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The Region of (In)Stability of a 2-Delay Equation Is Connected

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1. INTRODUCTION

Let $a, b, c \in \mathbb{R}$, $r_1, r_2 \in \mathbb{R}_{\geq}$. We consider the equation

$$\dot{x}(t) + ax(t) + bx(t - r_1) + cx(t - r_2) = 0. \tag{1}$$

Several authors (see, e.g., [2–6]) have investigated Eq. (1). Of special interest is the region of stability, i.e., conditions on the parameters a, b, c, r_1, r_2 (or some of them), which ensure asymptotic stability. Schoen and Geering [6] gave necessary and sufficient conditions in the case of fixed $r_1 = \frac{1}{2}r_2$, and under the assumption $|c| < \pi/r_2$. Mahaffy *et al.* [3] described the region of stability for fixed delays r_1, r_2 and variable a, b, c. They show how the region of stability evolves in the bc-plane changing the parameter a. Hale and Huang [2] gave a geometric description of the stable region in the $r_1 r_2$-plane for fixed a, b, c. They do this under the assumption that this region is connected. This seems reasonable, but to our knowledge has as yet not been proven.

We are interested in proving that both the region of stability and instability of (1) are connected. There are no general results known to us concerning these issues. The only partial result is a counterexample in [3] which shows that for fixed r_1, r_2, a and variable b, c the region of stability is not connected.

We shall show that both regions mentioned above are connected in the space of all parameters as well as for fixed delays. Also we prove that the region of instability in the $r_1 r_2$-plane, i.e., for fixed a, b, c, is connected. Unfortunately, as yet we have not been able to prove the assumption of

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Hale and Huang [2], i.e., that the stable region in the plane of delays for fixed parameters a, b, c is connected.

Actually our results are a little stronger than what we mentioned above. They imply that there are no "islands" of instability in the stable regions we consider, nor are there "islands" of stability in the region of instability, for variable a, b, c.

The techniques used to get these results are quite different for the two fundamental cases of a, b, c variable (and r_1, r_2 fixed or variable), and a, b, c fixed (and r_1, r_2 variable). In the former case we use various paths in the space of parameters a, b, c. In particular one which changes the real part of all roots of the characteristic equation in a controlled way. In the latter case we make a detailed investigation of the behaviour of purely imaginary roots of the characteristic equations as the delays r_1, r_2 vary.

We now introduce some notations we shall be using throughout this paper.

The characteristic equation of (1) is

$$\chi(\mu) = \chi(\mu; a, b, c, r_1, r_2) = \mu + a + be^{-r_1 \mu} + ce^{-r_2 \mu}.$$

We also write $\chi(\mu; a, b, c)$, $\chi(\mu; r_1, r_2)$ if we want to emphasize the dependence on certain parameters.

We are interested in the region of (asymptotic) stability

$$\begin{aligned} S &= \left\{(a, b, c, r_1, r_2) \in \mathbb{R}^3 \times \mathbb{R}^2_{\geq} : x(t) \equiv 0 \textit{ is asym. stable sol. of } (1)\right\} \\ &= \left\{(a, b, c, r_1, r_2) \in \mathbb{R}^3 \times \mathbb{R}^2_{\geq} : \operatorname{Re} \mu \geq 0 \Rightarrow \chi(\mu; a, b, c, r_1, r_2) \neq 0\right\}. \end{aligned}$$

If we are interested in the region of stability only with respect to the coefficients a, b, c fixing r_1, r_2, we write

$$S_{abc} \times \{(r_1, r_2)\} := S \cap \left(\mathbb{R}^3 \times \{(r_1, r_2)\}\right),$$

analogously for fixed a, b, c and varying r_1, r_2: $S_{r_1, r_2} = S_r \subset \mathbb{R}^2_{\geq}$.

Let $U \subset \mathbb{R}^3 \times \mathbb{R}^2_{\geq}$ denote the region where $x(t) \equiv 0$ is unstable. Similarly to what we did for the stable region, let $U_{abc} \subset \mathbb{R}^3$ resp. $U_r = U_{r_1, r_2} \subset \mathbb{R}^2_{\geq}$ denote the unstable regions we get fixing r_1, r_2 resp. a, b, c and varying the remaining parameters.

If we say Eq. (1) is (un-)stable, we always mean $x(t) \equiv 0$ is an asymptotically stable resp. unstable solution of (1).

