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Partial differential equations on time scales

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Abstract

Discrete and continuous formulations of partial differential operators are unified by a time scale formulation of partial differential operators. Results include an Euler–Lagrange equation for double integral variational problems on time scales and a Picone identity which implies a Sturm–Picone comparison theorem for second-order elliptic PDEs on time scales.

MSC: 35J

Keywords: Time scales; Measure chains; Delta derivatives; Elliptic partial differential equations; Euler–Lagrange equation; Picone identity; Sturm–Picone comparison theorem

1. Introduction

In order to unify results from the calculus of real numbers with results from the difference calculus, Hilger and Aulbach [8,13] generalized the definition of a derivative and of an integral to time scales in order to create the time scale calculus. A book on the subject of time scales, i.e., *measure chains*, by Kaymakçalan et al. [17] summarizes and organizes much of the time scale calculus. Other papers on time scales include joint and individual papers of Agarwal, Ahlbrandt, Bohner, Došlý, Erbe, Hilscher, Peterson, and Ridenhour [1–4,10,11,13–16]. A *time scale* $\mathbb{T}$ is defined to be any nonempty closed subset of the real numbers.

The above references provide motivation and formulation of delta derivatives on a time scale, properties of delta derivatives and integrals, and terminology. A function f on $\mathbb{T}$ is called *rd-continuous* ("right-dense continuous") on $\mathbb{T}$ if it is continuous at each right-dense point and maximal point of $\mathbb{T}$ and if its left-sided limit exists at left-dense points. Any continuous function is also rd-continuous. By

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C_{rd} we denote the set of all rd-continuous functions, while C^1_{rd} denotes the set of all Δ differentiable functions with rd-continuous derivatives.

2. Partial derivatives on several time scales

We introduce definitions of time scale derivatives and time scale integrals for functions of two variables as follows. Let $\mathbb{X}$ and $\mathbb{Y}$ be time scales. The *forward jump operators* $\sigma:\mathbb{X}\to\mathbb{X}$ and $\tau:\mathbb{Y}\to\mathbb{Y}$ are defined by $\sigma(x):=\inf\{s\in\mathbb{X}: s>x\}$ and $\tau(y):=\inf\{t\in\mathbb{Y}: t>y\}$ supplemented by $\sigma(x)=x$ if x is a maximal point of $\mathbb{X}$ and $\tau(y)=y$ if y is a maximal point of $\mathbb{Y}$. The *stepsize* (or "graininess" [8,13,17]) $h:\mathbb{X}\to\mathbb{R}$ is defined as $h(x)=\sigma(x)-x$. The *stepsize* $k:\mathbb{Y}\to\mathbb{R}$ is defined as $k(y)=\tau(y)-y$. We will use the notation $f^{\sigma}(x,y)=f(\sigma(x),y)$, $f^{\tau}(x,y)=f(x,\tau(y))$, and $f^{\sigma\tau}(x,y)=f(\sigma(x),\tau(y))$.

For $\mathbb{X}$ and $\mathbb{Y}$ bounded time scales, we let R denote the "rectangle" $R:=\mathbb{X}\times\mathbb{Y}$. Let R^i denote the set $R^i:=\mathbb{X}^i\times\mathbb{Y}^i$, where our $\mathbb{X}^i$ is the same as Hilger's $\mathbb{X}^{\kappa}$.

Because we will need notation for partial derivatives with respect to time scale variables x and y we employ lexigraphic ordering for consistency. (The Greek alphabet has Γ preceeding Δ.) As in [21] let $f^{\Gamma}(x,y)$ denote the time scale partial derivative with respect to x and let $f^{\Delta}(x,y)$ denote the time scale partial derivative with respect to y. Definitions of these partial derivatives are now given.

Let f be a real-valued function (or a matrix valued function) on $\mathbb{X}\times\mathbb{Y}$. At $(x,y)\in\mathbb{X}\times\mathbb{Y}$ we say that f has a "Γ partial derivative" $f^{\Gamma}(x,y)$ (with respect to x) if for each $\varepsilon>0$ there exists a neighborhood $\mathcal{U}_x$, (open in the relative topology of $\mathbb{X}$), of x such that

$$|f(\sigma(x),y)-f(s,y)-f^{\Gamma}(x,y)(\sigma(x)-s)|\leqslant\varepsilon|\sigma(x)-s| \tag{1}$$

for all $s\in\mathcal{U}_x$. At $(x,y)\in\mathbb{X}\times\mathbb{Y}$ we say that f has a "Δ partial derivative" $f^{\Delta}(x,y)\in\mathbb{R}$ (with respect to y) if for each $\varepsilon>0$ there exists a neighborhood $\mathcal{U}_y$ of y such that

$$|f(x,\tau(y))-f(x,t)-f^{\Delta}(x,y)(\tau(y)-t)|\leqslant\varepsilon|\tau(y)-t| \tag{2}$$

for all $t\in\mathcal{U}_y$. From single variable time scales, we have the useful formulas $f(\sigma(x),y)=f(x,y)+h(x)f^{\Gamma}(x,y)$ if $f^{\Gamma}(x,y)$ exists and $f(x,\tau(y))=f(x,y)+k(y)f^{\Delta}(x,y)$ if $f^{\Delta}(x,y)$ exists.

Let f be a real-valued (or matrix valued) function on $\mathbb{X}\times\mathbb{Y}$. The function f is called *rd-continuous in* y if for every $\alpha\in\mathbb{X}$, the function $f(\alpha,y)$ is rd-continuous on $\mathbb{Y}$. The function f is *rd-continuous in* x if for every $\beta\in\mathbb{Y}$ the function $f(x,\beta)$ is rd-continuous on $\mathbb{X}$.

Let CC_{rd} denote the set of functions $f(x,y)$ on $\mathbb{X}\times\mathbb{Y}$ with the properties

- f is rd-continuous in x,
- f is rd-continuous in y,
- if $(x_1,y_1)\in\mathbb{X}\times\mathbb{Y}$ with x_1 right-dense or maximal and y_1 right-dense or maximal, then f is continuous at (x_1,y_1),
- if x_1 and y_1 are both left-dense, then the limit of $f(x,y)$ exists as (x,y) approaches (x_1,y_1) along any path in the region $R_{\mathrm{LL}}(x_1,y_1)=\{(x,y): x\in[a,x_1]\cap\mathbb{X}, y\in[c,y_1]\cap\mathbb{Y}\}$.

Let CC^1_{rd} be the set of all functions in CC_{rd} for which both the Δ partial derivative and the Γ partial derivative exist and are in CC_{rd}.

3. Double integrals on time scales

Double integrals are defined as iterated integrals. If f has a Γ antiderivative A and A has a Δ antiderivative B, then

$$\int_c^d \int_a^b f(x,y)\Gamma x \Delta y := \int_c^d (A(b,y) - A(a,y))\Delta y$$
$$= B(b,d) - B(b,c) - B(a,d) + B(a,c), \tag{3}$$

where $A^{\Gamma} = f$, $B^{\Delta} = A$, and $(B^{\Delta})^{\Gamma} = A^{\Gamma} = f$. If f has a Δ antiderivative C and C has a Γ antiderivative D, then

$$\int_a^b \int_c^d f(x,y)\,\Delta y \Gamma x := \int_a^b (C(x,d) - C(x,c))\Gamma x$$
$$= D(b,d) - D(a,d) - D(b,c) + D(a,c), \tag{4}$$

where $C^{\Delta} = f$, $D^{\Gamma} = C$, and $(D^{\Gamma})^{\Delta} = C^{\Delta} = f$. The integration by parts formulas for double integrals are

$$\int_c^d \int_a^b f(\sigma(x),y) g^{\Gamma}(x,y)\Gamma x \Delta y$$
$$= -\int_c^d \int_a^b f^{\Gamma}(x,y) g(x,y)\Gamma x \Delta y + \int_c^d f(b,y)g(b,y)\Delta y - \int_c^d f(a,y)g(a,y)\,\Delta y \tag{5}$$

and

$$\int_a^b \int_c^d f(x,\tau(y)) g^{\Delta}(x,y)\Delta y \Gamma x = -\int_a^b \int_c^d f^{\Delta}(x,y) g(x,y)\Delta y \Gamma x$$
$$+ \int_a^b f(x,d)g(x,d)\Gamma x - \int_a^b f(x,c)g(x,c)\Gamma x. \tag{6}$$

In the continuous case [12], if f is Riemann integrable as a double integral on $\{a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, and if it is Riemann integrable in x for each y and Riemann integrable in y for each x, then

$$\int_a^b \mathrm{d}x \int_c^d f(x,y)\,\mathrm{d}y = \int_c^d \mathrm{d}y \int_a^b f(x,y)\,\mathrm{d}x. \tag{7}$$

Also, if f is continuous in $\{a < x < b, c < y < d\}$ and if both of the iterated Riemann integrals

$$\int_a^b \mathrm{d}x \int_c^d |f(x,y)|\,\mathrm{d}y \tag{8}$$

and

$$\int_c^d \mathrm{d}y \int_a^b |f(x,y)|\,\mathrm{d}x \tag{9}$$

exist, then [12] both of the following Riemann integrals exist and

$$\int_a^b \mathrm{d}x \int_c^d f(x,y)\,\mathrm{d}y = \int_c^d \mathrm{d}y \int_a^b f(x,y)\,\mathrm{d}x. \tag{10}$$

In the discrete case, if k, l, m, and n are integers with $k \leqslant l$ and $m \leqslant n$, then

$$\sum_{j=m}^{n}\sum_{i=k}^{l} a_{i,j} = \sum_{i=k}^{l}\sum_{j=m}^{n} a_{i,j}. \tag{11}$$

Thus the order of integration can be reversed in these important special cases of time scale double integrals. In some cases our proofs could be shortened if the integrands f in double integrals were restricted to the class $\mathscr{R}$, the *reversible class*, of functions f which are double integrable in both orders and have the property

$$\int_a^b \int_c^d f(x,y)\Gamma x \Delta y = \int_c^d \int_a^b f(x,y)\Delta y \Gamma x. \tag{12}$$

Since rd-continuous functions of one variable are integrable and double integrals are defined as iterated integrals, it follows that $f \in CC_{\mathrm{rd}}$ is double integrable in either order, although this condition alone has not been shown to imply that (12) holds. For full generality of our results we have given proofs which do not require f to be in $\mathscr{R}$ until our derivation of the Euler–Lagrange equation.

