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Communication

On trees with a maximum proper partial 0–1 coloring containing a maximum matching

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Abstract

I prove that in a tree in which the distance between any two endpoints is even, there is a maximum proper partial 0–1 coloring such that the edges colored by 0 form a maximum matching.

Keywords: Partial proper 0–1 coloring; Maximum partial proper 0–1 coloring; Maximum matching

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex x in G is denoted by $d_G(x)$. If $X \subseteq E(G)$ then a mapping $f : X \to \{0, 1\}$ is referred as a partial 0–1 coloring of the graph G. For $i = 0, 1$ and the partial 0–1 coloring f of the graph G, denote $f_i \equiv \{e \in X/f(e) = i\}$. The partial 0–1 coloring f is proper if the sets f_0 and f_1 are matchings of the graph G. Denote

$$\lambda(G) \equiv \max\{|f_0| + |f_1|/f \text{ is a proper partial 0– 1 coloring of the graph } G\}.$$

A proper partial 0–1 coloring f of the graph G is maximum if $|f_0| + |f_1| = \lambda(G)$. Set

$$\alpha(G) \equiv \max\{|f_i|/i = 0, 1 \text{ and } f \text{ is a maximum proper partial (shortly, MPP) 0– 1 coloring of the graph } G\}.$$

It is clear, that for every graph G $\alpha(G) \leqslant \beta(G)$, where $\beta(G)$ is the cardinality of a maximum matching of the graph G. In this paper I show that if G is a tree in which the distance between any two endpoints is even, the equality $\alpha(G) = \beta(G)$ holds. Nondefined terms and conceptions can be found in [1,2].

Lemma 1. *Let G be a graph, $u \in V(G)$, $w \in V(G)$, $(u, w) \in E(G)$, $d_G(u) = 1$. Then there is a MPP 0–1 coloring f of the graph G, such that $|f_0| = \alpha(G)$ and $(u, w) \in f_0$.*

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Proof. Let f be a MPP 0–1 coloring of the graph G with $|f_0| = \alpha(G)$. Suppose $(u, w) \notin f_0$.

Case 1: $(u, w) \notin f_1$. As f is a MPP 0–1 coloring of the graph G, there is a $(w, w') \in E(G)$, such that $(w, w') \in f_0$. Consider the mapping $g : f_1 \cup (f_0 \backslash \{(w, w')\}) \cup \{(u, w)\} \to \{0, 1\}$ defined in the following way:

$$g(e) = \begin{cases} 0 & \text{if } e \in (f_0 \backslash \{(w, w')\}) \cup \{(u, w)\}, \\ 1 & \text{if } e \in f_1. \end{cases}$$

It is clear that g is a MPP 0–1 coloring of the graph G, $(u, w) \in g_0$ and $|g_0| = |f_0| = \alpha(G)$.

Case 2: $(u, w) \in f_1$. As f is a MPP 0–1 coloring of the graph G, with $|f_0| = \alpha(G)$, then there is a $(w, w_1) \in f_0$. Consider the maximal alternating path $u, (u, w), w, (w, w_1), w_1, \ldots, w_{k-1}, (w_{k-1}, w_k), w_k$, where k is odd, $\{(u, w), (w_1, w_2), \ldots, (w_{k-2}, w_{k-1})\} \subseteq f_1$ and $\{(w, w_1), (w_2, w_3), \ldots, (w_{k-1}, w_k)\} \subseteq f_0$. Define a mapping $g : f_0 \cup f_1 \to \{0, 1\}$ as follows:

$$g(e) = \begin{cases} f(e) & \text{if } e \notin \{(u, w), (w, w_1), \ldots, (w_{k-1}, w_k)\}, \\ 1 - f(e) & \text{if } e \in \{(u, w), (w, w_1), \ldots, (w_{k-1}, w_k)\}. \end{cases}$$

Clearly, g is a MPP 0–1 coloring of the graph G with $(u, w) \in g_0$ and $|g_0| = |f_0| = \alpha(G)$. The proof is complete. □

Lemma 2. *Let G be a graph, $u \in V(G)$, $v \in V(G)$, $w \in V(G)$, $d_G(u) = d_G(v) = 1$, $(u, w) \in E(G)$, $(v, w) \in E(G)$. Then*

(a) *there is a MPP 0–1 coloring f of the graph G, such that $|f_0| = \alpha(G)$, $(u, w) \in f_0$ and $(v, w) \in f_1$;*
(b) $\lambda(G) = 2 + \lambda(G \backslash \{u, v, w\})$, $\alpha(G) = 1 + \alpha(G \backslash \{u, v, w\})$.