If we have only one delay (for example, if $r_2 = 0$, or $r_1 = r_2$, or $c = 0$) the exact region of stability is known (e.g., [1]). In this case the region of stability is connected: in the space of all parameters (say a, b, r), as well as for fixed a, b and variable r (then $S_r = [0, r_0(a, b)$ [is an interval), or for fixed r and variable a, b. The same holds for the region of instability.

2. FIXED DELAYS r_1, r_2

We start with a well known condition when (1) is stable resp. unstable independently of the delay (see, e.g., [3]).

LEMMA 1. *If* $a + b + c < 0$, *then* (1) *is unstable. If* $a > |b| + |c|$, *then* (1) *is stable.*

In the space of the coefficients a, b, c it is not difficult to show connectedness for both the stable region as well as the unstable region:

PROPOSITION 1. *S_{abc} and U_{abc} are connected and unbounded. Moreover we have*

$$\mathbb{R}^3 \setminus S_{abc} = \overline{U}_{abc}, \qquad \mathbb{R}^3 \setminus U_{abc} = \overline{S}_{abc}.$$

Proof. If $r_1 = 0$, or $r_2 = 0$, or $r_1 = r_2$, then we have only one delay and the conclusion holds (see [1]), as has already been mentioned. So assume for the rest of the proof $r_1, r_2 > 0$, $r_1 \neq r_2$.

We shall now define a curve $\Gamma(t)$ in the space of parameters a, b, c which corresponds to a shift t of the real part of all roots of the characteristic equation $\chi(\cdot; a, b, c)$: For fixed a_0, b_0, c_0 let Γ: $\mathbb{R} \to \mathbb{R}^3$, $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t), \Gamma_3(t)) = (a_0 - t, b_0 \exp(r_1 t), c_0 \exp(r_2 t))$. Then

$$\chi(x + iy; a_0, b_0, c_0) = \chi(x + t + iy; \Gamma(t)).$$

If we assume for a moment that $(a_0, b_0, c_0) \in S_{abc}$, then $\Gamma(] - \infty, 0]) \subset S_{abc}$, and $\Gamma_1(t) \to \infty$, $\Gamma_2(t), \Gamma_3(t) \to 0$ as $t \to -\infty$. This shows (a_0, b_0, c_0) to be connected in S_{abc} with the region $a > |b| + |c|$ which in turn is connected and unbounded.

To show U_{abc} to be connected we let one or two parameters depend on the remaining in such a way that the biggest root of $\chi(\cdot; a, b, c)$ remains fixed.

Assume $(a_0, b_0, c_0) \in U_{abc}$. If we change (a_0, b_0, c_0) slightly we can without loss of generality assume that there is a $\mu_0 = x_0 + iy_0 \in \mathbb{C}$, $x_0 > 0$, and $\chi(\mu_0; a_0, b_0, c_0) = 0$. There are two cases to handle.

Case $y_0 = 0$. We have $0 = x_0 + a_0 + b_0 \exp(-r_1 x_0) + c_0 \exp(-r_2 x_0)$. Set

$$a(b, c) := -x_0 - be^{-r_1 x_0} - ce^{-r_2 x_0}.$$

Then $\chi(\mu_0; a(b, c), b, c) \equiv 0$ and (a_0, b_0, c_0) can be connected within U_{abc} with $(-x_0, 0, 0)$ and thus with $(-1, 0, 0)$.

Case $y_0 > 0$. Now we have

$$0 = y_0 - be^{-r_1 x_0} \sin(r_1 y_0) - ce^{-r_2 x_0} \sin(r_2 y_0),$$

implying without loss of generality $\sin(r_1 y_0) \neq 0$.

Set

$$b(c) := e^{r_1 x_0} \frac{y_0 - c e^{-r_2 x_0} \sin(r_2 y_0)}{\sin(r_1 y_0)}$$

$$a(c) := -x_0 - b(c) e^{-r_1 x_0} \cos(r_1 y_0) - c e^{-r_2 x_0} \cos(r_2 y_0).$$

Then $\chi(\mu_0; a(c), b(c), c) \equiv 0$. Hence (a_0, b_0, c_0) is connected to

$$(a_1, b_1, 0) := \left(-x_0 - y_0 \frac{\cos(r_1 y_0)}{\sin(r_1 y_0)}, \frac{y_0 e^{r_1 x_0}}{\sin(r_1 y_0)}, 0 \right).$$

$c = 0$ implies we have only one delay, so $(a_1, b_1, 0)$ in turn is connected to $(-1, 0, 0)$ (see [1]).