Proposition 1. (a) *The functions σ and τ are rd-continuous.*
(b) *The functions h and k are rd-continuous.*
(c) *If $f \in CC_{\mathrm{rd}}$, then $f^{\sigma\tau} \in CC_{\mathrm{rd}}$, $f^{\sigma} \in CC_{\mathrm{rd}}$, and $f^{\tau} \in CC_{\mathrm{rd}}$.*

Proof. Parts (a) and (b) follow from single time scale facts. (c) Let $f \in CC_{\mathrm{rd}}$ and let $(x_1, y_1) \in R$. Then f, f^{σ}, f^{τ}, and $f^{\sigma\tau}$ are all rd-continuous in both x and y. If x_1 and y_1 are both right-dense or maximal, then f is continuous at (x_1, y_1), σ is continuous at x_1, and τ is continuous at y_1. Thus f, f^{σ}, f^{τ}, and $f^{\sigma\tau}$ are all continuous at (x_1, y_1). If x_1 and y_1 are both left-dense, then the left-sided limit of σ at x_1 exists, the left-sided limit of τ at y_1 exists, and the limit of $f(x, y)$ as (x, y) approaches (x_1, y_1) along any path in R_{LL} exists. Thus the limit of f, f^{σ}, f^{τ}, and $f^{\sigma\tau}$ all exist as (x, y) approaches (x_1, y_1) along any path in R_{LL}.

Proposition 2. *Let f and g be real-valued functions on $\mathbb{X} \times \mathbb{Y}$ or matrix valued functions such that the product fg is defined.*
(a) *If f is continuous, then $f \in CC_{\mathrm{rd}}$.*
(b) *If f is continuous and $g \in CC_{\mathrm{rd}}$, then $fg \in CC_{\mathrm{rd}}$.*

Proof. (a) This follows from the definition of continuous.
(b) Let $(x_1, y_1) \in R$. If x_1 is right-dense, then f and g are continuous in x at any point $(x_1, y) \in R$, so fg is continuous in x at any point $(x_1, y) \in R$. If y_1 is right-dense, then f and g are continuous in y at any point $(x, y_1) \in R$ and fg is continuous in y at any point $(x, y_1) \in R$. If both x_1 and y_1 are right-dense, then f and g are continuous at (x_1, y_1) and fg is continuous at (x_1, y_1). If x_1 and y_1 are both maximal, then f and g are continuous at (x_1, y_1) and fg is continuous at (x_1, y_1). If x_1 is left-dense, then the left-sided limits in x of f and g exist for any point $(x_1, y) \in R$, so the

left-sided limit in x of fg exists for any point $(x_1, y) \in R$. If y_1 is left-dense, then the left-sided limits in y of f and g exist for any point $(x, y_1) \in R$, so the left-sided limit in y of fg exists for any point $(x, y_1) \in R$. If both x_1 and y_1 are left-dense, then the limits of both f and g exist as (x, y) approaches (x_1, y_1) along any path in R_{LL}. Thus the limit of fg exists as (x, y) approaches (x_1, y_1) along any path in R_{LL}. □

Proposition 3. *Suppose each of the time scales $\mathbb{X}$ and $\mathbb{Y}$ contain at least two points. Let $f \in CC_{\mathrm{rd}}$. Let $x_0 \in \mathbb{X}^i$ and $y_0 \in \mathbb{Y}^i$. Then*

(i) $\int_{x_0}^{\sigma(x_0)} f(x, y)\Gamma x = h(x_0) f(x_0, y)$,

(ii) $\int_{y_0}^{\tau(y_0)} f(x, y)\Delta y = k(y_0) f(x, y_0)$,

(iii) $\int_{y_0}^{\tau(y_0)} \int_{x_0}^{\sigma(x_0)} f(x, y)\Gamma x \Delta y = \int_{x_0}^{\sigma(x_0)} \int_{y_0}^{\tau(y_0)} f(x, y)\Delta y = h(x_0)k(y_0) f(x_0, y_0)$.

Proof. Since the integral in the left side of (i) exists, there exists a function A such that $A^{\Gamma} = f$. Then

$$A(\sigma(x_0), y) - A(x_0, y) = h(x_0)A^{\Gamma}(x_0, y) = h(x_0) f(x_0, y)$$

and

$$\int_{x_0}^{\sigma(x_0)} f(x, y)\Gamma x = A(\sigma(x_0), y) - A(x_0, y) = h(x_0) f(x_0, y).$$

The proof of (ii) is similar. Item (iii) follows from (i) and (ii). □

Recall that Hilger [13] used $\rho(x)$ for the *left jump* function on $\mathbb{X}$. We will use the notation $v(y)$ for the left jump function on the time scale $\mathbb{Y}$.

Proposition 4. *Suppose $f \in CC_{\mathrm{rd}}$.*

(i) *If $f(x, y_0) = 0$ for all time scale points x in $[x_1, x_2]^i$, then*

$$\int_{y_0}^{\tau(y_0)} \int_{x_1}^{x_2} f(x, y)\Gamma x \Delta y = 0.$$

(ii) *If $f(x, v(y_0)) = 0$ for all time scale points x in $[x_1, x_2]^i$, then*

$$\int_{v(y_0)}^{y_0} \int_{x_1}^{x_2} f(x, y)\Gamma x \Delta y = 0.$$

Proof. For (i), let $F(y) := \int_{x_1}^{x_2} f(x, y)\Gamma x$. Then $F(y_0) = \int_{x_1}^{x_2} f(x, y_0)\Gamma x = 0$ and

$$\int_{y_0}^{\tau(y_0)} \int_{x_1}^{x_2} f(x, y)\Gamma x \Delta y = \int_{y_0}^{\tau(y_0)} F(y)\Delta y = k(y_0)F(y_0) = 0.$$

Part (ii) follows from (i) by relabeling $\tau(y_0)$ as y_1 and y_0 as $v(y_0)$. □

Proposition 5. (i) *If* $a < c < b$ *and* $f \in C_{\mathrm{rd}}$, *then*

$$\int_a^b f(x)\Gamma x = \int_a^c f(x)\Gamma x + \int_c^b f(x)\Gamma x.$$

(ii) *If* P *is any rectangular partition of* $[a,b] \times [c,d]$ *into a finite number of subrectangles* R_{ij} *and* $f \in C_{\mathrm{rd}}$, *then*

$$\int_c^d \int_a^b f(x,y)\Gamma x \Delta y = \sum_{R_{i,j} \in P} \int\int_{R_{ij}} f(x,y)\Gamma x \Delta y.$$

Proposition 6. *Assume that* $\mathbb{X}$ *is a bounded time scale which contains at least two points. Let* $a := \min \mathbb{X}$, $b = \max \mathbb{X}$, *and denote* $\mathbb{X}$ *by* $[a,b]$. *Suppose that* f *and* g *are in* C_{rd} *on* $\mathbb{X}$. *Then*

(i) *If* $|f(x)| \leqslant g(x)$ *on*$[a,b]^i$, *then* $|\int_a^b f(x)\Gamma x| \leqslant \int_a^b g(x)\Gamma x$.
(ii) $|\int_a^b f(x)\Gamma x| \leqslant \int_a^b |f(x)|\Gamma x$.
(iii) *If* $f(x) \geqslant 0$ *on* $[a,b]^i$, *then*$\int_a^b f(x)\Gamma x \geqslant 0$.
(iv) *If* $x_0 \in [a,b]^i$ *is such that*$f(x_0) > 0$, *then there exist time scale points* x_1, x_2 *in* $[a,b]$ *with* $x_1 < x_2$ *and* $x_1 \leqslant x_0 \leqslant x_2$ *such that* $\int_{x_1}^{x_2} f(x)\Gamma x > 0$.
(v) *If* $f(x) \geqslant 0$ *on* $[a,b]^i$ *and there exists a time scale point* $x_0 \in [a,b]^i$ *such that* $f(x_0) > 0$, *then* $\int_a^b f(x)\Gamma x > 0$.

Proof. Result (i) is given by Hilger [13]. Part (iv) corrects Agarwal and Bohner's Lemma 2, part (4) [2, p. 679], where they assert that

(4) If $t_1 \in \mathbb{T}^\kappa$ and $f(t_1) > 0$, then there exists $t_2 \in \mathbb{T}$ with $\int_{t_1}^{t_2} f(t)\Delta t > 0$; □

Note that there is no such t_2 in the continuous example of $f(t) := 1$ on the real interval $\mathbb{T} = [0,1]$ if t_1 is chosen as 1.

Proposition 7. *Consider bounded time scales* $\mathbb{X} = [a,b]$ *and* $\mathbb{Y} = [c,d]$, *each of which contain at least two points, and* f, g *in* CC_{rd} *on* $R := \mathbb{X} \times \mathbb{Y}$.

