Simplest miniversal deformations of matrices, matrix pencils, and contragredient matrix pencils

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Abstract

For a family of linear operators $A(\vec{\lambda}) : U \to U$ over $\mathbb{C}$ that smoothly depend on parameters $\vec{\lambda} = (\lambda_1, \ldots, \lambda_k)$, V.I. Arnold obtained the simplest normal form of their matrices relative to a smoothly depending on $\vec{\lambda}$ change of a basis in U. We solve the same problem for a family of linear operators $A(\vec{\lambda}) : U \to U$ over $\mathbb{R}$, for a family of pairs of linear mappings $A(\vec{\lambda}) : U \to V$, $B(\vec{\lambda}) : U \to V$ over $\mathbb{C}$ and $\mathbb{R}$, and for a family of pairs of counter linear mappings $A(\vec{\lambda}) : U \to V$, $B(\vec{\lambda}) : V \to U$ over $\mathbb{C}$ and $\mathbb{R}$. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

All matrices and representations are considered over a field $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. We base on ideas and methods from Arnold's paper [1], extending them on quiver representations.

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Systems of linear mappings are conveniently studied if we consider them as representations of a quiver. A *quiver* is a directed graph, its *representation A* over $\mathbb{F}$ is given by assigning to each vertex i a finite-dimensional vector space A_i over $\mathbb{F}$ and to each arrow $\alpha : i \to j$ a linear mapping $A_\alpha : A_i \to A_j$. For example, the problems of classifying representations of the quivers

are the problems of classifying, respectively, linear operators $A : U \to U$ (its solution is the Jordan normal form), pairs of linear mappings $A : U \to V$, $B : U \to V$ (the matrix pencil problem, solved by Kronecker), and pairs of counter linear mappings $A : U \to V$, $B : V \to U$ (the contragredient matrix pencil problem, solved in [2] and studied in detail in [3]).

Studying families of quiver representations smoothly depending on parameters, we can independently reduce each representation to canonical form, but then we lose the smoothness (and even the continuity) relative to the parameters. It leads to the problem of reducing to normal form by a smoothly depending on parameters change of bases not only the matrices of a given representation, but of an arbitrary family of representations close to it. This normal form is obtained from the normal form of matrices of the given representation by adding to some of their entries holomorphic functions of the parameters that are zero for the zero value of parameters. The number of these entries must be minimal to obtain the simplest normal form.

This problem for representations of the quiver $\cdot\circlearrowleft$ over $\mathbb{C}$ was solved by Arnold [1] (see also [4, Section 30]). We solve it for holomorphically depending on parameters representations of the quiver $\cdot\circlearrowleft$ over $\mathbb{R}$ and representations of the quivers $\cdot\rightrightarrows\cdot$ and $\cdot\rightleftarrows\cdot$ both over $\mathbb{C}$ and over $\mathbb{R}$. In the obtained simplest normal forms, all the summands to entries are independent parameters. A normal form with the minimal number of independent parameters, but not of the summands to entries, was obtained in [5] (see also [4, Section 30E]) for representations of the quiver $\cdot\circlearrowleft$ over $\mathbb{R}$ and in [6] (partial cases were considered in [7,8]) for representations of the quiver $\cdot\rightrightarrows\cdot$ over $\mathbb{C}$.

2. Deformations of quiver representations

Let Q be a quiver with vertices $1, \ldots, t$. Its *matrix representation A* of dimension $\vec{n} = (n_1, \ldots, n_t) \in \{0, 1, 2, \ldots\}^{\mathrm{t}}$ over $\mathbb{F}$ is given by assigning a matrix $A_\alpha \in \mathbb{F}^{n_j \times n_i}$ to each arrow $\alpha : i \to j$. Denote by $\mathscr{R}(\vec{n}, \mathbb{F})$ the vector space of all matrix representations of dimension $\vec{n}$ over $\mathbb{F}$. An *isomorphism* $S : A \to B$ of $A, B \in \mathscr{R}(\vec{n}, \mathbb{F})$ is given by a sequence $S = (S_1, \ldots, S_t)$ of non-singular matrices $S_i \in \mathrm{Gl}(n_i, \mathbb{F})$ such that $B_\alpha = S_j A_\alpha S_i^{-1}$ for each arrow $\alpha : i \to j$.

By an $\mathbb{F}$-*deformation* of $A \in \mathscr{R}(\vec{n}, \mathbb{F})$ is meant a parametric matrix representation $\mathscr{A}(\lambda_1, \ldots, \lambda_k)$ (or for short $\mathscr{A}(\vec{\lambda})$, where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_k)$), whose

entries are convergent in a neighborhood of $\vec{0}$ power series of variables (they are called *parameters*) $\lambda_1, \ldots, \lambda_k$ over $\mathbb{F}$ such that $\mathscr{A}(\vec{0}) = A$.

Two deformations $\mathscr{A}(\vec{\lambda})$ and $\mathscr{B}(\vec{\lambda})$ of $A \in \mathscr{R}(\vec{n}, \mathbb{F})$ are called *equivalent* if there exists a *deformation* $\mathscr{I}(\vec{\lambda})$ (its entries are convergent in a neighborhood of $\vec{0}$ power series and $\mathscr{I}(\vec{0}) = I$) of the identity isomorphism $I = (I_{n_1}, \ldots, I_{n_t})$: $A \to A$ such that

$$\mathscr{B}_\alpha(\vec{\lambda}) = \mathscr{I}_j(\vec{\lambda})\mathscr{A}_\alpha(\vec{\lambda})\mathscr{I}_i^{-1}(\vec{\lambda}), \quad \alpha : i \to j,$$

in a neighborhood of $\vec{0}$.

A deformation $\mathscr{A}(\lambda_1, \ldots, \lambda_k)$ of A is called *versal* if every deformation $\mathscr{B}(\mu_1, \ldots, \mu_l)$ of A is equivalent to a deformation $\mathscr{A}(\varphi_1(\vec{\mu}), \ldots, \varphi_k(\vec{\mu}))$, where $\varphi_i(\vec{\mu})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = \vec{0}$. A versal deformation $\mathscr{A}(\lambda_1, \ldots, \lambda_k)$ of A is called *miniversal* if there is no versal deformation having less than k parameters.

For a matrix representation $A \in \mathscr{R}(\vec{n}, \mathbb{F})$ and a sequence $C = (C_1, \ldots, C_t)$, $C_i \in \mathbb{F}^{n_i \times n_i}$, we define the matrix representation $[C, A] \in \mathscr{R}(\vec{n}, \mathbb{F})$ as follows:

$$[C, A]_\alpha = C_j A_\alpha - A_\alpha C_i, \quad \alpha : i \to j.$$

A miniversal deformation $\mathscr{A}(\lambda_1, \ldots, \lambda_k)$ of A will be called *simplest* if it is obtained from A by adding to certain k of its entries, respectively, λ_1 to the first, λ_2 to the second , ..., and λ_k to the kth. The next theorem is a simple conclusion of a well-known fact.