Proof. (a) By Lemma 1, there is a MPP 0–1 coloring f of the graph G, such that $|f_0| = \alpha(G)$ and $(u, w) \in f_0$. Suppose $(v, w) \notin f_1$, then there is a $(w, w') \in E(G)$, such that $(w, w') \in f_1$. Consider a mapping $g : f_0 \cup (f_1 \backslash \{(w, w')\}) \cup \{(v, w)\} \to \{0, 1\}$ defined in the following way:

$$g(e) = \begin{cases} 1 & \text{if } e \in (f_1 \backslash \{(w, w')\}) \cup \{(v, w)\}, \\ 0 & \text{if } e \in f_0. \end{cases}$$

Clearly, g is a MPP 0–1 coloring of the graph G with $(u, w) \in g_0$, $(v, w) \in g_1$ and $|g_0| = \alpha(G)$.

(b) Let $w_1, \ldots, w_r$ be vertices of the graph G such that $d_G(w) = r + 2$ $(r \geqslant 0)$, $u \notin \{w_1, \ldots, w_r\}$, $v \notin \{w_1, \ldots, w_r\}$, $(w, w_i) \in E(G)$ for $i = 1, \ldots, r$, and f be a MPP 0–1 coloring f of the graph G, such that $|f_0| = \alpha(G)$, $(u, w) \in f_0$, $(v, w) \in f_1$. As $(w, w_i) \notin f_0 \cup f_1$ for $i = 1, \ldots, r$, we have

$$\lambda(G) = \lambda(G \backslash \{(w, w_1), \ldots, (w, w_r)\}) = 2 + \lambda(G \backslash \{u, v, w\}),$$

$$\alpha(G) = \alpha(G \backslash \{(w, w_1), \ldots, (w, w_r)\}) = 1 + \alpha(G \backslash \{u, v, w\}).$$

The proof is complete. □

Corollary. *Let G be a graph, $U = \{u_0, u_1, u_2, u_3, u_4\}$ be a subset of the set of vertices of G satisfying the conditions: $d_G(u_0) = d_G(u_4) = 1$, $d_G(u_1) = d_G(u_3) = 2$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3, 4$. Then the following is true:*

$$\lambda(G) = \lambda(G \backslash U) + 4, \quad \alpha(G) \geqslant 2 + \alpha(G \backslash U).$$

Proof. Lemma 2 implies $\lambda(G) = 2 + \lambda(G \backslash \{u_0, u_4\}) = \lambda(G \backslash U) + 4$, therefore $\alpha(G) \geqslant 2 + \alpha(G \backslash U)$. □

Theorem. *Let G be a tree in which the distance between any two endpoints is even. Then the equality $\alpha(G) = \beta(G)$ holds.*

Proof. Clearly, the statement of the theorem is true for the case $|E(G)| \leqslant 6$. Assume that it holds for trees with $|E(G)| \leqslant t - 1$, and let us prove that it will hold for the case $|E(G)| = t$, where $t \geqslant 7$.

Case 1: There is $U = \{u_0, u_1, u_2, u_3\} \subseteq V(G)$, such that $d_G(u_0) = 1$, $d_G(u_1) = d_G(u_2) = 2$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3$. Set $G' = G \backslash \{u_0, u_1\}$. Clearly, $\beta(G) = \beta(G') + 1$. As $d_G(u_0) = 1$, $d_G(u_1) = 2$ and $d_{G\backslash\{u_0\}}(u_1) = 1$, $d_{G\backslash\{u_0\}}(u_2) = 2$, we have $\lambda(G) = 1 + \lambda(G\backslash\{u_0\}) = \lambda(G') + 2$, thus if g is a MPP 0–1 coloring of tree G', such that $|g_0| = \alpha(G')$ and $(u_2, u_3) \in g_0$, then the mapping $f : g_0 \cup g_1 \cup \{(u_0, u_1), (u_1, u_2)\} \to \{0, 1\}$ defined as

$$f(e) = \begin{cases} g(e) & \text{if } e \notin \{(u_0, u_1), (u_1, u_2)\}, \\ 1 & \text{if } e = (u_1, u_2), \\ 0 & \text{if } e = (u_0, u_1) \end{cases}$$

is a MPP 0–1 coloring of the tree G, therefore $\alpha(G) \geqslant |f_0| = 1 + |g_0| = 1 + \alpha(G')$. As the distance between any two endpoints of G' is even and $|E(G')| < t$, we have $\alpha(G') = \beta(G')$, therefore

$$\alpha(G) \geqslant 1 + \alpha(G') = 1 + \beta(G') = \beta(G) \quad \text{or} \quad \alpha(G) = \beta(G).$$