(i) *If* $|f(x,y)| \leqslant g(x,y)$ *on* R^i, *then*$|\int_c^d \int_a^b f(x,y)\Gamma x \Delta y| \leqslant \int_c^d \int_a^b g(x,y)\Gamma x \Delta y$.
(ii) $|\int_c^d \int_a^b f(x,y)\Gamma x \Delta y| \leqslant \int_c^d \int_a^b |f(x,y)|\Gamma x \Delta y$.
(iii) *If* $f(x,y) \geqslant 0$ *on* R^i, *then*$\int_c^d \int_a^b f(x,y)\Gamma x \Delta y \geqslant 0$.
(iv) *If* $(x_0,y_0) \in R^i$ *is such that* $f(x_0,y_0) > 0$, *then there exist time scale points* u_1, u_2 *in* $[a,b]$, v_1, v_2 *in* $[c,d]$ *with* $u_1 < u_2$, $v_1 < v_2$, $u_1 \leqslant x_0 \leqslant u_2$, *and* $v_1 \leqslant y_0 \leqslant v_2$ *such that* $\int_{v_1}^{v_2} \int_{u_1}^{u_2} f(x,y)\Gamma x \Delta y > 0$.
(v) *If* $f(x,y) \geqslant 0$ *on* R^i *and* $f(x_0,y_0) > 0$ *for some* $(x_0,y_0) \in R^i$, *then* $\int_c^d \int_a^b f(x,y)\Gamma x \Delta y > 0$.

Proof of (iv). Suppose each of the time scales $\mathbb{X}$ and $\mathbb{Y}$ contain at least two points. Let $f \in CC_{\mathrm{rd}}$. Assume $(x_0,y_0) \in \mathbb{X}^i \times \mathbb{Y}^i$ is such that $f(x_0,y_0) > 0$. In each of the following four cases we show existence of a point (x_i,y_i) such that the integral over the region with corners at (x_0,y_0) and (x_i,y_i) is positive. Each of the points (x_i,y_i) is in the ith quadrant relative to (x_0,y_0).

1. We establish the first quadrant result. If x_0 is not a maximal point of $\mathbb{X}$ and y_0 is not a maximal point of $\mathbb{Y}$, then there exists a point (x_1, y_1) in $\mathbb{X}\times\mathbb{Y}$ with $x_0 < x_1$ and $y_0 < y_1$ such that $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\Gamma x\Delta y > 0$.

Assume that x_0 and y_0 are not maximal.

Case (i): x_0 is right-scattered. Choose $x_1 := \sigma(x_0)$. If y_0 is right-scattered and $y_1 := \tau(y_0)$ then part (iii) of Proposition 3 gives

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\Gamma x\Delta y = h(x_0)k(y_0)f(x_0,y_0) > 0.$$

Next, assume x_0 is right-scattered and y_0 is right-dense. Then part (i) of Proposition 3 gives

$$\int_{x_0}^{x_1} f(x,y)\Gamma x = h(x_0)f(x_0,y).$$

Since $h(x_0)f(x_0,y)$ is an rd-continuous function of y, there exists $y_1\in\mathbb{Y}$ with $y_0 < y_1$ such that $f(x_0,y) > 0$ for $y_0 \leqslant y \leqslant y_1$ and $\int_{y_0}^{y_1} h(x_0)f(x_0,y)\Delta y > 0$. Thus

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\Gamma x\Delta y = \int_{y_0}^{y_1} h(x_0)f(x_0,y)\Delta y > 0.$$

Case (ii): x_0 is right-dense.

The proof for the case where x_0 is right-dense and y_0 is right-scattered proceeds by selecting $y_1 := \tau(y_0)$ and noting that since $f(x,y_0)$ is continuous at x_0, there exists an x_1 with $x_1 > x_0$ such that $f(x,y_0) > 0$ on $[x_0,x_1]$. Since

$$\int_{y_0}^{\tau(y_0)} g(y)\Delta y = g(y_0)k(y_0)$$

for any $g(y)$ which is defined at y_0, the choice of $g(y) := \int_{x_0}^{x_1} f(x,y)\Gamma x$ gives

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\Gamma x\Delta y = \int_{y_0}^{y_1} g(y)\Delta y = g(y_0)k(y_0) = k(y_0)\int_{x_0}^{x_1} f(x,y_0)\Gamma x > 0.$$

If both x_0 and y_0 are right-dense, then f is continuous at (x_0,y_0). Then there exists a $\delta > 0$ such that $f(x,y) \geqslant f(x_0,y_0)/2$ on

$$\{(x,y)\colon (x,y)\in\mathbb{X}\times\mathbb{Y}, x_0 \leqslant x \leqslant x_0+\delta, y_0 \leqslant y \leqslant y_0+\delta\}.$$

Then there exists $x_1\in\mathbb{X}\cap(x_0,x_0+\delta]$ and $y_1\in\mathbb{Y}\cap(y_0,y_0+\delta]$ such that $f(x,y) \geqslant f(x_0,y_0)/2$ for $(x,y)\in\{(x,y)\colon x_0\leqslant x\leqslant x_1, y_0\leqslant y\leqslant y_1, (x,y)\in\mathbb{X}\times\mathbb{Y}\}$. Thus

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\Gamma x\Delta y \geqslant \int_{y_0}^{y_1}\int_{x_0}^{x_1} (f(x_0,y_0)/2)\Gamma x\Delta y > \int_{y_0}^{y_1}\int_{x_0}^{x_1} 0\Gamma x\Delta y = 0.$$

2. Next we establish the second quadrant result. If x_0 is a maximal point of $\mathbb{X}$ and y_0 is not a maximal point of $\mathbb{Y}$, then there exists a point (x_2,y_2) in $\mathbb{X}\times\mathbb{Y}$ with $x_2 < x_0$ and $y_0 < y_2$ such that $\int_{y_0}^{y_2}\int_{x_2}^{x_0} f(x,y)\Gamma x\Delta y > 0$.

Assume x_0 is maximal and hence is not isolated. Then $f(\cdot,y_0)$ is continuous at x_0 and there exists a point x_2 in $[a,x_0)$ such that $f(x,y_0) > 0$ on $[x_2,x_0]$.

Consider the case of y_0 right-scattered. Set $y_2 = \tau(y_0)$ for

$$\int_{y_0}^{y_2}\int_{x_2}^{x_0} f(x,y)\Gamma x\Delta y = \int_{y_0}^{y_2}\int_{x_2}^{x_0} f(x,y_0)\Gamma x\Delta y > 0.$$

In the case of y_0 right-dense, $f(x,y)$ is continuous at (x_0,y_0) and there exists (x_2,y_2) such that $f(x,y)>0$ on $[x_2,x_0]\times[y_0,y_2]$.

3. Next we establish the third quadrant result. If x_0 is a maximal point of $\mathbb{X}$ and y_0 is a maximal point of $\mathbb{Y}$ then there exists a point (x_3,y_3) in $\mathbb{X}\times\mathbb{Y}$ with $x_3<x_0$ and $y_3<y_0$ such that $\int_{y_3}^{y_0}\int_{x_3}^{x_0} f(x,y)\Gamma x\Delta y>0$.

Assume that x_0 is maximal (hence not isolated) and y_0 is maximal (hence not isolated). Then x_0 and y_0 are left-dense, f is continuous at (x_0,y_0), and there exists a point (x_3,y_3), $x_3<x_0$, $y_3<y_0$, such that $f(x,y)>0$ on $[x_3,x_0]\times[y_3,y_0]$.

4. Next we establish the fourth quadrant result. If x_0 is not a maximal point of $\mathbb{X}$ and y_0 is a maximal point of $\mathbb{Y}$, then there exists a point (x_4,y_4) in $\mathbb{X}\times\mathbb{Y}$ with $x_0<x_4$ and $y_4<y_0$ such that $\int_{y_4}^{y_0}\int_{x_0}^{x_4} f(x,y)\Gamma x\Delta y>0$.

Assume x_0 is not a maximum and y_0 is a maximum (hence not isolated). If x_0 is right-dense, then $f(x,y)$ is continuous at (x_0,y_0) and there exists a point (x_4,y_4), $x_0<x_4$, $y_4<y_0$, such that $f(x,y)>0$ on $[x_0,x_4]\times[y_4,y_0]$. If x_0 is right-scattered, set $x_4=\sigma(x_0)$. Then

$$\int_{x_0}^{x_4} f(x,y)\Gamma x = f(x_0,y)h(x_0) > 0$$

for $y_4 \leqslant y \leqslant y_0$ for some y_4 in $[c,y_0)$. Hence

$$\int_{y_4}^{y_0}\int_{x_0}^{x_4} f(x,y)\Gamma x\Delta y > 0. \qquad \square$$

A proof of the continuous case of the following lemma is found in [7].

Lemma 8 (the fundamental lemma). *If $R=[a,b]\times[c,d]$, M is continuous on $(R^i)^i$, and*

$$\int_c^d\int_a^b M(x,y)\zeta^{\sigma\tau}(x,y)\Gamma x\Delta y = 0 \tag{13}$$

for any C^1 function $\zeta(x,y)$ which vanishes on the boundary of R, then the function $M(x,y)$ is 0 at every time scale point (x,y) in $(R^i)^i$.

1-d Proof. We first establish the single integral result that if

$$\int_a^b M(x)\eta^\sigma(x)\Gamma x = 0$$

for all $\eta\in C^1$ with $\eta(a)=0=\eta(b)$, then $M(x)\equiv 0$ on $([a,b]^i)^i$.