In both cases (a_0, b_0, c_0) is connected to $(-1, 0, 0)$, and thus to $(-a, 0, 0)$, $a > 0$ (see Lemma 1). So U_{abc} is connected and unbounded.

If now $(a_0, b_0, c_0) \in \partial S_{abc} \cup \partial U_{abc}$, then there is a $y_0 \geq 0$, such that $\chi(iy_0) = 0$, and $\chi(\cdot)$ has no roots with positive real part. Using the curve $\Gamma(t)$ defined above we see $(a_0, b_0, c_0) \in \partial S_{abc} \cap \partial U_{abc}$, which implies $\mathbb{R}^3 \setminus S_{abc} = \overline{U}_{abc}$, $\mathbb{R}^3 \setminus U_{abc} = \overline{S}_{abc}$.

As a corollary we get the connectedness in the space of all parameters:

COROLLARY 1. *S and U are connected and unbounded, and* $(\mathbb{R}^3 \times \mathbb{R}^2_{\geq}) \setminus S = \overline{U}$, $(\mathbb{R}^3 \times \mathbb{R}^2_{\geq}) \setminus U = \overline{S}$.

Proof. Proposition 1 and Lemma 1.

3. FIXED PARAMETERS a, b, c

In this section a, b, c are fixed real numbers. As already has been mentioned, in the case of only one delay both S_r and U_r are connected, so we will always assume $b \neq 0 \neq c$.

We shall show that U_r is connected and unbounded (see Proposition 2).

Since the proof is very technical, and there are often various cases to be looked at, we start by giving a rough sketch of the proof to come, for the two most important cases. At the same time we introduce some notations we shall use throughout this section.

$(r_1, r_2) \in U_r$ if the characteristic function $\chi(\cdot; r_1, r_2)$ has a root μ with $\operatorname{Re} \mu > 0$. So as usual we look at those (r_1, r_2) for which $\chi(,; r_1, r_2)$ has a purely imaginary root iy, $y \geq 0$.

We introduce new variables (s_1, s_2) and (σ, τ) which we shall use in the proofs to come.

As others before, define

$$s_j := r_j y, \qquad j = 1, 2. \tag{2}$$

Then $\chi(iy; r_1, r_2) = 0$ is equivalent to

$$0 = a + b\cos s_1 + c\cos s_2 \tag{3}$$

$$0 = y - b\sin s_1 - c\sin s_2. \tag{4}$$

The solution (s_1, s_2, y) of Eqs. (3), (4) is locally a real analytic curve, but it is not always true that $s_1 = s_1(s_2)$ or vice versa. If we rotate the $s_1 s_2$-plane by $\frac{\pi}{4}$ introducing new variables (σ, τ)

$$\sigma := \frac{1}{2}(s_1 - s_2) = \frac{1}{2}(r_1 - r_2)y, \qquad \tau := \frac{1}{2}(s_1 + s_2) = \frac{1}{2}(r_1 + r_2)y$$
$$\Leftrightarrow r_1 = \frac{\sigma + \tau}{y}, \qquad r_2 = \frac{\tau - \sigma}{y}, \tag{5}$$

then the solution to (3), (4) expressed in terms of σ and τ has (locally) the form $(\sigma, \tau, y) = (\sigma, \tau(\sigma), y(\sigma))$; i.e., the solution (r_1, r_2, y) of $\chi(iy; r_1, r_2) = 0$ is a curve $(r_1, r_2, y) = (r_1, r_2, y)(\sigma) = (r_1(\sigma), r_2(\sigma), y(\sigma))$ (see Lemma 2). We shall call it, or just the part $(r_1, r_2)(\sigma)$, the solution curve.

Abusing notation, we shall use the same name $(y, r_1, r_2$, etc.) regardless of the variables it depends on. For example, $y = y(s_1, s_2) = y(\sigma)$, using (4) and expressing y as a function of s_1, s_2, or σ, respectively.

Moreover, on the left-hand side (as σ increases) of such a solution curve the corresponding root μ of χ satisfies $\operatorname{Re}\mu > 0$; on the right-hand side it is $\operatorname{Re}\mu < 0$ (for details see Lemma 2). This means that locally the points (r_1, r_2) on the left-hand side of a solution curve always belong to U_r.

We shall frequently use this fact. In particular we shall use without further comment that if a continuous curve $\Gamma(t)$ consists piecewise of solution curves $(r_1, r_2)(\sigma)$ which cross at the "glueing points" of Γ, then going parallel to these $(r_1, r_2)(\sigma)$ just within the respective left-hand sides, one gets a curve $\tilde{\Gamma}$ arbitrarily near to Γ which lies entirely in U_r.