Theorem 2.1. *Let* $\mathscr{A}(\vec{\lambda}) = A + \mathscr{B}(\vec{\lambda})$, $\vec{\lambda} = (\lambda_1, \ldots, \lambda_k)$, *be an* $\mathbb{F}$-*deformation of a matrix representation* $A \in \mathscr{R}(\vec{n}, \mathbb{F})$, $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, *where* k *entries of* $\mathscr{B}(\vec{\lambda})$ *are the independent parameters* $\lambda_1, \ldots, \lambda_k$ *and the other entries are zeros. Then* $\mathscr{A}(\vec{\lambda})$ *is a simplest miniversal deformation of* A *if and only if*

$$\mathscr{R}(\vec{n}, \mathbb{F}) = \mathscr{P}_\mathscr{A} \oplus \mathscr{T}_A,$$

where $\mathscr{P}_\mathscr{A}$ *is the* k-*dimensional vector space of all* $\mathscr{B}(\vec{a})$, $\vec{a} \in \mathbb{F}^k$, *and* $\mathscr{T}_A$ *is the vector space of all* $[C, A]$, $C \in \mathbb{F}^{n_1 \times n_1} \times \cdots \times \mathbb{F}^{n_t \times n_t}$.

Proof. Two subspaces of a vector space V are *transversal* if their sum is equal to V. The class of all isomorphic to $A \in \mathscr{R}(\vec{n}, \mathbb{F})$ matrix representations may be considered as the orbit A^G of A under the following action of the group $G = \mathrm{GL}(n_1, \mathbb{F}) \times \cdots \times \mathrm{GL}(n_t, \mathbb{F})$ on the space $\mathscr{R}(\vec{n}, \mathbb{F})$:

$$A_\lambda^S = S_j A_\lambda S_i^{-1}, \quad \lambda : i \to j,$$

for all $A \in \mathscr{R}(\vec{n}, \mathbb{F})$, $S = (S_1, \ldots, S_t) \in G$, and arrows λ. A deformation $\mathscr{A}(\vec{\lambda})$ of a matrix representation $A \in \mathscr{R}(\vec{n}, \mathbb{F})$ is called a *transversal to the orbit* A^G *at the point* A if the space $\mathscr{R}(\vec{n}, \mathbb{F})$ is the sum of the space $\mathscr{A}_* \mathbb{F}^k$ (that is, of the image of

the linearization $\mathscr{A}_*$ of $\mathscr{A}(\vec{\lambda})$ near A; the linearization means that only first derivatives matter) and of the tangent space to the orbit A^G at the point A. The following, for example, ([9, Section 1.6] and [1]): *a transversal (of the minimal dimension) to the orbit is a (mini)versal deformation.*

It proves the theorem since $\mathscr{P}_\mathscr{A}$ is the space $\mathscr{A}_* \mathbb{F}^k$ and $\mathscr{T}_A$ is the tangent space to the orbit A^G at the point A; the last follows from

$$A_\lambda^{I+\varepsilon C} = (I + \varepsilon C_j)A_\lambda(I + \varepsilon C_i)^{-1} = (I + \varepsilon C_j)A_\lambda(I - \varepsilon C_i + \varepsilon^2 C_i - \cdots)$$

$$= A_\lambda + \varepsilon(C_j A_\lambda - A_\lambda C_i) + \varepsilon^2 \cdots$$

for all $C = (C_1, \ldots, C_t)$, $C_i \in \mathbb{F}^{n_i \times n_i}$, small ε, and arrows $\lambda : i \to j$. $\square$

Corollary 2.1. *There exists a simplest miniversal $\mathbb{F}$-deformation for every matrix representation over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$.*

Proof. Let $A \in \mathscr{R}(\vec{n}, \mathbb{F})$, let $T_1, \ldots, T_r$ be a basis of the space $\mathscr{T}_A$, and let $E_1, \ldots, E_l$ be the basis of $\mathscr{R}(\vec{n}, \mathbb{F})$ consisting of all matrix representations of dimension $\vec{n}$ such that each of theirs has one entry equaling 1 and the others equaling 0. Removing from the sequence $T_1, \ldots, T_r, E_1, \ldots, E_l$ every representation that is a linear combination of the preceding representations, we obtain a new basis $T_1, \ldots, T_r$ of the space $\mathscr{R}(\vec{n}, \mathbb{F})$. By Theorem 2.1, the deformation

$$\mathscr{A}(\lambda_1, \ldots, \lambda_k) = A + \lambda_1 E_{i_1} + \cdots + \lambda_k E_{i_k}$$

is a simplest miniversal deformation of A since $E_{i_1}, \ldots, E_{i_k}$ is a basis of $\mathscr{P}_\mathscr{A}$ and $\mathscr{R}(\vec{n}, \mathbb{F}) = \mathscr{P}_\mathscr{A} \oplus \mathscr{T}_A$. $\square$

By a set of canonical representations of a quiver Q, we mean an arbitrary set of "nice" matrix representations such that every class of isomorphic representations contains exactly one representation from it. Clearly, it suffices to study deformations of the canonical representations.

Arnold [1] obtained a simplest miniversal deformation of the Jordan matrices (i.e., canonical representations of the quiver $\cdot\!\circlearrowright$). In the rest of the paper, we obtain the simplest miniversal deformations of canonical representations of the quiver $\cdot\!\circlearrowright$ over $\mathbb{R}$ and of the quivers $\cdot\!\rightrightarrows\!\cdot$ and $\cdot\!\leftrightarrows\!\cdot$ both over $\mathbb{C}$ and over $\mathbb{R}$.

Remark 2.1. Arnold [1] proposed an easy method to obtain a miniversal (but not a simplest miniversal) deformation of a matrix under similarity by solving a certain system of linear equations. The method is of considerable current use (see [6–8,10]). Although we do not use it in the next sections, now we show how to extend this method to quiver representations.

The space $\mathscr{R}(\vec{n}, \mathbb{F})$ may be considered as a Euclidean space with scalar product

$$\langle A, B \rangle = \sum_{\alpha \in Q_1} \operatorname{tr}(A_\alpha B_\alpha^*),$$

where Q_1 is the set of arrows of Q and B_α^* is the adjoint of B_α.

Let $A \in \mathscr{R}(\vec{n}, \mathbb{F})$ and let $T_1, \ldots, T_k$ be a basis of the orthogonal complement $\mathscr{T}_A^\perp$ to the tangent space $\mathscr{T}_A$. The deformation

$$\mathscr{A}(\lambda_1, \ldots, \lambda_k) = A + \lambda_1 T_1 + \cdots + \lambda_k T_k \tag{1}$$

is a miniversal deformation (since it is a transversal of the minimal dimension to the orbit of A) called an orthogonal miniversal deformation.

For every arrow $\alpha : i \to j$, we denote $b(\alpha) := i$ and $e(\alpha) := j$.

By the proof of Theorem 2.1, $B \in \mathscr{T}_A^\perp$ if and only if $\langle B, [C, A] \rangle = 0$ for all $C \in \mathbb{F}^{n_1 \times n_1} \times \cdots \times \mathbb{F}^{n_t \times n_t}$. Then

$$\langle B, [C, A] \rangle = \sum_{\alpha \in Q_1} \operatorname{tr}(B_\alpha (C_{e(\alpha)} A_\alpha - A_\alpha C_{b(\alpha)})^*)$$

$$= \sum_{\alpha \in Q_1} \operatorname{tr}(B_\alpha A_\alpha^* C_{e(\alpha)}^* - B_\alpha C_{b(\alpha)}^* A_\alpha^*) = \sum_{i=1}^{t} \operatorname{tr}(S_i C_i^*) = 0,$$

where

$$S_i := \sum_{e(\alpha)=i} B_\alpha A_\alpha^* - \sum_{b(\alpha)=i} A_\alpha^* B_\alpha.$$

Taking $C_i = S_i$ for all vertices $i = 1, \ldots, t$, we obtain $S_i = 0$.