Case A: We establish that M is zero at left-dense points in $([a,b]^i)^i$. Assume $M(x_0)>0$ where x_0 is left-dense and in $([a,b]^i)^i$. (Hence $x_0\neq a$.) By continuity there exists a $\delta>0$ such that $M(x)\geqslant M(x_0)/2$ for time scale points x with $|x-x_0|\leqslant\delta$. Then there exists an increasing sequence

u_k, $k=1,\ldots,$ of time scale points in $[x_0-\delta,x_0)$ such that u_k converges to x_0 from below. Then $\sigma(u_1)\leqslant u_2<x_0$ and we define η by

$$\eta(x):=(x-\sigma(u_1))^2(\sigma(x_0)-x)^2$$

for $\sigma(u_1)\leqslant x\leqslant\sigma(x_0)$, and $\eta(x):=0$, otherwise. Then $\eta^\sigma(x)=0$ for $x\leqslant u_1$ and $\eta^\sigma(x)=0$ for $x\geqslant x_0$. Since $u_1\leqslant\sigma(u_1)\leqslant u_2<u_3\leqslant\sigma(u_3)<x_0$, we have $\eta(\sigma(u_3))>0$; hence $M(u_3)\eta^\sigma(u_3)>0$, and Proposition 7 gives the contradiction

$$0=\int_a^b M(x)\eta^\sigma(x)\Gamma x=\int_{u_1}^{x_0} M(x)\eta^\sigma(x)\Gamma x>0.$$

Case B: We establish that M is zero at right-dense points in $([a,b]^i)^i$. The proof differs from Case A only in that we have a decreasing sequence of time scale points v_k in $(x_0,x_0+\delta]$ which converges from above to x_0. Then $\sigma(v_1)>x_0$ and we define η by

$$\eta(x):=(x-\sigma(x_0))^2(\sigma(v_1)-x)^2,$$

for $\sigma(x_0)\leqslant x\leqslant\sigma(v_1)$, and $\eta(x):=0$, otherwise. Then $M(v_3)\eta^\sigma(v_3)>0$ and Proposition 7 gives the contradiction

$$0=\int_a^b M(x)\eta^\sigma(x)\Gamma x=\int_{x_0}^{v_1} M(x)\eta^\sigma(x)\Gamma x>0.$$

Case C: x_0 is left-scattered, right-scattered and $\sigma(x_0)<\sigma^2(x_0):=\sigma(\sigma(x_0))$ or $a=x_0<\sigma(x_0)<\sigma^2(x_0)$. Assume $M(x_0)>0$. Define η by $\eta(\sigma(x_0)):=1$ and $\eta(x):=0$, otherwise. Then since $\rho(x_0)$, the left jump of x_0, satisfies $\rho(x_0)<x_0$ or $a=x_0=\rho(x_0)$, we have the contradiction

$$0=\int_a^b M(x)\eta^\sigma(x)\Gamma x=\int_{\rho(x_0)}^{x_0} M(x)\eta^\sigma(x)\Gamma x$$

$$+\int_{x_0}^{\sigma(x_0)} M(x)\eta^\sigma(x)\Gamma x=0+h(x_0)M(x_0)>0.$$

Case D: x_0 is left-scattered, right-scattered and $\sigma(x_0)=\sigma^2(x_0)$. Then x_0 in $(R^i)^i$ implies $\sigma(x_0)<b$. Thus $\sigma(x_0)$ is right-dense and by Case B, $M(\sigma(x_0))=0$. Assume $M(x_0)>0$. Continuity of M at $\sigma(x_0)$ implies existence of a $\delta>0$ such that $\delta<\sigma(x_0)-x_0$ and $|M(x)|\leqslant 1$ on $[\sigma(x_0),\sigma(x_0)+\delta]$. Let v_k, $k=1,\ldots,$ be a decreasing sequence of time scale points in $(\sigma(x_0),\sigma(x_0)+\delta)$ which converge to $\sigma(x_0)$ from above. For $k=2,\ldots,$ let $\gamma_k:=(\sigma(x_0)-x_0)^2(\sigma(x_0)-\sigma(v_k))^2$ and define η_k by

$$\eta_k(x):=(1/\gamma_k)(x-x_0)^2(x-\sigma(v_k))^2$$

on $[x_0,\sigma(v_k)]$, and $\eta_k(x):=0$, otherwise. Then $\eta_k(\sigma(x))$ is 0 outside $[\rho(x_0),v_k]$, we have $\eta_k(\sigma(x_0))=1$, and

$$\int_{x_0}^{\sigma(x_0)} M(x)\eta_k^\sigma(x)\Gamma x=h(x_0)M(x_0)\eta_k(\sigma(x_0))=h(x_0)M(x_0).$$

For $k=2,\ldots,$ we have

$$0<\sigma(v_k)-\sigma(x_0)\leqslant v_{k+1}-\sigma(x_0)<\delta<\sigma(x_0)-x_0.$$

Since $\sigma(x_0)$ is closer to $\sigma(v_k)$ than to x_0, the function $\eta_k(x)$ has its maximum value at a point in $[x_0,\sigma(x_0)]$ and since $\eta_k(\sigma(x_0))=1$, we have $|M(x)\eta_k^\sigma(x)|\leqslant 1$ on $[\sigma(x_0),v_k]$ and $|\int_{x_0}^{v_k} M(x)\eta_k^\sigma(x)\Gamma x|\leqslant v_k-\sigma(x_0)$. Thus

$$0=\int_a^b M(x)\eta_k^\sigma(x)\Gamma x=\int_{\rho(x_0)}^{x_0} M(x)\eta^\sigma(x)\Gamma x+\int_{x_0}^{\sigma(x_0)} M(x)\eta^\sigma(x)\Gamma x+\int_{\sigma(x_0)}^{v_k} M(x)\eta^\sigma(x)\Gamma x$$

$$=0+h(x_0)M(x_0)+\int_{\sigma(x_0)}^{v_k} M(x)\eta^\sigma(x)\Gamma x.$$

As $k\to\infty$, the last integral goes to 0, since the integrand is bounded in absolute value and the length of the interval goes to 0. Thus we obtain in the limit that $0=h(x_0)M(x_0)>0$ which is a contradiction to $M(x_0)>0$. Thus Case D is established.

Note that $x_0=a$ is included in Case B if right-dense and in Case C if right-scattered. Also $x_0=b$ is included in Case A if left-dense and is not in $([a,b]^i)^i$ if left-scattered. Thus the one-dimensional Fundamental Lemma is established. □

2-d Proof. First relabel the previous cases A, B, C, D on x_0 as cases a, b, c, d on y_0. Relabel x_0 as y_0, $\eta(x)$ by $\phi(y)$, $\sigma(x)$ by $\tau(y)$, $\rho(x)$, the backward jump by $v(y)$, u_i by s_i, v_k by t_j and define $\zeta(x,y)$ by

$$\zeta(x,y):=\eta(x)\phi(y).$$

Cases on (x_0,y_0) of Aa, Ab, Ac, Ba, Bb, Bb, Ca, Cb, Cc follow directly from showing that the integrand $M(x,y)\zeta^{\sigma\tau}(x,y)$ is nonnegative on R^i and positive at some point of R^i. Then Proposition 7 yields a contradiction. Cases involving Case D or Case d require a limiting argument.

Case Da is the case of x_0 left- and right-scattered with $\sigma(x_0)=\sigma^2(x_0)$ and y_0 left-dense. (Hence $x_0>a$ and $y_0>c$.) Then $M(\sigma(x_0),y_0)=0$ by Case Ba. Assume $M(x_0,y_0)>0$. Continuity of M implies existence of a $\delta>0$ such that $M(x,y)\geqslant M(x_0,y_0)/2$ for $|x-x_0|\leqslant\delta$, $|y-y_0|\leqslant\delta$ and $|M(x,y)|\leqslant 1$ for $|x-\sigma(x_0)|\leqslant\delta$ and $|y-y_0|\leqslant\delta$. Since y_0 is left-dense, there exists an increasing sequence $s_k\in[y_0-\delta,y_0)$ converging to y_0 from below. Define ϕ by

$$\phi(y):=(y-\tau(s_1))^2(\tau(y_0)-y)^2$$

for $\tau(s_1)\leqslant y\leqslant\tau(y_0)$ and $\phi(y):=0$, otherwise. For v_k and $\eta_k(x)$ as in Case D, set $\zeta_k(x,y):=\eta_k(x)\phi(y)$. Then $\zeta_k^{\sigma\tau}$ is 0 outside $[\rho(x_0),v_k]\times[s_1,y_0]$ and

$$0=\int_c^d\int_a^b M(x,y)\zeta_k^{\sigma\tau}(x,y)\Gamma x\Delta y=\int_{s_1}^{y_0}\int_{\rho(x_0)}^{v_k} M(x,y)\zeta_k^{\sigma\tau}(x,y)\Gamma x\Delta y$$

$$=0+\int_{s_1}^{y_0}\int_{x_0}^{\sigma(x_0)} M(x,y)\zeta_k^{\sigma\tau}(x,y)\Gamma x\Delta y+\int_{s_1}^{y_0}\int_{\sigma(x_0)}^{v_k} M(x,y)\zeta_k^{\sigma\tau}(x,y)\Gamma x\Delta y$$

$$\to\int_{s_1}^{y_0}\int_{x_0}^{\sigma(x_0)} M(x,y)\zeta_k^{\sigma\tau}(x,y)\Gamma x\Delta y>0\quad\text{as } k\to\infty.$$

Case Ad is dual to Case Da and follows by reversing the roles of the x and y axis, i.e., flip the transparency of the region of integration for Case Da about the 45° line for case Ad.