Here, as in the rest of this article, when we say two curves cross each other, we mean that they intersect non-tangentially. When we say two solution curves can be connected within U_r, we mean there is a continuous curve $\gamma: [\alpha, \beta] \to \mathbb{R}^2_{\geq}$, such that $\gamma(\alpha), \gamma(\beta)$ lie on both solution curves, respectively, and $\gamma(]\alpha, \beta[) \subset U_r$.

We need to know more in detail what the solution curves look like. To that end we use the variables s_1, s_2 and Eqs. (3), (4). Obviously any translates of a solution (s_1, s_2) by multiples of 2π in any direction are also a solution of (3), (4). So we only have to solve them for (s_1, s_2) in a suitable chosen square of length 2π.

Typically the solutions of (3) form a curve $(s_1, s_2)(t)$. Depending on the parameters a, b, c, this curve can essentially have one of two forms: it can

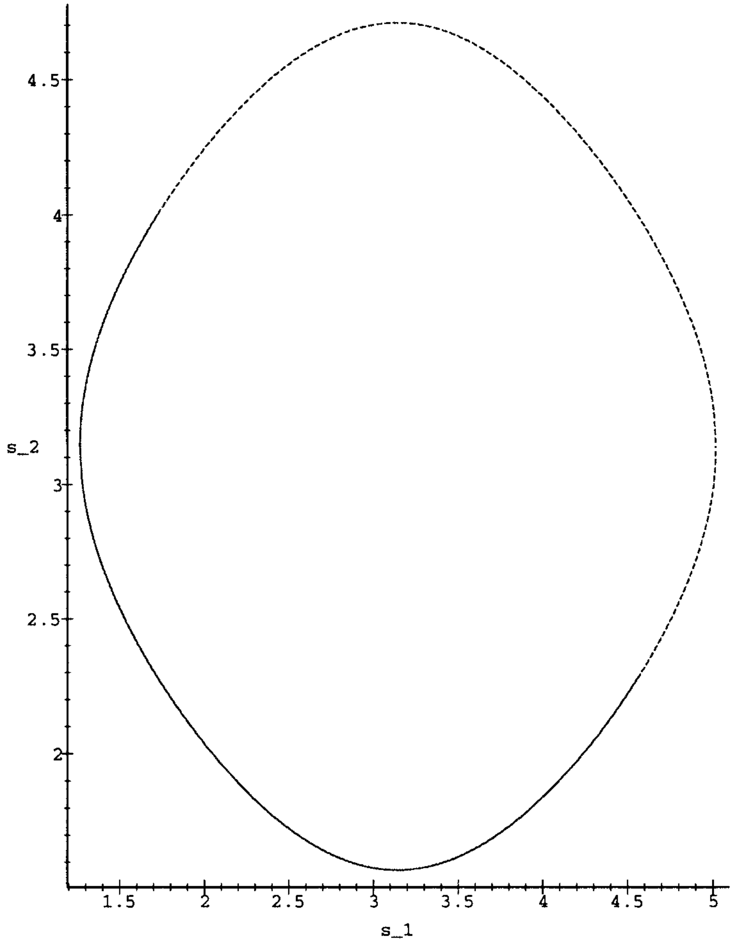

FIG. 1. Example for case (i) of Lemma 3. The drawn (dotted) curve is $(s_1, s_2)(\sigma)$ with $y(\sigma) > 0$ ($y(\sigma) < 0$). Here $a = b = 1$, $c = 1.3$, $n_b = n_c = 1$.

be a closed simple curve (see Fig. 1 for a typical example), or it has the form $s_2 = s_2(s_1)$, $s_1 \in \mathbb{R}$ (resp. $s_1(s_2)$, see Fig. 3 for an example). Actually in the latter case there are always two such curves, namely $s_2(s_1)$ and $2\pi - s_2(s_1)$, but defining $y = y(s_1)$ by (4), for one of these we have $y(s_1) > 0$ while for the other $y(s_1) < 0$ (see Lemma 3), which we disregard because only a positive y gives positive r_1, r_2.

Similarly in the former case of a closed curve $(s_1, s_2)(t)$, for half of the curve the corresponding $y(t)$ is positive, while for the other half it is negative. Again only the former part is of interest to us (see Lemma 3 for a detailed list of possible cases, and some further information about the curves $(s_1, s_2)(t)$). Having a solution $(s_1(t), s_2(t), y(t))$ of Eqs. (3), (4), $r_j(t) := s_j(t)/y(t)$, $j = 1, 2$, satisfies $\chi(iy(t); r_1(t), r_2(t)) = 0$. That is, out of a curve $(s_1(t), s_2(t))$ we get a solution curve $(r_1, r_2)(t)$ by pointwise scaling by $\frac{1}{y(t)}$.