Therefore, every orthogonal miniversal deformation of A has the form (1), where $T_1, \ldots, T_k$ is a fundamental system of solutions of the system of homogeneous matrix equations

$$\sum_{e(\alpha)=i} X_\alpha A_\alpha^* = \sum_{b(\alpha)=i} A_\alpha^* X_\alpha, \quad i = 1, \ldots, t,$$

with unknowns $T = \{ X_\alpha \mid \alpha \in Q_1 \}$.

3. Deformations of matrices

In this section, we obtain a simplest miniversal $\mathbb{R}$-deformation of a real matrix under similarity.

Let us denote

$$J_r^{\mathbb{C}}(\lambda) = J_r(\lambda) := \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}, \quad J_r := J_r(0); \tag{2}$$

and, for $\lambda = a + bi \in \mathbb{C}$ $(b \geqslant 0)$, denote $J_r^{\mathbb{R}}(\lambda) := J_r(\lambda)$ if $b = 0$ and

$$J_r^{\mathbb{R}}(\lambda) := \begin{bmatrix} T_{ab} & I_2 & & \\ & T_{ab} & \ddots & \\ & & \ddots & I_2 \\ & & & T_{ab} \end{bmatrix} \quad \text{if } b > 0, \quad \text{where } T_{ab} := \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad (3)$$

(the size of $J_r(\lambda)$, $J_r^{\mathbb{C}}(\lambda)$ and $J_r^{\mathbb{R}}(\lambda)$ is $r \times r$).

Clearly, every square matrix over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ is similar to a matrix of the form

$$\oplus_i \Phi^{\mathbb{F}}(\lambda_i), \quad \lambda_i \neq \lambda_j, \text{ if } i \neq j, \tag{4}$$

uniquely determined up to permutations of summands, where

$$\Phi^{\mathbb{F}}(\lambda_i) := \operatorname{diag}(J_{s_{i1}}^{\mathbb{F}}(\lambda_i), \, J_{s_{i2}}^{\mathbb{F}}(\lambda_i), \ldots), \quad s_{i1} \geqslant s_{i2} \geqslant \cdots \tag{5}$$

Let

$$\mathscr{H} = [H_{ij}] \tag{6}$$

be a parametric block matrix with $p_i \times q_j$ blocks H_{ij} of the form

$$H_{ij} = \begin{bmatrix} * \\ \vdots & 0 \\ * \end{bmatrix} \quad \text{if } p_i \leqslant q_j, \qquad H_{ij} = \begin{bmatrix} & 0 \\ * \cdots * \end{bmatrix} \quad \text{if } p_i > q_j, \tag{7}$$

where the stars denote independent parameters.

Arnold [1] (see also [4, Section 30]) proved that one of the simplest miniversal $\mathbb{C}$-deformations of the matrix (4) for $\mathbb{F} = \mathbb{C}$ is $\oplus_i(\Phi^{\mathbb{C}}(\lambda_i) + \mathscr{H}_i)$, where $\mathscr{H}_i$ is of the form (6). Galin [5] (see also [4, Section 30E]) showed that one of the miniversal $\mathbb{R}$-deformations of the matrix (4) for $\mathbb{F} = \mathbb{R}$ is $\oplus_i(\Phi^{\mathbb{R}}(\lambda_i) + \mathscr{H}_{\lambda_i})$, where $\mathscr{H}_\lambda$ $(\lambda \in \mathbb{R})$ is of the form (6) and $\mathscr{H}_\lambda$ $(\lambda \notin \mathbb{R})$ is obtained from a matrix of the form (6) by the replacement of its entries $\alpha + \beta i$ with 2×2 blocks $T_{\alpha\beta}$ (see (3)). For example, a real 4×4 matrix with two Jordan 2×2 blocks with eigenvalues $x \pm iy$ $(y \neq 0)$ has a miniversal $\mathbb{R}$-deformation

$$\begin{bmatrix} x & y & 1 & 0 \\ -y & x & 0 & 1 \\ 0 & 0 & x & y \\ 0 & 0 & -y & x \end{bmatrix} + \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ -\beta_1 & \alpha_1 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 & 0 \\ -\beta_2 & \alpha_2 & 0 & 0 \end{bmatrix} \tag{8}$$

with the parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$. We prove that a simplest miniversal $\mathbb{R}$-deformation of this matrix may be obtained by the replacement of the second column $(\beta_1, \alpha_1, \beta_2, \alpha_2)^{\mathrm{T}}$ in (8) with $(0, 0, 0, 0)^{\mathrm{T}}$.

Theorem 3.1 (Arnold [1] for $\mathbb{F} = \mathbb{C}$). *One of the simplest miniversal $\mathbb{F}$-deformations of the canonical matrix (4) under similarity over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ is $\oplus_i(\Phi^{\mathbb{F}}(\lambda_i) + \mathscr{H}_i)$, where $\mathscr{H}_i$ is of the form (6).*

Proof. Let A be the matrix (4). By Theorem 2.1, we must prove that for every $M \in \mathbb{F}^{m \times m}$ there exists $S \in \mathbb{F}^{m \times m}$ such that

$$M + SA - AS = N, \tag{9}$$

where N is obtained from $\oplus_i\mathscr{H}_i$ by replacing its stars with elements of $\mathbb{F}$ and is uniquely determined by M. The matrix A is block-diagonal with diagonal blocks of the form $J_r^{\mathbb{F}}(\lambda)$. We apply the same partition into blocks to M and N and rewrite the equality (9) for blocks:

$$M_{ij} + S_{ij}A_j - A_iS_{ij} = N_{ij}.$$

The theorem follows from the next lemma. $\quad\square$

Lemma 3.1. *For given $J_p^{\mathbb{F}}(\lambda), J_q^{\mathbb{F}}(\mu)$, and for every matrix $M \in \mathbb{F}^{p \times q}$ there exists a matrix $S \in \mathbb{F}^{p \times q}$ such that $M + SJ_q^{\mathbb{F}}(\mu) - J_p^{\mathbb{F}}(\lambda)S = 0$ if $\lambda \neq \mu$, and $M + SJ_q^{\mathbb{F}}(\mu) - J_p^{\mathbb{F}}(\lambda)S = H$ if $\lambda = \mu$, where H is of the form (7) with elements from $\mathbb{F}$ instead of the stars; moreover, H is uniquely determined by M.*

Proof. If $\lambda \neq \mu$, then $SJ_q^{\mathbb{F}}(\mu)$ and $J_p^{\mathbb{F}}(\lambda)S$ have no common eigenvalue, the matrix S exists (cf. ([11, Section 8])).

Let $\lambda = \mu$ and let $\mathbb{F} = \mathbb{C}$ or $\lambda \in \mathbb{R}$. Put $C := SJ_q^{\mathbb{F}}(\lambda) - J_p^{\mathbb{F}}(\lambda)S = SJ_q - J_pS$. As is easily seen, C is an arbitrary matrix $[c_{ij}]$ (for a suitable S) satisfying the condition: if its diagonal $C_t = \{c_{ij} \mid i - j = t\}$ contains both an entry from the first column and an entry from the last row, then the sum of entries of this diagonal is equal to zero. It proves the lemma in this case.