Case Db differs only from Case Da in that y_0 is right-dense and uses a decreasing sequence t_j converging from above to y_0. Define ϕ by

$$\phi(y) := (y - \tau(y_0))^2(\tau(t_1) - y)^2$$

for $\tau(y_0) \leqslant y \leqslant \tau(t_1)$ and $\phi(y) := 0$, otherwise. For v_k and $\eta_k(x)$ as in Case D, set $\zeta_k(x, y) := \eta_k(x)\phi(y)$. Then

$$0 = \int_c^d \int_a^b M(x, y)\zeta_k^{\sigma\tau}(x, y)\Gamma x\Delta y = \int_{y_0}^{t_1} \int_{\rho(x_0)}^{v_k} M(x, y)\zeta_k^{\sigma\tau}(x, y)\Gamma x\Delta y$$

$$= 0 + \int_{y_0}^{t_1} \int_{x_0}^{\sigma(x_0)} M(x, y)\zeta_k^{\sigma\tau}(x, y)\Gamma x\Delta y + \int_{y_0}^{t_1} \int_{\sigma(x_0)}^{v_k} M(x, y)\zeta_k^{\sigma\tau}(x, y)\Gamma x\Delta y$$

$$\to \int_{y_0}^{t_1} \int_{x_0}^{\sigma(x_0)} M(x, y)\zeta_k^{\sigma\tau}(x, y)\Gamma x\Delta y > 0, \quad \text{as } k \to \infty.$$

Case Bd is dual to Case Db.

Case Dc is the case of x_0 left- and right-scattered with $\sigma(x_0) = \sigma^2(x_0)$ and y_0 left- and right-scattered with $\tau(y_0) < \tau^2(y_0)$. Define $\eta_k(x)$ as in Case D and define ϕ by $\phi(\tau(y_0)) := 1$ and $\phi(y) := 0$, otherwise. The point $(\sigma(x_0), y_0)$ satisfies Case Bc; hence $M(\sigma(x_0), y_0) = 0$. Then for δ and v_k chosen properly we have

$$0 = \int_c^d \int_a^b M(x, y)\zeta_k^{\sigma\tau}(x, y)\Gamma x\Delta y = \int_{v(y_0)}^{\tau(y_0)} \int_{\rho(x_0)}^{v_k} M(x, y)\zeta_k^{\sigma\tau}(x, y)\Gamma x\Delta y$$

$$= \int_{v(y_0)}^{y_0} \int_{\rho(x_0)}^{v_k} M(x, y)\zeta_k^{\sigma\tau}(x, y)\Gamma x\Delta y + \int_{y_0}^{\tau(y_0)} \int_{\rho(x_0)}^{v_k} M(x, y)\zeta_k^{\sigma\tau}(x, y)\Gamma x\Delta y.$$

Proposition 4 reduces integrals over $[v(y_0), y_0]$ to 0 because $\zeta^{\sigma\tau}(x, v(y_0)) = 0$. Thus

$$0 = \int_{y_0}^{\tau(y_0)} \int_{\rho(x_0)}^{x_0} M\zeta_k^{\sigma\tau}\Gamma x\Delta y + \int_{y_0}^{\tau(y_0)} \int_{x_0}^{\sigma(x_0)} M\zeta_k^{\sigma\tau}\Gamma x\Delta y + \int_{y_0}^{\tau(y_0)} \int_{\sigma(x_0)}^{v_k} M\zeta_k^{\sigma\tau}\Gamma x\Delta y$$

$$= 0 + M(x_0, y_0)\eta_k(\sigma(x_0))\phi(\tau(y_0))h(x_0)k(y_0) + \int_{y_0}^{\tau(y_0)} \int_{\sigma(x_0)}^{v_k} M\zeta_k^{\sigma\tau}\Gamma x\Delta y$$

$$\to M(x_0, y_0)h(x_0)k(y_0) > 0 \quad \text{as } k \to \infty.$$

Case Cd is dual to Case Dc.

The final case, that of Case Dd is the case of x_0 left- and right-scattered with $\sigma(x_0)=\sigma^2(x_0)$ and y_0 left- and right-scattered with $\tau(y_0)=\tau^2(y_0)$. Then $M(\sigma(x_0),y_0)=0$, since $(\sigma(x_0),y_0)$ is Case Bb. Also, $M(x_0,\tau(y_0))=0$, since $(x_0,\tau(y_0))$ is Case Db and $M(\sigma(x_0),\tau(y_0))=0$, since $(\sigma(x_0),\tau(y_0))$ is Case Bb. Assume $M(x_0,y_0)>0$. Thus for proper choices of δ, a sequence v_k converging from above to $\sigma(x_0)$, and a sequence t_j converging to $\tau(y_0)$ from above, we have for $\zeta_{jk}(x,y):=\eta_k(x)\phi_j^{\tau}(y)$ the resulting contradiction

$$0=\int_c^d\int_a^b M\zeta_{jk}^{\sigma\tau}\Gamma x\Delta y=\int_{v(y_0)}^{t_j}\int_{\rho(x_0)}^{v_k} M\zeta_{jk}^{\sigma\tau}\Gamma x\Delta y=\int_{y_0}^{t_j}\int_{\rho(x_0)}^{v_k} M\zeta_{jk}^{\sigma\tau}\Gamma x\Delta y$$

$$=\int_{y_0}^{\tau(y_0)}\int_{\rho(x_0)}^{x_0} M\zeta_{jk}^{\sigma\tau}\Gamma x\Delta y+\int_{y_0}^{\tau(y_0)}\int_{x_0}^{\sigma(x_0)} M\zeta_{jk}^{\sigma\tau}\Gamma x\Delta y+\int_{y_0}^{\tau(y_0)}\int_{\sigma(x_0)}^{v_k} M\zeta_{jk}^{\sigma\tau}\Gamma x\Delta y$$

$$+\int_{\tau(y_0)}^{t_j}\int_{\rho(x_0)}^{x_0} M\zeta_{jk}^{\sigma\tau}\Gamma x\Delta y+\int_{\tau(y_0)}^{t_j}\int_{x_0}^{\sigma(x_0)} M\zeta_{jk}^{\sigma\tau}\Gamma x\Delta y+\int_{\tau(y_0)}^{t_j}\int_{\sigma(x_0)}^{v_k} M\zeta_{jk}^{\sigma\tau}\Gamma x\Delta y$$

$$\to M(x_0,y_0)h(x_0)k(y_0)>0 \quad \text{as } j,\ k\to\infty. \qquad \square$$

4. Double integral calculus of variations

Bohner's [9] single integral variational calculus on time scales is now extended to double integral variational calculus on time scales. Discrete variational theory was summarized in Chap. 4 and 5 of Ref. [6]. Consider time scales $\mathbb{X}=[a,b]$ and $\mathbb{Y}=[c,d]$. Let $R=[a,b]\times[c,d]$. Consider a functional J defined by

$$J(z)=\int_c^d\int_a^b L(x,y,z(\sigma(x),\tau(y)),z^{\Gamma}(x,\tau(y)),z^{\Delta}(\sigma(x),y))\Gamma x\Delta y, \tag{14}$$

where $L(x,y,z,p,q)$ is a continuous function defined on $\mathbb{X}\times\mathbb{Y}\times\mathbb{R}^3$, (where $\mathbb{R}$ is the reals), and is C^2 in the last three variables. Following Bohner, we define the norm

$$\|f\|=\max_{(x,y)\in R^i}|f(\sigma(x),\tau(y))|+\max_{(x,y)\in R^i}|(f(x,\tau(y)))^{\Gamma}|+\max_{(x,y)\in R^i}|(f(\sigma(x),y))^{\Delta}|.$$

Consider the collection of all $z(x,y)\in CC^1_{\mathrm{rd}}$ in R which become a given continuous function on the boundary of R. We call such functions *admissible*. A function $\hat{z}\in CC^1_{\mathrm{rd}}$ is called a *weak local minimum* for $J(z)$ of Eq. (14) provided there exists $\delta>0$ such that $J(\hat{z})\leqslant J(z)$ for all $z\in CC^1_{\mathrm{rd}}$ with z admissible and $\|z-\hat{z}\|<\delta$. A function $\zeta\in CC^1_{\mathrm{rd}}$ is called an *admissible variation* provided $\zeta=0$ on the boundary of R.

Theorem 9 (Euler–Lagrange). *Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded time scales each containing at least three points. Set $a:=\min\mathbb{X}$, $b:=\max\mathbb{X}$, $c:=\min\mathbb{Y}$, and $d:=\max\mathbb{Y}$. Let $R=\mathbb{X}\times\mathbb{Y}=[a,b]\times[c,d]$.*

Consider a functional J defined by (14). *Assume that L has the properties*

$$\int_c^d \int_a^b L_z \Gamma x \Delta y = \int_a^b \int_c^d L_z \Delta y \Gamma x, \tag{15}$$

$$\int_c^d \int_a^b L_p \Gamma x \Delta y = \int_a^b \int_c^d L_p \Delta y \Gamma x, \tag{16}$$

$$\int_c^d \int_a^b L_q \Gamma x \Delta y = \int_a^b \int_c^d L_q \Delta y \Gamma x. \tag{17}$$

If $\hat{z}$ is a weak local minimum for $J(z)$ of Eq. (14) *such that for functions $p(x,y) := \hat{z}^{\Gamma}(x,\tau(y))$ and $q(x,y) := \hat{z}^{\Delta}(\sigma(x),y)$, the function $\hat{z}(\sigma(x),\tau(y))$, is continuous on R^i, $L_p(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)$, $L_q(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)$, and $L_z(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)$, are continuous on R^i, and the partial derivatives $(L_p(x,y,\hat{z}(\sigma(x),\tau(y)),p,q))^{\Gamma}$, $(L_q(x,y,\hat{z}(\sigma(x),\tau(y)),p,q))^{\Delta}$ exist and are continuous on $(R^i)^i$, then the Euler–Lagrange equation*

$$L_z(x,y,\hat{z}(\sigma(x),\tau(y)),p,q) = (L_p(x,y,\hat{z}(\sigma(x),\tau(y)),p,q))^{\Gamma} + (L_q(x,y,\hat{z}(\sigma(x),\tau(y)),p,q))^{\Delta}, \tag{18}$$

holds for $(x,y) \in (R^i)^i$.

