], \tag{4.1}$$

$$y(0) + \sum_{k=1}^{m} c_k y(t_k) = y_0, \tag{4.2}$$

where $F : [0,\sigma(b)] \times \mathbb{R} \to 2^{\mathbb{R}}/\{\emptyset\}$ is a multivalued map defined by

$$(t,x) \to F(t,x) := \left[\frac{x^2}{x^2+2} + t, \frac{x^2}{x^2+1} + t + 1\right].$$

It is clear that F satisfies (H2)–(H4). Let $f \in \left[\frac{x^2}{x^2+2} + t, \frac{x^2}{x^2+1} + t + 1\right]$, then we have

$$|f| \leq \max\left(\frac{x^2}{x^2+2} + t, \frac{x^2}{x^2+1} + t + 1\right) \leq 2 + \sigma(b)$$

for each $(t,x) \in [0,\sigma(b)] \times \mathbb{R}$. Therefore, $\|F(t,x)\| \leq 2 + \sigma(b)$. Assume that (H1) holds. From Theorem 3.1, we conclude that the problem (4.1) and (4.2) have at least one solution on $[0,\sigma(b)]$. But this result cannot be deduced from those that are discussed in [4,5].

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