Let $\lambda = \mu$, $\mathbb{F} = \mathbb{R}$ and $\lambda = a + bi$, $b > 0$. Then $p = 2m$ and $q = 2n$ for certain m and n. We must prove that every $2m \times 2n$ matrix M can be reduced to a uniquely determined matrix H of the form (7) (with real numbers instead of the stars) by transformations

$$M \mapsto M + SJ_{2n}^{\mathbb{R}}(\lambda) - J_{2m}^{\mathbb{R}}(\lambda)S, \quad S \in \mathbb{F}^{2m \times 2n}. \tag{10}$$

Let us partition M and S into 2×2 blocks M_{ij} and S_{ij}, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. For every 2×2 matrix $P = [p_{ij}]$, define (see (3))

$$P' := PT_{01} - T_{01}P = \begin{bmatrix} -p_{12} - p_{21} & p_{11} - p_{22} \\ p_{11} - p_{22} & p_{12} + p_{21} \end{bmatrix}.$$

By (3), the transformation (10) has the form $M \mapsto M + S(T_{ab} \oplus \cdots \oplus T_{ab}) - (T_{ab} \oplus \cdots \oplus T_{ab})S + SJ_{2n}^2 - J_{2m}^2S = M + b[S(T_{01} \oplus \cdots \oplus T_{01}) - (T_{01} \oplus \cdots \oplus T_{01})S] + SJ_{2n}^2 - J_{2m}^2S$, that is

$$M \longmapsto \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \hdotsfor{4} \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix} + b \begin{bmatrix} S'_{11} & S'_{21} & \cdots & S'_{1n} \\ S'_{21} & S'_{22} & \cdots & S'_{2n} \\ \hdotsfor{4} \\ S'_{m1} & S'_{m2} & \cdots & S'_{mn} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & S_{11} & \cdots & S_{1,n-1} \\ 0 & S_{21} & \cdots & S_{2,n-1} \\ \hdotsfor{4} \\ 0 & S_{m1} & \cdots & S_{m,n-1} \end{bmatrix} - \begin{bmatrix} S_{21} & S_{22} & \cdots & S_{2n} \\ \hdotsfor{4} \\ S_{m1} & S_{m2} & \cdots & S_{mn} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{11}$$

Let first $m \leqslant n$. If $m > 1$, we make $M_{mn} = 0$ selecting S'_{mn} and $S_{m,n-1}$. To preserve it, we must further take the transformations (11) with S satisfying $bS'_{mn} + S_{m,n-1} = 0$; that is,

$$S'_{mn} = -b^{-1} S_{m,n-1} \quad \text{and} \quad S_{m,n-1} = \begin{bmatrix} -\alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

with arbitrary α and β.

Selecting

$$S'_{m,n-1} = \begin{bmatrix} -2\beta & -2\alpha \\ -2\alpha & 2\beta \end{bmatrix}$$

and $S_{m,n-2}$, we make $M_{m,n-1} = 0$. To preserve it, we must take $bS'_{m,n-1} + S_{m,n-2} = 0$; that is, $S'_{m,n-1} = -b^{-1} S_{m,n-2}$ and

$$S_{m,n-2} = \begin{bmatrix} -\alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

with arbitrary α and β; and so on until obtain $M_{m2} = \cdots = M_{mn} = 0$. To preserve theirs, we must take

$$S_{m1} = \begin{bmatrix} -\alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

with arbitrary α and β and suitable

$$S_{M_{m1}+b} \begin{bmatrix} -2\beta & -2\alpha \\ -2\alpha & 2\beta \end{bmatrix},$$

we make

$$M_{m1} = \begin{bmatrix} \gamma & 0 \\ \delta & 0 \end{bmatrix},$$

where γ and δ are uniquely determined.

We have reduced the last strip of M to the form

$$[M_{m1} \cdots M_{mn}] = \begin{bmatrix} \gamma & 0 & \cdots & 0 \\ \delta & 0 & \cdots & 0 \end{bmatrix}. \tag{12}$$

To preserve it, we must take $S_{m1} = \cdots = S_{m,n-1} = S'_{mn} = 0$ since the number of zeros in $M_{m1}, \ldots, M_{mn}$ is equal to the number of parameters in $S_{m1}, \ldots, S_{m,n-1}, S'_{mn}$.

The next to last strip of M transforms as follows:

$$[M_{m-1,1} \cdots M_{m-1,n}] \mapsto [M_{m-1,1} \cdots M_{m-1,n}] + b[S'_{m-1,1} \cdots S'_{m-1,n}]$$
$$+ [0S_{m-1,1} \cdots S_{m-1,n-1}] - [0 \cdots 0S_{mn}].$$

In the same way, we reduce it to the form

$$[M_{m-1,1} \cdots M_{m-1,n}] = \begin{bmatrix} \tau & 0 & \cdots & 0 \\ v & 0 & \cdots & 0 \end{bmatrix}$$

taking, say, $S_{mn} = 0$. We must prove that τ and v are uniquely determined for all S_{mn} such that $S'_{mn} = 0$. It may be proved as for the γ and δ from (12) since the next to last horizontal strip of M, without the last block, is transformed as the last strip: $[M_{m-1,1} \cdots M_{m-1,n-1}] \mapsto [M_{m-1,1} \cdots M_{m-1,n-1}] + b[S'_{m-1,1} \cdots S'_{m-1,n-1}] + [0S_{m-1,1} \cdots S_{m-1,n-2}]$ (recall that $m \leqslant n$, so this equality is not empty for $m > 1$).

We repeat this procedure until reduce M to the form (7).

If $m > n$, we reduce M to the form (7) starting with the first vertical strip. $\square$

4. Deformations of matrix pencils

The canonical form problem for pairs of matrices $A, B \in \mathbb{F}^{m \times n}$ under transformations of simultaneous equivalence

$$(A, B) \mapsto (SAR^{-1}, SBR^{-1}), \quad S \in \mathrm{GL}(m, \mathbb{F}), \ \mathbb{R} \in \mathrm{GL}(\mathbb{n}, \mathbb{F})$$

(that is, for representations of the quiver $\cdot \leftrightarrows \cdot$) was solved by Kronecker: each pair is uniquely, up to permutation of summands, reduced to a direct sum of pairs of the form (see (2) and (3))

$$(I, J_r^{\mathbb{F}}(\lambda)), \ (J_r, I), \ (F_r, K_r), \ (F_r^{\mathrm{T}}, K_r^{\mathrm{T}}), \tag{13}$$

where $\lambda = a + bi \in \mathbb{C}$ ($b \geqslant 0$ if $\mathbb{F} = \mathbb{R}$) and

$$F_r = \begin{bmatrix} 1 & & 0 \\ 0 & \ddots & \\ & \ddots & 1 \\ 0 & & 0 \end{bmatrix}, \quad K_r = \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix} \tag{14}$$

are matrices of size $r \times (r-1)$, $r \times (r-1)$, $r \geqslant 1$.

A miniversal, but not a simplest miniversal, deformation of the canonical pairs of matrices under simultaneous similarity was obtained in [6], partial cases were considered in [7,8].