Proof. Let ζ be an admissible variation, i.e., a real-valued function on $R = \mathbb{X} \times \mathbb{Y}$ such that ζ^{Γ} and ζ^{Δ} both exist, are in $CC_{\rm rd}$, and ζ is zero on the boundary of the rectangle R. Let $Z(x,y,\varepsilon) = z(\sigma(x),\tau(y)) + \varepsilon\zeta(\sigma(x),\tau(y))$, $P(x,y,\varepsilon) = p(x,\tau(y)) + \varepsilon\zeta^{\Gamma}(x,\tau(y))$, and $Q(x,y,\varepsilon) = q(\sigma(x),y) + \varepsilon\zeta^{\Delta}(\sigma(x),y)$. Define

$$\Phi(\varepsilon) = \int_c^d \int_a^b L(x,y,Z,P,Q) \Gamma x \Delta y. \tag{19}$$

Then

$$\Phi'(\varepsilon) = \int_c^d \int_a^b \frac{\partial}{\partial \varepsilon} L(x,y,Z,P,Q) \Gamma x \Delta y \tag{20}$$

follows in a similar way to Bohner's proof [9] in the case of one independent variable.

Since $\hat{z}$ is a local minimum of (14) and $\varepsilon = 0$ is an interior point of the domain of Φ, $\Phi'(0) = 0$. Thus setting $\varepsilon = 0$ yields

$$\begin{aligned} 0 = \Phi'(0) &= \int_c^d \int_a^b L_z(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)\zeta(\sigma(x),\tau(y)) \Gamma x \Delta y \\ &\quad + \int_c^d \int_a^b L_p(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)\zeta^{\Gamma}(x,\tau(y)) \Gamma x \Delta y \\ &\quad + \int_c^d \int_a^b L_q(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)\zeta^{\Delta}(\sigma(x),y) \Gamma x \Delta y. \end{aligned}$$

Using formula (5) for integration by parts with respect to x, we have

$$\int_c^d \int_a^b L_p(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)\zeta^{\Gamma}(x,\tau(y))\Gamma x\Delta y$$

$$= -\int_c^d \int_a^b (L_p(x,y,\hat{z}(\sigma(x),\tau(y)),p,q))^{\Gamma}\zeta(\sigma(x),\tau(y))\Gamma x\Delta y$$

$$+\int_c^d L_p(b,y,\hat{z}(\sigma(b),\tau(y)),\hat{z}^{\Gamma}(b,\tau(y)),\hat{z}^{\Delta}(\sigma(b),y))\zeta(b,y)\Delta y$$

$$-\int_c^d L_p(a,y,\hat{z}(\sigma(a),\tau(y)),\hat{z}^{\Gamma}(a,\tau(y)),\hat{z}^{\Delta}(\sigma(a),y))\zeta(a,y)\Delta y$$

$$= -\int_c^d \int_a^b (L_p(x,y,\hat{z}(\sigma(x),\tau(y)),p,q))^{\Gamma}\zeta(\sigma(x),\tau(y))\Gamma x\Delta y$$

and formula (6) for integration by parts with respect to y, we have

$$\int_c^d \int_a^b L_q(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)\zeta^{\Delta}(\sigma(x),y)\Gamma x\Delta y$$

$$= -\int_a^b \int_c^d (L_q(x,y,\hat{z}(\sigma(x),\tau(y)),p,q))^{\Delta}\zeta(\sigma(x),\tau(y))\Delta y\Gamma x$$

$$+\int_a^b L_q(x,y,\hat{z}(\sigma(x),\tau(d))\hat{z}^{\Gamma}(\sigma(x),d),\hat{z}^{\Delta}(x,\tau(d)))\zeta(x,d)\Gamma x$$

$$-\int_a^b L_q(x,y,\hat{z}(\sigma(x),\tau(c))\hat{z}^{\Gamma}(\sigma(x),c),\hat{z}^{\Delta}(x,\tau(c)))\zeta(x,c)\Gamma x$$

$$= -\int_a^b \int_c^d (L_q(x,y,\hat{z}(\sigma(x),\tau(y)),p,q))^{\Delta}\zeta(\sigma(x),\tau(y))\Delta y\Gamma x.$$

Thus

$$\int_c^d \int_a^b L_z(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)\zeta(\sigma(x),\tau(y))\Gamma x\Delta y$$

$$+\int_c^d \int_a^b L_p(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)\zeta^{\Gamma}(x,\tau(y))\Gamma x\Delta y$$

$$+\int_c^d\int_a^b L_q(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)\zeta^{\Delta}(\sigma(x),y)\Gamma x\Delta y$$

$$=\int_c^b\int_a^b (L_z(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)-L_p(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)^{\Gamma}$$

$$-L_q(x,y,\hat{z}(\sigma(x),\tau(y)),p,q)^{\Delta})\zeta(\sigma(x),\tau(y))\Gamma x\Delta y=0. \tag{21}$$

By Lemma 8, the Euler–Lagrange equation (18) holds for $(x,y)\in(R^i)^i$.

The discrete version, given in symmetric form (i.e., formally self-adjoint form) by Ahlbrandt and Harmsen in [5], of the two variable Euler–Lagrange equation uses the notation $z(i,j):=z(x_i,y_j)$ and

$$p_{i,j}=\frac{1}{\Delta x_{i-1}}[z(x_i,y_j)-z(x_{i-1},y_j)]\equiv\frac{\Delta_i z_{i-1,j}}{\Delta x_{i-1}}, \tag{22}$$

$$q_{i,j}=\frac{1}{\Delta y_{j-1}}[z(x_i,y_j)-z(x_i,y_{j-1})]\equiv\frac{\Delta_j z_{i,j-1}}{\Delta y_{j-1}} \tag{23}$$

in the discrete Euler–Lagrange equation

$$L_z(x_{i-1},y_{j-1},x_i,y_j,z_{i,j},p_{i,j},q_{i,j})=\frac{1}{\Delta x_{i-1}}\Delta_i L_p(x_{i-1},y_{j-1},x_i,y_j,z_{i,j},p_{i,j},q_{i,j})$$

$$+\frac{1}{\Delta y_{j-1}}\Delta_j L_q(x_{i-1},y_{j-1},x_i,y_j,z_{i,j},p_{i,j},q_{i,j}). \tag{24}$$

5. A double integral Picone identity

Bohner and Agarwal [2] derived a single independent variable Picone identity on time scales. This can be generalized to a two independent variable Picone identity for elliptic operators on time scales which will reduce to the continuous Picone identity on a rectangle. Then this Picone identity can be used to prove a two-variable Sturm–Picone comparison theorem which generalizes Kreith's result [18] in the continuous case except for the fact that Kreith's result allows nonrectangular domains.

Theorem 10 (a Picone identity). *Let $\mathbb{X}$ and $\mathbb{Y}$ be time scales. Suppose that $m(x,y)$ and $M(x,y)$ are continuous 2×2 positive definite diagonal matrices defined on $\mathbb{X}\times\mathbb{Y}$ with m and M in CC^1_{rd}. Let $P(x,y)$ and $p(x,y)$ be in CC_{rd}. Suppose that $u(x,y)$ and $v(x,y)$ are solutions of*

$$(m_{11}(u^{\tau})^{\Gamma})^{\Gamma}+(m_{22}(u^{\sigma})^{\Delta})^{\Delta}+pu^{\sigma\tau}=0 \tag{25}$$

and

$$(M_{11}(v^{\tau})^{\Gamma})^{\Gamma}+(M_{22}(v^{\sigma})^{\Delta})^{\Delta}+Pv^{\sigma\tau}=0, \tag{26}$$

on $\mathbb{X}\times\mathbb{Y}$, respectively. Also assume that $v(x,y)$ is never 0 for all $(x,y)\in\mathbb{X}\times\mathbb{Y}$. Then

$$(u^\tau m_{11}(u^\tau)^\Gamma)^\Gamma + (u^\sigma m_{22}(u^\sigma)^\Delta)^\Delta - \left(\frac{(u^\tau)^2}{v^\tau}M_{11}(v^\tau)^\Gamma\right)^\Gamma - \left(\frac{(u^\sigma)^2}{v^\sigma}M_{22}(v^\sigma)^\Delta\right)^\Delta$$

$$= (P-p)(u^{\sigma\tau})^2 + [(u^\tau)^\Gamma \quad (u^\sigma)^\Delta](m-M)[(u^\tau)^\Gamma \quad (u^\sigma)^\Delta]^{\mathrm{T}}$$

$$+\begin{pmatrix}(u^\tau)^\Gamma - \frac{u^\tau}{v^\tau}(v^\tau)^\Gamma\\ (u^\sigma)^\Delta - \frac{u^\sigma}{v^\sigma}(v^\sigma)^\Delta\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\frac{v^\tau}{v^{\sigma\tau}}M_{11} & 0\\ 0 & \frac{v^\sigma}{v^{\sigma\tau}}M_{22}\end{pmatrix}\begin{pmatrix}(u^\tau)^\Gamma - \frac{u^\tau}{v^\tau}(v^\tau)^\Gamma\\ (u^\sigma)^\Delta - \frac{u^\sigma}{v^\sigma}(v^\sigma)^\Delta\end{pmatrix}. \tag{27}$$