Case 2: There is $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_4) = d_G(u_6) = 1$, $d_G(u_1) = d_G(u_3) = d_G(u_5) = 2$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3, 4$, $(u_2, u_5) \in E(G)$, $(u_5, u_6) \in E(G)$. Set $G' = G\backslash\{u_5, u_6\}$. Clearly, $\beta(G) = \beta(G') + 1$. From Corollary follows that $\lambda(G) = \lambda(G\backslash\{u_0, u_1, u_2, u_3, u_4\}) + 4$, therefore $\lambda(G) = \lambda(G\backslash\{(u_2, u_5)\}) = \lambda(G') + 1$ and $\alpha(G) \geqslant 1 + \alpha(G')$. Note that the distance between any two endpoints of the tree G' is even and $|E(G')| < t$, thus the equality $\alpha(G') = \beta(G')$ holds, and therefore

$$\alpha(G) \geqslant 1 + \alpha(G') = 1 + \beta(G') = \beta(G) \quad \text{or} \quad \alpha(G) = \beta(G).$$

Case 3: There is $U = \{u_0, u_1, u_2\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_2) = 1$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2$. Let $D_1, \ldots, D_r$ be the connected components of $G\backslash U$. Clearly, $\beta(G) = 1 + \sum_{i=1}^{r} \beta(D_i)$. Note that for $i = 1, \ldots, r$ D_i is a tree for which $|E(D_i)| < t$ and the distance between any two endpoints is even, thus $\alpha(D_i) = \beta(D_i)$, therefore, by Lemma 2, we have

$$\alpha(G) = 1 + \alpha(G\backslash U) = 1 + \sum_{i=1}^{r} \alpha(D_i) = 1 + \sum_{i=1}^{r} \beta(D_i) = \beta(G).$$

Case 4: There is $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_6) = 1$, $d_G(u_1) = d_G(u_3) = d_G(u_5) = 2$, $d_G(u_2) = 3$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3, 4, 6$, $(u_2, u_5) \in E(G)$. Set $G' = G\backslash\{u_0, u_1\}$. Clearly, $\beta(G) = \beta(G') + 1$. As $|E(G')| < t$ and the distance between any two endpoints of the tree G' is even, the equality $\alpha(G') = \beta(G')$ holds.

Lemma 1 implies that there is a MPP 0–1 coloring g of the tree $G\backslash\{u_0, u_1, u_2, u_5, u_6\}$ such that $(u_3, u_4) \in g_0$. Consider the mapping $f : g_0 \cup g_1 \cup \{(u_0, u_1), (u_2, u_3), (u_2, u_5), (u_5, u_6)\} \to \{0, 1\}$ defined as follows:

$$f(e) = \begin{cases} g(e) & \text{if } e \notin \{(u_0, u_1), (u_2, u_3), (u_2, u_5), (u_5, u_6)\}, \\ 0 & \text{if } e \in \{(u_0, u_1), (u_2, u_5)\}, \\ 1 & \text{if } e \in \{(u_2, u_3), (u_5, u_6)\}. \end{cases}$$

Corollary implies that f is a MPP 0–1 coloring of the tree G, therefore $\lambda(G) = \lambda(G\backslash\{(u_1, u_2)\}) = \lambda(G') + 1$ and $\alpha(G) \geqslant 1 + \alpha(G') = \beta(G') + 1 = \beta(G)$ or $\alpha(G) = \beta(G)$.

Case 5: There is $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_4) = d_G(u_6) = 1$, $d_G(u_1) = d_G(u_3) = 2$, $d_G(u_2) = d_G(u_5) = 3$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3, 4, 6$, $(u_2, u_5) \in E(G)$, $(u_5, u_7) \in E(G)$. Set $G' = G\backslash\{u_0, u_1, u_2, u_3, u_4\}$. Note that $\beta(G) = \beta(G') + 2$. As $|E(G')| < t$ and the distance between any two endpoints of the tree G' is even, the equality $\alpha(G') = \beta(G')$ holds. From Corollary we have

$$\alpha(G) \geqslant 2 + \alpha(G') = 2 + \beta(G') = \beta(G) \quad \text{or} \quad \alpha(G) = \beta(G).$$

Case 6: There is $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_4) = d_G(u_5) = d_G(u_9) = 1$, $d_G(u_1) = d_G(u_3) = d_G(u_6) = d_G(u_8) = 2$, $d_G(u_2) = d_G(u_7) = 3$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3, 4, 6, 7, 8, 9$, $(u_2, u_{10}) \in E(G)$, $(u_7, u_{10}) \in E(G)$. Set $G' = G\backslash\{u_0, u_1, u_2, u_3, u_4\}$. Clearly $\beta(G) = \beta(G') + 2$. As $|E(G')| < t$ and the distance between any two endpoints of the tree G' is even, the equality $\alpha(G') = \beta(G')$ holds, therefore from Corollary we have

$$\alpha(G) \geqslant 2 + \alpha(G') = 2 + \beta(G') = \beta(G) \quad \text{or} \quad \alpha(G) = \beta(G).$$

As every tree G, in which the distance between any two endpoints is even, and $|E(G)| \geqslant 7$, satisfies at least one of the conditions of the six cases considered above, the proof of the theorem is complete. □

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