Denote by $0^\uparrow$ ($0^\downarrow$, $0^\leftarrow$, $0^\rightarrow$) a matrix, in which all entries are zero except for the entries of the first row (the last row, the first column, the last column) that are independent parameters; and denote by Z the $p \times q$ matrix, in which the first $\max\{q - p, 0\}$ entries of the first row are independent parameters and the other entries are zeros:

$$0^\uparrow = \begin{bmatrix} * & \cdots & * \\ 0 & \cdots & 0 \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots} \\ 0 & \cdots & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} * & \cdots & * & 0 & \cdots & 0 \\ & & & & \ddots & \\ & \mathbf{0} & & 0 & \cdots & 0 \end{bmatrix}. \tag{15}$$

Theorem 4.1. *Let*

$$(A, B) = \bigoplus_{i=1}^{l}(F_{p_i}, K_{p_i}) \oplus (I, C) \oplus (D, I) \oplus \bigoplus_{i=1}^{r}(F_{q_i}^{\mathrm{T}}, K_{q_i}^{\mathrm{T}}) \tag{16}$$

be a canonical pair of matrices under simultaneous equivalence over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, *where* C *is of the form* (4), $D = \Phi^{\mathbb{F}}(0)$ *(see* (5)*), and* [2] $p_1 \leqslant \cdots \leqslant p_l$, $q_1 \geqslant \cdots \geqslant q_r$. *Then one of the simplest miniversal* $\mathbb{F}$-*deformations of* (A, B) *has the form* $(\mathscr{A}, \mathscr{B}) =$

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where $\tilde{C}$ and $\tilde{D}$ are simplest miniversal $\mathbb{F}$-deformations of C and D under similarity (for instance, given by Theorem 2.1).

Let us denote by $S^{\succ}$ ($S^{\prec}$, $S^{\curlyvee}$, $S^{\curlywedge}$) the matrix that is obtained from a matrix S by removing of its first column (last column, first row, last row), and denote by S (S, $S_{\triangledown}$, $S_{\triangle}$) the matrix that is obtained from a matrix S by connecting of the zero column to the right (zero column to the left, zero row at the bottom, zero row at the top).

The following equalities hold for every $p \times q$ matrix S:

$$SF_q = S^{\prec} \quad SK_q = S^{\succ} \quad SF_{q+1}^{\mathrm{T}} = S \quad SK_{q+1}^{\mathrm{T}} = S \quad SJ_q = S^{\prec}$$

$$F_{p+1}S = S_{\triangledown} \quad K_{p+1}S = S_{\triangle} \quad F_p^{\mathrm{T}}S = S^{\curlywedge} \quad K_p^{\mathrm{T}}S = S^{\curlyvee} \quad J_pS = S_{\triangledown}^{\curlyvee}$$

Proof of Theorem 4.1. By Theorem 2.1, we must prove that for every $M, N \in \mathbb{F}^{m \times n}$ there exist $S \in \mathbb{F}^{m \times m}$ and $R \in \mathbb{F}^{n \times n}$ such that

$$(M, N) + (SA - AR, SB - BR) = (P, Q), \tag{17}$$

where (P, Q) is obtained from $(\mathscr{A}, \mathscr{B}) - (A, B)$ by replacing the stars with elements of $\mathbb{F}$ and is uniquely determined by (M, N). The matrices A and B have the block-diagonal form: $A = A_1 \oplus A_2 \oplus \cdots$, $B = B_1 \oplus B_2 \oplus \cdots$, where $\mathscr{P}_i = (A_i, B_i)$ are direct summands of the form (13). We apply the same partition into blocks to M and N and rewrite the equality (17) for blocks:

$$(M_{ij}, N_{ij}) + (S_{ij}A_j - A_iR_{ij}, S_{ij}B_j - B_iR_{ij}) = (P_{ij}, Q_{ij}).$$

Therefore, for every pair of summands $\mathscr{P}_i = (A_i, B_i)$ and $\mathscr{P}_j = (A_j, B_j)$, $i \leqslant j$, we must prove that

(a) the pair (M_{ij}, N_{ij}) can be reduced to the pair (P_{ij}, Q_{ij}) by transformations $(M_{ij}, N_{ij}) \mapsto (M_{ij}, N_{ij}) + (\triangle M_{ij}, \triangle N_{ij})$, where

$$\triangle M_{ij} := SA_j - A_iR, \quad \triangle N_{ij} := SB_j - B_iR$$

with arbitrary R and S; moreover, (P_{ij}, Q_{ij}) is uniquely determined (more exactly, its entries on the places of stars are uniquely determined) by (M_{ij}, N_{ij}); and, if $i < j$,

(b) the pair (M_{ji}, N_{ji}) can be reduced to the pair (P_{ji}, Q_{ji}) by transformations $(M_{ji}, N_{ji}) \mapsto (M_{ji}, N_{ji}) + (\triangle M_{ji}, \triangle N_{ji})$, where

$$\triangle M_{ji} := SA_i - A_jR, \quad \triangle N_{ji} := SB_i - B_jR$$

with arbitrary R and S; moreover, (P_{ji}, Q_{ji}) is uniquely determined by (M_{ji}, N_{ji}).

Case 1: $\mathscr{P}_i = (F_p, K_p)$ and $\mathscr{P}_j = (F_q, K_q)$, $p \leqslant q$.

(a) We have $\triangle M_{ij} = SF_q - F_pR = S^\prec - R_\triangledown$. Adding $\triangle M_{ij}$, we make $M_{ij} = 0$; to preserve it, we must further take S and R for which $\triangle M_{ij} = 0$, i.e. $S = [R_\triangledown \,\vdots\,]$, where the points denote an arbitrary column. Further, $\triangle N_{ij} = SK_q - K_pR = S^\succ - R_\triangle = [R_\triangledown \,\vdots\,]^\succ - R_\triangle = [X_\triangledown \,\vdots\,] - [\,\vdots X]_\triangle$, where $X := R^\succ$. Clearly, $\triangle N_{ij}$ is an arbitrary matrix $[\delta_{\alpha\beta}]$ that satisfies the condition: if its diagonal $D_t = \{\delta_{\alpha\beta} \mid \alpha - \beta = t\}$ contains an entry from the first row and does not contain an entry from the last column, then the sum of entries of this diagonal is equal to zero. Adding $\triangle N_{ij}$, we make $N_{ij} = Z$, where Z is of the form (15) but with elements of $\mathbb{F}$ instead of the stars. If $i = j$, then $p = q$, $N_{ii} = Z$ has size $p \times (p - 1)$, so $N_{ii} = 0$ (see (15)).

(b) We have $\triangle M_{ji} = SF_p - F_qR$ and $\triangle N_{ji} = SK_p - K_qR$; so we analogously make $M_{ji} = 0$ and $N_{ji} = Z$. But since Z has size $q \times (p - 1)$ and $p \leqslant q$, $N_{ji} = Z = 0$ (see (15)).

Case 2: $\mathscr{P}_i = (F_p, K_p)$ and $\mathscr{P}_j = (I, J_q^\mathbb{F}(\lambda))$.

(a) We have $\triangle M_{ij} = S - F_pR = S - R_\triangledown$. Make $M_{ij} = 0$; to preserve it, we must further take $S = R_\triangledown$. Then $\triangle N_{ij} = SJ_q^\mathbb{F}(\lambda) - K_pR = (RJ_q^\mathbb{F}(\lambda))_\triangledown - R_\triangle$. Using the last row of R, we make the last row of N_{ij} equaling zero, then the next to the last row equaling zero, and so on until reduce N_{ij} to the form $0^\uparrow$ (with elements of $\mathbb{F}$ instead of the stars).