Proof. The proof is based on Kreith's proof [18] with the modifications for derivatives on time scales used by Agarwal and Bohner [2]. Calculating the first two terms of the left-hand side of Eq. (27) yields

$$(u^\tau m_{11}(u^\tau)^\Gamma)^\Gamma + (u^\sigma m_{22}(u^\sigma)^\Delta)^\Delta$$

$$= (u^\tau)^\Gamma m_{11}(u^\tau)^\Gamma + u^{\sigma\tau}(m_{11}(u^\tau)^\Gamma)^\Gamma + (u^\sigma)^\Delta m_{22}(u^\sigma)^\Delta + u^{\sigma\tau}(m_{22}(u^\sigma)^\Delta)^\Delta$$

$$= u^{\sigma\tau}((m_{11}(u^\tau)^\Gamma)^\Gamma + (m_{22}(u^\sigma)^\Delta)^\Delta) + m_{11}((u^\tau)^\Gamma)^2 + m_{22}((u^\sigma)^\Delta)^2$$

$$= -\,p(u^{\sigma\tau})^2 + m_{11}((u^\tau)^\Gamma)^2 + m_{22}((u^\sigma)^\Delta)^2. \tag{28}$$

To calculate the second pair of terms of the left-hand side of Eq. (27) we need to first calculate

$$\left(\frac{(u^\tau)^2}{v^\tau}M_{11}(v^\tau)^\Gamma\right)^\Gamma + \left(\frac{(u^\sigma)^2}{v^\sigma}M_{22}(v^\sigma)^\Delta\right)^\Delta$$

$$+\frac{v^\tau}{v^{\sigma\tau}}M_{11}\left((u^\tau)^\Gamma - \frac{u^\tau}{v^\tau}(v^\tau)^\Gamma\right)^2 + \frac{v^\sigma}{v^{\sigma\tau}}M_{22}\left((u^\sigma)^\Delta - \frac{u^\sigma}{v^\sigma}(v^\sigma)^\Delta\right)^2$$

$$= \frac{(u^{\sigma\tau})^2}{v^{\sigma\tau}}(M_{11}(v^\tau)^\Gamma)^\Gamma + \frac{(u^{\sigma\tau})^2}{v^{\sigma\tau}}(M_{22}(v^\sigma)^\Delta)^\Delta + M_{11}\left(\frac{u^\tau(u^\tau)^\Gamma(v^\tau)^\Gamma}{v^{\sigma\tau}} + \frac{u^{\sigma\tau}(u^\tau)^\Gamma(v^\tau)^\Gamma}{v^{\sigma\tau}}\right.$$

$$\left.-\frac{(u^\tau)^2((v^\tau)^\Gamma)^2}{v^\tau v^{\sigma\tau}} + \frac{v^\tau}{v^{\sigma\tau}}((u^\tau)^\Gamma)^2 - \frac{2(u^\tau)^\Gamma(v^\tau)^\Gamma u^\tau}{v^{\sigma\tau}} + \frac{(u^\tau)^2((v^\tau)^\Gamma)^2}{v^{\sigma\tau}v^\tau}\right)$$

$$+M_{22}\left(\frac{u^\sigma(u^\sigma)^\Delta(v^\sigma)^\Delta}{v^{\sigma\tau}} + \frac{u^{\sigma\tau}(u^\sigma)^\Delta(v^\sigma)^\Delta}{v^{\sigma\tau}}\right.$$

$$\left.-\frac{(u^\sigma)^2((v^\sigma)^\Delta)^2}{v^\sigma v^{\sigma\tau}} + \frac{v^\sigma}{v^{\sigma\tau}}((u^\sigma)^\Delta)^2 - \frac{2(u^\sigma)^\Delta(v^\sigma)^\Delta u^\sigma}{v^{\sigma\tau}} + \frac{(u^\sigma)^2((v^\sigma)^\Delta)^2}{v^\sigma v^{\sigma\tau}}\right)$$

$$= \frac{(u^{\sigma\tau})^2}{v^{\sigma\tau}}((M_{11}(v^{\tau})^{\Gamma})^{\Gamma} + (M_{22}(v^{\sigma})^{\Delta})^{\Delta})$$

$$+ M_{11}\left(\frac{v^{\tau}}{v^{\sigma\tau}}((u^{\tau})^{\Gamma})^2 + (u^{\sigma\tau} - u^{\tau})\frac{(u^{\tau})^{\Gamma}(v^{\tau})^{\Gamma}}{v^{\sigma\tau}}\right) + M_{22}\left(\frac{v^{\sigma}}{v^{\sigma\tau}}((u^{\sigma})^{\Delta})^2 + (u^{\sigma\tau} - u^{\sigma})\frac{(u^{\sigma})^{\Delta}(v^{\sigma})^{\Delta}}{v^{\sigma\tau}}\right)$$

$$= \frac{(u^{\sigma\tau})^2}{v^{\sigma\tau}}(-Pv^{\sigma\tau}) + M_{11}\left(\frac{v^{\tau}}{v^{\sigma\tau}}((u^{\tau})^{\Gamma})^2 + \frac{h(u^{\tau})^{\Gamma}(u^{\tau})^{\Gamma}(v^{\tau})^{\Gamma}}{v^{\sigma\tau}}\right)$$

$$+ M_{22}\left(\frac{v^{\sigma}}{v^{\sigma\tau}}((u^{\sigma})^{\Delta})^2 + \frac{k(u^{\sigma})^{\Delta}(u^{\sigma})^{\Delta}(v^{\sigma})^{\Delta}}{v^{\sigma\tau}}\right)$$

$$= -P(u^{\sigma\tau})^2 + M_{11}\left(\frac{v^{\tau} + h(v^{\tau})^{\Gamma}}{v^{\sigma\tau}}\right)((u^{\tau})^{\Gamma})^2 + M_{22}\left(\frac{v^{\sigma} + k(v^{\sigma})^{\Delta}}{v^{\sigma\tau}}\right)((u^{\sigma})^{\Delta})^2$$

$$= -P(u^{\sigma\tau})^2 + M_{11}\frac{v^{\sigma\tau}}{v^{\sigma\tau}}((u^{\tau})^{\Gamma})^2 + M_{22}\frac{v^{\sigma\tau}}{v^{\sigma\tau}}((u^{\sigma})^{\Delta})^2$$

$$= -P(u^{\sigma\tau})^2 + M_{11}((u^{\tau})^{\Gamma})^2 + M_{22}((u^{\sigma})^{\Delta})^2.$$

Thus the negative of the second pair of terms of the left-hand side of Eq. (27) are determined by

$$\left(\frac{(u^{\tau})^2}{v^{\tau}}M_{11}(v^{\tau})^{\Gamma}\right)^{\Gamma} + \left(\frac{(u^{\sigma})^2}{v^{\sigma}}M_{22}(v^{\sigma})^{\Delta}\right)^{\Delta}$$

$$= -P(u^{\sigma\tau})^2 + M_{11}((u^{\tau})^{\Gamma})^2 + M_{22}((u^{\sigma})^{\Delta})^2 - \frac{v^{\tau}}{v^{\sigma\tau}}M_{11}\left((u^{\tau})^{\Gamma} - \frac{u^{\tau}}{v^{\tau}}(v^{\tau})^{\Gamma}\right)^2$$

$$-\frac{v^{\sigma}}{v^{\sigma\tau}}M_{22}\left((u^{\sigma})^{\Delta} - \frac{u^{\sigma}}{v^{\sigma}}(v^{\sigma})^{\Delta}\right)^2. \tag{29}$$

The left-hand side of the Picone Identity, Eq. (27), is obtained by subtracting (29) from (28) for

$$(u^{\tau}m_{11}(u^{\tau})^{\Gamma})^{\Gamma} + (u^{\sigma}m_{22}(u^{\sigma})^{\Delta})^{\Delta} - \left(\frac{(u^{\tau})^2}{v^{\tau}}M_{11}(v^{\tau})^{\Gamma}\right)^{\Gamma} - \left(\frac{(u^{\sigma})^2}{v^{\sigma}}M_{22}(v^{\sigma})^{\Delta}\right)^{\Delta}$$

$$= -p(u^{\sigma\tau})^2 + m_{11}((u^{\tau})^{\Gamma})^2 + m_{22}((u^{\sigma})^{\Delta})^2 - (-P(u^{\sigma\tau})^2 + M_{11}((u^{\tau})^{\Gamma})^2 + M_{22}((u^{\sigma})^{\Delta})^2$$

$$-\frac{v^{\tau}}{v^{\sigma\tau}}M_{11}\left((u^{\tau})^{\Gamma} - \frac{u^{\tau}}{v^{\tau}}(v^{\tau})^{\Gamma}\right)^2 - \frac{v^{\sigma}}{v^{\sigma\tau}}M_{22}\left((u^{\sigma})^{\Delta} - \frac{u^{\sigma}}{v^{\sigma}}(v^{\sigma})^{\Delta}\right)^2\Bigg)$$

$$= (P - p)(u^{\sigma\tau})^2 + ((u^\tau)^\Gamma, (u^\sigma)^\Delta)(m - M)((u^\tau)^\Gamma, (u^\sigma)^\Delta)^{\mathrm{T}}$$

$$+ \frac{v^\tau}{v^{\sigma\tau}} M_{11} \left((u^\tau)^\Gamma - \frac{u^\tau}{v^\tau}(v^\tau)^\Gamma \right)^2 + \frac{v^\sigma}{v^{\sigma\tau}} M_{22} \left((u^\sigma)^\Delta - \frac{u^\sigma}{v^\sigma}(v^\sigma)^\Delta \right)^2$$

$$= (P - p)(u^{\sigma\tau})^2 + ((u^\tau)^\Gamma, (u^\sigma)^\Delta)(m - M)((u^\tau)^\Gamma, (u^\sigma)^\Delta)^{\mathrm{T}}$$

$$+ \begin{pmatrix} (u^\tau)^\Gamma - \frac{u^\tau}{v^\tau}(v^\tau)^\Gamma \\ (u^\sigma)^\Delta - \frac{u^\sigma}{v^\sigma}(v^\sigma)^\Delta \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \frac{v^\tau}{v^{\sigma\tau}} M_{11} & 0 \\ 0 & \frac{v^\sigma}{v^{\sigma\tau}} M_{22} \end{pmatrix} \begin{pmatrix} (u^\tau)^\Gamma - \frac{u^\tau}{v^\tau}(v^\tau)^\Gamma \\ (u^\sigma)^\Delta - \frac{u^\sigma}{v^\sigma}(v^\sigma)^\Delta \end{pmatrix}$$

which gives the Picone Identity, Eq. (27). This result can be extended in a natural way to 3×3 diagonal matrices. That result can be found in [21].