(b) We have $\triangle M_{ji} = SF_p - R = S^\prec - R$. Make $\triangle M_{ji} = 0$, then $R = S^\prec$; $\triangle N_{ji} = SK_p - J_q^\mathbb{F}(\lambda)R = S^\succ - (J_q^\mathbb{F}(\lambda)S)^\prec$. We make $N_{ji} = 0$ starting with the last row (with the last horizontal strip if $\mathbb{F} = \mathbb{R}$ and $\lambda \notin \mathbb{R}$).

Case 3: $\mathscr{P}_i = (F_p, K_p)$ and $\mathscr{P}_j = (J_q, I)$.

(a) We have $\triangle N_{ij} = S - K_pR$, make $N_{ij} = 0$, then $S = K_pR = R_\triangle$; $\triangle M_{ij} = SJ_q - F_pR = (RJ_q)_\triangle - R_\triangledown$. Reduce M_{ij} to the form $0^\downarrow$ starting with the first row.

(b) We have $\triangle N_{ji} = SK_p - R$, make $\triangle N_{ji} = 0$, then $R = SK_p = S^\succ$; $\triangle M_{ji} = SF_p - J_qR = S^\prec - (J_qS)^\succ$. We make $M_{ji} = 0$ starting with the last row.

Case 4: $\mathscr{P}_i = (F_p, K_p)$ and $\mathscr{P}_j = (F_q^\mathrm{T}, K_q^\mathrm{T})$.

(a) We have $\triangle M_{ij} = SF_q^\mathrm{T} - F_pR = S - R_\triangledown$. Reduce M_{ij} to the form $0^\rightarrow$, then $(S - R_\triangledown)^\prec = S - R_\triangledown^\prec = 0$, $S = R_\triangledown^\prec$. Put $X := R^\prec$, then $S = X_\triangledown$ and $R = [X \,\vdots\,]$, where the points denote an arbitrary row. Further, $\triangle N_{ij} = SK_q^\mathrm{T} - K_pR = S - R_\triangle = (X_\triangledown) - [X \,\vdots\,]_\triangle$. Clearly, $\triangle N_{ij}$ is an arbitrary matrix $[\delta_{\alpha\beta}]$ that satisfies the condition: if its secondary diagonal $D_t = \{\delta_{\alpha\beta} \mid \alpha + \beta = t\}$ contains an entry from the first row, then the sum of entries of this diagonal is equal to zero. Adding $\triangle N_{ij}$, we reduce N_{ij} to the form $0^\uparrow$.

(b) We have $\triangle M_{ji} = SF_p - F_q^\mathrm{T}R = S^\prec - R^\wedge$. Make $M_{ji} = 0$, then $S = [R^\wedge \,\vdots\,]$. Further, $\triangle N_{ji} = SK_p - K_q^\mathrm{T}R = S^\succ - R^\curlyvee = [R^\wedge \,\vdots\,]^\succ - R^\curlyvee$, make $N_{ji} = 0$ starting with the last column.

Case 5: $\mathscr{P}_i = (I, J_p^\mathbb{F}(\lambda))$ and $\mathscr{P}_j = (I, J_q^\mathbb{F}(\mu))$.

(a) We have $\triangle M_{ij} = S - R$. Make $M_{ij} = 0$, then $S = R$; $\triangle N_{ij} = SJ_q^\mathbb{F}(\mu) - J_p^\mathbb{F}(\lambda)R$. Using Lemma 3.1, we make $N_{ij} = 0$ if $\lambda \neq \mu$ and $N_{ij} = H$ if $\lambda = \mu$.

(b) We have $\triangle M_{ji} = S - R$ and $\triangle N_{ji} = SJ_p^{\mathbb{F}}(\lambda) - J_q^{\mathbb{F}}(\mu)R$. As in Case 5(a), make $M_{ji} = 0$, $N_{ji} = 0$ if $\lambda \neq \mu$ and $N_{ji} = H$ if $\lambda = \mu$.

Case 6: $\mathscr{P}_i = (I, J_p^{\mathbb{F}}(\lambda))$ and $\mathscr{P}_j = (J_q, I)$.

(a) We have $\triangle M_{ij} = SJ_q - R = S^{\prec} - R$. Make $M_{ij} = 0$, then $R = S^{\prec}$; $\triangle N_{ij} = S - J_p^{\mathbb{F}}(\lambda)R = S - (J_p^{\mathbb{F}}(\lambda)S)^{\prec}$. We make $N_{ij} = 0$ starting with the first column.

(b) We have $\triangle M_{ji} = S - J_qR$, make $M_{ji} = 0$, then $S = R_{\triangledown}^{\curlyvee}$; $\triangle N_{ji} = SJ_p^{\mathbb{F}}(\lambda) - R = (RJ_p^{\mathbb{F}}(\lambda))_{\triangledown}^{\curlyvee} - R$. We make $N_{ji} = 0$ starting with the last row.

Case 7: $\mathscr{P}_i = (I, J_p^{\mathbb{F}}(\lambda))$ and $\mathscr{P}_j = (F_q^{\mathrm{T}}, K_q^{\mathrm{T}})$.

(a) We have $\triangle M_{ij} = SF_q^{\mathrm{T}} - R$. Make $M_{ij} = 0$, then $R = S$; $\triangle N_{ij} = SK_q^{\mathrm{T}} - J_p^{\mathbb{F}}(\lambda)R = S - (J_p^{\mathbb{F}}(\lambda)S)$. We reduce N_{ij} to the form $0^{\leftarrow}$ starting with the last row (with the last horizontal strip if $\mathbb{F} = \mathbb{R}$ and $\lambda \notin \mathbb{R}$).

(b) We have $\triangle M_{ji} = S - F_q^{\mathrm{T}}R$, make $M_{ji} = 0$, then $S = R^{\curlywedge}$, $\triangle N_{ji} = SJ_p^{\mathbb{F}}(\lambda) - K_q^{\mathrm{T}}R = (RJ_p^{\mathbb{F}}(\lambda))^{\curlywedge} - R^{\curlyvee}$. We make $N_{ji} = 0$ starting with the first column (with the first vertical strip if $\mathbb{F} = \mathbb{R}$ and $\lambda \notin \mathbb{R}$).

Case 8: $\mathscr{P}_i = (J_p, I)$ and $\mathscr{P}_j = (J_q, I)$. Interchanging the matrices in each pair, we reduce this case to Case 5.

Case 9: $\mathscr{P}_i = (J_p, I)$ and $\mathscr{P}_j = (F_q^{\mathrm{T}}, K_q^{\mathrm{T}})$.

(a) We have $\triangle N_{ij} = SK_q^{\mathrm{T}} - R$. Make $N_{ij} = 0$, then $R = S$; $\triangle M_{ij} = SF_q^{\mathrm{T}} - J_pR = S - (J_pS)$. We reduce M_{ij} to the form $0^{\rightarrow}$ starting with the first column.

(b) We have $\triangle N_{ji} = S - K_q^{\mathrm{T}}R$, make $N_{ji} = 0$, then $S = R^{\curlyvee}$, $\triangle M_{ji} = SJ_p - F_q^{\mathrm{T}}R = (RJ_p)^{\curlyvee} - R^{\curlywedge}$. We make $M_{ji} = 0$ starting with the first column.

Case 10: $\mathscr{P}_i = (F_p^{\mathrm{T}}, K_p^{\mathrm{T}})$ and $\mathscr{P}_j = (F_q^{\mathrm{T}}, K_q^{\mathrm{T}})$, $p \geqslant q$.)