6. A Sturm–Picone comparison theorem

Picone identities on time scales can be used to prove Sturm–Picone comparison theorems on time scales. Because of discrete cases, second-order equations are only satisfied on $(R^i)^i$. Before using the Picone identity for Sturmian theory let us start by considering compact time scales $\mathbb{X}$ and $\mathbb{Y}$, each of which contain at least six points. Let $a = \min \mathbb{X}$ and $c = \min \mathbb{Y}$ and suppose $b \in (\mathbb{X}^i)^i$ and $d \in (\mathbb{Y}^i)^i$ are such that $[a, b]$ and $[c, d]$ each contain at least four time scale points. Let R be the time scale region $[a, b] \times [c, d]$. Introduce some notation. Let R^{o} denote the set

$$R^{\mathrm{o}} = \{(x, y) \in \mathbb{X} \times \mathbb{Y}: a < x < b, c < y < d\} \tag{30}$$

and R^{j}, the *jump set*, is defined by

$$R^{\mathrm{j}} = \{(x, y) \in \mathbb{X} \times \mathbb{Y}: (x, y), (\sigma(x), y), (x, \tau(y)), (\sigma(x), \tau(y)) \in R^{\mathrm{o}}\}. \tag{31}$$

Define R^{s}, the *shadow set*, by

$$R^{\mathrm{s}} = \{(x, y) \in \mathbb{X} \times \mathbb{Y}: a \leqslant x \leqslant \sigma(b), c \leqslant y \leqslant \tau(d)\}. \tag{32}$$

Note that R, R^{j}, R^{o}, and R^{s} are all nonempty.

The continuous case of the following Sturm–Picone comparison theorem is a result of Kreith [18].

Theorem 11. *Assume that $\mathbb{X}$ and $\mathbb{Y}$ are compact time scales each containing at least six points. Let a, b, c, d be as above. Assume that $u(x, y)$ and $v(x, y)$ satisfy* (25), *and* (26), *respectively*, *on R with $u(x, y) = 0$ on the boundary of the rectangle R and $u > 0$ in R^{o}. Suppose that in R*

(i) *$P(x, y) \geqslant p(x, y)$, $P \not\equiv p$ in R^{j};*
(ii) *$0 < \xi^{\mathrm{T}} M \xi \leqslant \xi^{\mathrm{T}} m \xi$ for all real vectors $\xi \neq 0$.*

Then $v(x, y)$ has a zero in R^{s}.

Proof. Assume $v(x, y) > 0$ on R^s. Since $u = 0$ on the boundary of R,

$$\int_c^d \int_a^b (u^\tau m_{11}(u^\tau)^\Gamma)^\Gamma \Gamma x \Delta y$$

$$= \int_c^d \{[u(x, \tau(y))m_{11}(x, y)(u(x, \tau(y)))^\Gamma]_{x=a}^{x=b}\}\Delta y$$

$$= \int_c^d \{u(b, \tau(y))m_{11}(b, y)(u(b, \tau(y)))^\Gamma - u(a, \tau(y))m_{11}(a, y)(u(a, \tau(y)))^\Gamma\}\Delta y = 0, \qquad (33)$$

$$\int_c^d \int_a^b (u^\sigma m_{22}(u^\sigma)^\Delta)^\Delta \Gamma x \Delta y$$

$$= \int_a^b \int_c^d (u^\sigma m_{22}(u^\sigma)^\Delta)^\Delta \Delta y \Gamma x$$

$$= \int_a^b \{[(u(\sigma(x), y)m_{22}(x, y)(u(\sigma(x), y))^\Delta]_{y=c}^{y=d}\}\Gamma x$$

$$= \int_a^b \{(u(\sigma(x), d)m_{22}(x, d)(u(\sigma(x), d))^\Delta - (u(\sigma(x), c)m_{22}(x, c)(u(\sigma(x), c))^\Delta\}\Gamma x = 0, \qquad (34)$$

$$\int_c^d \int_a^b \left(\frac{(u^\tau)^2}{v^\tau} M_{11}(v^\tau)^\Gamma\right)^\Gamma \Gamma x \Delta y$$

$$= \int_c^d \left\{\left[\frac{(u(x, \tau(y)))^2}{v(x, \tau(y))} M_{11}(x, y)(v(x, \tau(y)))^\Gamma\right]_{x=a}^{x=b}\right\}\Delta y$$

$$= \int_c^d \left\{\frac{(u(b, \tau(y)))^2}{v(b, \tau(y))} M_{11}(b, y)(v(b, \tau(y)))^\Gamma - \frac{(u(a, \tau(y)))^2}{v(a, \tau(y))} M_{11}(a, y)(v(a, \tau(y)))^\Gamma\right\}\Delta y = 0, \qquad (35)$$

and

$$\int_c^d \int_a^b \left(\frac{(u^\sigma)^2}{v^\sigma} M_{22}(v^\sigma)^\Delta\right)^\Delta \Gamma x \Delta y$$

$$= \int_a^b \int_c^d \left(\frac{(u^\sigma)^2}{v^\sigma} M_{22}(v^\sigma)^\Delta\right)^\Delta \Delta y \Gamma x$$

$$= \int_a^b \left\{ \left[\frac{(u(\sigma(x),y))^2}{v(\sigma(x),y)} M_{22}(x,y)(v(\sigma(x),y))^{\Delta} \right]_{y=c}^{y=d} \right\} \Gamma x$$

$$= \int_a^b \left\{ \frac{(u(\sigma(x),d))^2}{v(\sigma(x),d)} M_{22}(x,d)(v(\sigma(x),d))^{\Delta} - \frac{(u(\sigma(x),c))^2}{v(\sigma(x),c)} M_{22}(x,c)(v(\sigma(x),c))^{\Delta} \right\} \Gamma x = 0. \tag{36}$$

Then by the Picone identity (27) and (33), (34), (35), (36), we have

$$0 = \int_c^d \int_a^b (u^{\tau} m_{11}(u^{\tau})^{\Gamma})^{\Gamma} + (u^{\sigma} m_{22}(u^{\sigma})^{\Delta})^{\Delta} - \left(\frac{(u^{\tau})^2}{v^{\tau}} M_{11}(v^{\tau})^{\Gamma} \right)^{\Gamma} - \left(\frac{(u^{\sigma})^2}{v^{\sigma}} M_{22}(v^{\sigma})^{\Delta} \right)^{\Delta} \Gamma x \Delta y$$

$$= \int_c^d \int_a^b (P-p)(u^{\sigma\tau})^2 + [(u^{\tau})^{\Gamma} \quad (u^{\sigma})^{\Delta}](m-M)[(u^{\tau})^{\Gamma} \quad (u^{\sigma})^{\Delta}]^{\mathrm{T}}$$

$$+ \begin{bmatrix} (u^{\tau})^{\Gamma} - \frac{u^{\tau}}{v^{\tau}}(v^{\tau})^{\Gamma} \\ (u^{\sigma})^{\Delta} - \frac{u^{\sigma}}{u^{\sigma}}(v^{\sigma})^{\Delta} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{v^{\tau}}{v^{\sigma\tau}} M_{11} & 0 \\ 0 & \frac{v^{\sigma}}{v^{\sigma\tau}} M_{22} \end{bmatrix} \begin{bmatrix} (u^{\tau})^{\Gamma} - \frac{u^{\tau}}{v^{\tau}}(v^{\tau})^{\Gamma} \\ (u^{\sigma})^{\Delta} - \frac{u^{\sigma}}{u^{\sigma}}(v^{\sigma})^{\Delta} \end{bmatrix} \Gamma x \Delta y$$

$$\geqslant \int_c^d \int_a^b \left\{ (P-p)(u^{\sigma\tau})^2 + [(u^{\tau})^{\Gamma} \quad (u^{\sigma})^{\Delta}](m-M)[(u^{\tau})^{\Gamma} \quad (u^{\sigma})^{\Delta}]^{\mathrm{T}} \right\} \Gamma x \Delta y.$$

Since $P \not\equiv p$ and $P \geqslant p$ on R^{j} there exists a point $(x_1, y_1) \in R^{\mathrm{j}}$ such that $(P-p)(x_1, y_1) > 0$. Since $(x_1, y_1) \in R^{\mathrm{j}}$, $(\sigma(x_1), \tau(y_1)) \in R^{\mathrm{o}}$. Then $(\sigma(x_1), \tau(y_1)) \in R^{\mathrm{o}}$ and $u((\sigma(x_1), \tau(y_1)) > 0$ so $(u((\sigma(x_1), \tau(y_1)))^2 > 0$. Thus the term

$$(P-p)(x_1, y_1)(u((\sigma(x_1), \tau(y_1)))^2$$

is positive and as a consequence of Proposition 7, the last integral of the previous display is positive. Thus we have a contradiction. Therefore v must have a zero on R^{s}.

The above Picone identity (27) may be generalized to 2×2 positive-definite symmetric matrices. Further work in extending Kreith's papers [19,20] may be found in [21]

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