(a) We have $\triangle M_{ij} = SF_q^{\mathrm{T}} - F_p^{\mathrm{T}}R$ and $\triangle N_{ij} = SK_q^{\mathrm{T}} - K_p^{\mathrm{T}}R$, so $(\triangle M_{ij})^{\mathrm{T}} = (-R^{\mathrm{T}})F_p - F_q(-S^{\mathrm{T}})$ and $(\triangle N_{ij})^{\mathrm{T}} = (-R^{\mathrm{T}})K_p - K_q(-S^{\mathrm{T}})$. Reasoning as in Case 1(a), we make $M_{ij}^{\mathrm{T}} = 0$ and $N_{ij}^{\mathrm{T}} = Z$, that is $M_{ij} = 0$ and $N_{ij} = Z^{\mathrm{T}}$ ($N_{ij} = 0$ if $i = j$).

(b) We have $\triangle M_{ji} = SF_p^{\mathrm{T}} - F_q^{\mathrm{T}}R$ and $\triangle N_{ji} = SK_p^{\mathrm{T}} - K_q^{\mathrm{T}}R$, so we analogously make $M_{ji} = 0$ and $N_{ji} = Z^{\mathrm{T}}$. Since the size of Z^{T} is $(q - 1) \times p$ and $p \geqslant q$, by (15) we have $Z^{\mathrm{T}} = 0$. $\square$

5. Deformations of contragredient matrix pencils

The canonical form problem for pairs of matrices $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times m}$ under transformations of contragredient equivalence

$$(A, B) \mapsto (SAR^{-1}, RBS^{-1}), \quad S \in \mathrm{GL}(m, \mathbb{F}), \quad \mathbb{R} \in \mathrm{GL}(\mathfrak{n}, \mathbb{F}),$$

(i.e., for representations of the quiver $\rightleftarrows$) was solved in [2,3]: each pair is uniquely, up to permutation of cells $J_r^{\mathbb{F}}(\lambda)$ in $\oplus_i \Phi^{\mathbb{F}}(\lambda_i)$, reduced to a direct sum

$$\underset{j}{\oplus}(I,C) \oplus \overset{t_1}{\underset{j=1}{\oplus}}(I_{r_{1j}},J_{r_{1j}}) \oplus \overset{t_2}{\underset{j=1}{\oplus}}(J_{r_{2j}},I_{r_{2j}}) \oplus \overset{t_3}{\underset{j=1}{\oplus}}(F_{r_{3j}},G_{r_{3j}}) \oplus \overset{t_4}{\underset{j=1}{\oplus}}(G_{r_{4j}},F_{r_{4j}}) \qquad (18)$$

(we use the notation (14) and put $G_r := K_r^{\mathrm{T}}$), where C is of the form (4) and $r_{i1} \geqslant r_{i2} \geqslant \cdots \geqslant r_{it_i}$.

Theorem 5.1. *One of the simplest miniversal $\mathbb{F}$-deformations of the canonical pair* (18) *under contragredient equivalence over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ is the direct sum of $(I, \tilde{C})$ ($\tilde{C}$ is a simplest miniversal $\mathbb{F}$-deformation of C under similarity, see Theorem* 2.1) *and*

$$\left(\left[\begin{array}{c|c|cc} \oplus_j I_{r_{1j}} & 0 & 0 & \\ \hline 0 & \oplus_j J_{r_{2j}} + \mathcal{H} & \mathcal{H} & \\ \hline 0 & \mathcal{H} & P_3 & \mathcal{H} \\ & & 0 & Q_4 \end{array} \right], \left[\begin{array}{c|c|cc} \oplus_j J_{r_{1j}} + \mathcal{H} & \mathcal{H} & \mathcal{H} & \\ \hline \mathcal{H} & \oplus_j I_{r_{2j}} & 0 & \\ \hline \mathcal{H} & 0 & Q_3 & 0 \\ & & \mathcal{H} & P_4 \end{array} \right] \right),$$

where

$$P_l = \begin{bmatrix} F_{r_{l1}} + H & H & & \cdots & H \\ & F_{r_{l2}} + H & \ddots & & \vdots \\ & & & \ddots & H \\ 0 & & & & F_{r_{lt_l}} + H \end{bmatrix},$$

$$Q_l = \begin{bmatrix} G_{r_{l1}} & & & 0 \\ H & G_{r_{l2}} & & \\ \vdots & \ddots & \ddots & \\ H & \cdots & H & G_{r_{lt_l}} \end{bmatrix}$$

($l = 3, 4$), $\mathcal{H}$ and H are matrices of the form (6) *and* (7), *the stars denote independent parameters.*

Proof. Let (A, B) be the canonical pair of matrices (18). By Theorem 2.1, we must prove that for every $M \in \mathbb{F}^{m \times n}$, $N \in \mathbb{F}^{n \times m}$ there exist $S \in \mathbb{F}^{m \times m}$ and $R \in \mathbb{F}^{n \times n}$ such that

$$(M, N) + (SA - AR, RB - BS) = (P, Q),$$

or, in the block form,

$$(M_{ij}, N_{ij}) + (S_{ij}A_j - A_iR_{ij}, R_{ij}B_j - B_iS_{ij}) = (P_{ij}, Q_{ij}),$$

where (P, Q) is obtained from $(\mathscr{A}, \mathscr{B}) - (A, B)$ by replacing stars with complex numbers and is uniquely determined by (M, N).

Therefore, for every pair of summands $\mathscr{P}_i = (A_i, B_i)$ and $\mathscr{P}_j = (A_j, B_j)$, $i \leqslant j$, from the decomposition (18), we must prove that

(a) the pair (M_{ij}, N_{ij}) can be reduced to the pair (P_{ij}, Q_{ij}) by transformations $(M_{ij}, N_{ij}) \mapsto (M_{ij}, N_{ij}) + (\triangle M_{ij}, \triangle N_{ij})$, where

$$\triangle M_{ij} = SA_j - A_iR, \quad \triangle N_{ij} = RB_j - B_iS$$

with arbitrary R and S; moreover, (P_{ij}, Q_{ij}) is uniquely determined (more exactly, its entries on the places of stars are uniquely determined) by (M_{ij}, N_{ij}); and, if $i < j$,

(b) the pair (M_{ji}, N_{ji}) can be reduced to the pair (P_{ji}, Q_{ji}) by transformations $(M_{ji}, N_{ji}) \mapsto (M_{ji}, N_{ji}) + (\triangle M_{ji}, \triangle N_{ji})$, where

$$\triangle M_{ji} = SA_i - A_jR, \quad \triangle N_{ji} = RB_i - B_jS$$

with arbitrary R and S; moreover, (P_{ji}, Q_{ji}) is uniquely determined by (M_{ji}, N_{ji}).

Case 1: $\mathscr{P}_i = (I, J_p^{\mathbb{F}}(\lambda))$ and $\mathscr{P}_j = (I, J_q^{\mathbb{F}}(\mu))$.

(a) We have $\triangle M_{ij} = S - R$. Make $M_{ij} = 0$, then $S = R$; $\triangle N_{ij} = RJ_q^{\mathbb{F}}(\mu) - J_p^{\mathbb{F}}(\lambda)S$. Using Lemma 3.1, we make $N_{ij} = 0$ if $\lambda \neq \mu$, and $N_{ij} = H$ (see (7)) if $\lambda = \mu$.

(b) We have $\triangle M_{ji} = S - R$ and $\triangle N_{ji} = RJ_p^{\mathbb{F}}(\lambda) - J_q^{\mathbb{F}}(\mu)S$. As in Case 1(a), make $M_{ji} = 0$, then $N_{ji} = 0$ if $\lambda \neq \mu$ and $N_{ji} = H$ if $\lambda = \mu$.

Case 2: $\mathscr{P}_i = (I, J_p^{\mathbb{F}}(\lambda))$ and $\mathscr{P}_j = (J_q, I)$.

(a) We have $\triangle M_{ij} = SJ_q - R$. Make $M_{ij} = 0$, then $R = SJ_q$, $\triangle N_{ij} = R - J_p^{\mathbb{F}}(\lambda)S = SJ_q - J_p^{\mathbb{F}}(\lambda)S$. Using Lemma 3.1, we make $N_{ij} = 0$ if $\lambda \neq 0$ and $N_{ij} = H$ if $\lambda = 0$.

(b) We have $\triangle M_{ji} = S - J_qR$. Make $M_{ji} = 0$, then $S = J_qR$, $\triangle N_{ji} = RJ_p^{\mathbb{F}}(\lambda) - S = RJ_p^{\mathbb{F}}(\lambda) - J_qR$. We make $N_{ji} = 0$ if $\lambda \neq 0$ and $N_{ji} = H$ if $\lambda = 0$.

Case 3: $\mathscr{P}_i = (I, J_p^{\mathbb{F}}(\lambda))$ and $\mathscr{P}_j = (F_q, G_q)$.

(a) We have $\triangle M_{ij} = SF_q - R = S^{\prec} - R$. Make $M_{ij} = 0$, then $R = S^{\prec}$, $\triangle N_{ij} = RG_q - J_p^{\mathbb{F}}(\lambda)S = S^{\prec} - J_p^{\mathbb{F}}(\lambda)S = SJ_q - J_p^{\mathbb{F}}(\lambda)S$. Using Lemma 3.1, we make $N_{ij} = 0$ if $\lambda \neq 0$ and $N_{ij} = H$ if $\lambda = 0$.

(b) We have $\triangle M_{ji} = S - F_qR = S - R_{\triangledown}$. Make $M_{ji} = 0$, then $S = R_{\triangledown}$, $\triangle N_{ji} = RJ_p^{\mathbb{F}}(\lambda) - G_qS = RJ_p^{\mathbb{F}}(\lambda) - R_{\triangledown}^{\curlyvee} = RJ_p^{\mathbb{F}}(\lambda) - J_{q-1}R$. We make $N_{ji} = 0$ if $\lambda \neq 0$ and $N_{ji} = H$ if $\lambda = 0$.

Case 4: $\mathscr{P}_i = (I, J_p^{\mathbb{F}}(\lambda))$ and $\mathscr{P}_j = (G_q, F_q)$.

(a) We have $\triangle M_{ij} = SG_q - R$. Make $M_{ij} = 0$, then $R = S$, $\triangle N_{ij} = RF_q - J_p^{\mathbb{F}}(\lambda)S = S^{\prec} - J_p^{\mathbb{F}}(\lambda)S = SJ_{q-1} - J_p^{\mathbb{F}}(\lambda)S$. Using Lemma 3.1, we make $N_{ij} = 0$ if $\lambda \neq 0$ and $N_{ij} = H$ if $\lambda = 0$.

(b) We have $\triangle M_{ji} = S - G_qR = S - R^{\curlyvee}$. Make $M_{ji} = 0$, then $S = R^{\curlyvee}$, $\triangle N_{ji} = RJ_p^{\mathbb{F}}(\lambda) - F_qS = RJ_p^{\mathbb{F}}(\lambda) - R_{\triangledown}^{\curlyvee} = RJ_p^{\mathbb{F}}(\lambda) - J_qR$. We make $N_{ji} = 0$ if $\lambda \neq 0$ and $N_{ji} = H$ if $\lambda = 0$.

Case 5: $\mathscr{P}_i = (J_p, I)$ and $\mathscr{P}_j = (J_q, I)$. Interchanging the matrices in each pair, we reduce this case to Case 1.

Case 6: $\mathscr{P}_i = (J_p, I)$ and $\mathscr{P}_j = (F_q, G_q)$. Interchanging the matrices in each pair, we reduce this case to Case 4.

Case 7: $\mathscr{P}_i = (J_p, I)$ and $\mathscr{P}_j = (G_q, F_q)$. Interchanging the matrices in each pair, we reduce this case to Case 3.

Case 8: $\mathscr{P}_i = (F_p, G_p)$ and $\mathscr{P}_j = (F_q, G_q)$, $i \leqslant j$ (and hence $p \geqslant q$).

(a) We have $\triangle N_{ij} = RG_q - G_pS = R - S^{\curlyvee}$. Make $N_{ij} = 0$, then $R = S^{\curlyvee}$. Further, $\triangle M_{ij} = SF_q - F_pR = S^{\prec} - R_{\triangledown}$, so $(\triangle M_{ij})^{\curlyvee} = (S^{\curlyvee})^{\prec} - R^{\curlyvee}_{\triangledown} = R^{\prec} - R^{\curlyvee}_{\triangledown} = RJ_{q-1} - J_{p-1}R$ and the first row of $\triangle M_{ij}$ is arbitrary (due to the first row of S). We make the first row of M_{ij} equaling zero. Following the proof of Lemma 3.1 and taking into account that $p \geq q$, we make all entries of the $(p-1) \times (q-1)$ matrix $M^{\curlyvee}_{ij}$ equaling zero except for the last row and obtain $M_{ij} = H$.

(b) We have $i < j$, $\triangle M_{ji} = SF_p - F_qR = S^{\prec} - R_{\triangledown}$. Make $M_{ji} = 0$, then $S^{\prec} = R_{\triangledown}$. Further, $\triangle N_{ji} = RG_p - G_qS = R - S^{\curlyvee}$, $(\triangle N_{ji})^{\prec} = R^{\prec} - R^{\curlyvee}_{\triangledown} = RJ_{p-1} - J_{q-1}R$ and the last column of $\triangle N_{ji}$ is arbitrary (due to the last column of S). We make the last column of $\triangle N_{ji}$ equaling zero. By Lemma 3.1 and the inequality $p \geqslant q$, we make all entries of the $(q-1) \times (p-1)$ matrix $N^{\prec}_{ji}$ equaling zero except for the first column and obtain $N_{ji} = H$.

Case 9: $\mathscr{P}_i = (F_p, G_p)$ and $\mathscr{P}_j = (G_q, F_q)$.

(a) We have $\triangle N_{ij} = RF_q - G_pS = R^{\prec} - S^{\curlyvee}$. Make $N_{ij} = 0$, then $R^{\prec} = S^{\curlyvee}$, i.e. $R = X^{\curlyvee}$ and $S = X^{\prec}$ for an arbitrary X. Further, $\triangle M_{ij} = SG_q - F_pR = S - R_{\triangledown} = X^{\prec} - X^{\curlyvee}_{\triangledown}$, we make $M_{ij} = H$.

(b) We have $\triangle M_{ji} = SF_p - G_qR$ and $\triangle N_{ji} = RG_p - F_qS$. So we analogously make $M_{ji} = 0$ and $N_{ji} = H$.

Case 10: $\mathscr{P}_i = (G_p, F_p)$ and $\mathscr{P}_j = (G_q, F_q)$, $i \leqslant j$. Interchanging the matrices in each pair, we reduce this case to Case 8. $\quad\square$

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