The behaviour of $(r_1, r_2)(t)$ is quite different for both cases of $(s_1, s_2)(t)$ being either one-half of a closed curve or a function of s_1. We shall look at them separately, starting with the latter case which is the easier one.

If $s_2 = s_2(s_1)$, $y = y(s_1)$ is a solution of (3), (4) with $y(s_1) > 0$ for all $s_1 \in \mathbb{R}$, then $r_1 = s_1/y(s_1) \to \infty$, as $s_1 \to \infty$. We already mentioned that locally the left-hand side of any solution curve $(r_1, r_2)(t)$ belongs to U_r, so U_r is unbounded. It is also connected, since for decreasing $s_1, (r_1(s_1), r_2(s_1))$ eventually crosses the r_2-axis. But the set of r_2 such that $(0, r_2) \in U_r$ is connected, because there is only one delay.

If $(s_1, s_2) = (s_1, s_2)(\sigma)$, $\sigma \in I$, is one-half of a closed simple curve, then at the endpoints of I we have $y(\sigma) = 0$. This means the corresponding solution curves $(r_1, r_2)(\sigma)$ tend to infinity, as σ approaches an endpoint of I. The left-hand side of the solution curve belongs to U_r, which therefore is unbounded.

The more difficult part is to prove that U_r is connected.

Every point in U_r is obviously connected to a point on the boundary of U_r, that is, an (r_1, r_2) on one solution curve. Since the left-hand side of these solution curves belong to U_r, it is sufficient to show that all solution curves are connected; i.e., between any two given solution curves there is a continuous path Γ connecting them, which consists piecewise of solution curves, and at those points of Γ where two pieces join, the corresponding solution curves cross each other.

This is done in Lemma 5, where we construct such continuous curves Γ. Each Γ intersects each straight line $t \mapsto t(\cos\varphi, \sin\varphi)$, $0 < \varphi < \frac{1}{2}\pi$ fixed, and the minimum of Γ can be chosen arbitrarily big. As already mentioned, the solution curves $(r_1, r_2)(\sigma)$, $\sigma \in I$, tend to infinity as σ approaches an endpoint of I, so any two given solution curves intersect a suitably chosen curve Γ (see Lemma 6).

This was roughly the proof we will present in this section.

As has already been said, we start by proving the set of $(r_1, r_2) \in \mathbb{R}^2_>$ for which $\chi(\cdot; r_1, r_2) = 0$ has a purely imaginary root, is locally a curve $(r_1, r_2)(\sigma)$, and the left-hand side of it (as σ increases) belongs to U_r.

LEMMA 2. *Assume $a + b + c > 0$ and $\chi(iy_0; \rho_1, \rho_2) = 0$, $y_0 \geq 0$, $\rho_1, \rho_2 > 0$. Then there is an open interval I containing $\sigma_0 := \frac{1}{2}(\rho_1 - \rho_2)y_0$, an open neighborhood $V \subset \mathbb{R}^3_>$ of (ρ_1, ρ_2, y_0), and real analytic maps $y, r_1, r_2\colon I \to V$, such that*

$$(r_1(\sigma_0), r_2(\sigma_0), y(\sigma_0)) = (\rho_1, \rho_2, y_0)$$

and for $(r_1, r_2, y) \in V$

$$\chi(iy; r_1, r_2) = 0 \Leftrightarrow \exists \sigma \in I:$$
$$(r_1, r_2, y) = (r_1, r_2, y)(\sigma) = \big(r_1(\sigma), r_2(\sigma), y(\sigma)\big).$$

Moreover, we have $\sigma \equiv \frac{1}{2}(r_1(\sigma) - r_2(\sigma))y(\sigma)$.

There is a neighborhood $V_r \subset \mathbb{R}^2_>$ *of* (ρ_1, ρ_2), *such that the curve* $(r_1, r_2)(\sigma) = (r_1(\sigma), r_2(\sigma))$ *divides* V_r *into two parts*: *the left-hand side resp. right-hand side as* σ *increases. For* (r_1, r_2) *in the left-hand side* $\chi(\cdot; r_1, r_2)$ *has a root* $\mu = \mu(r_1, r_2)$ *with* $\operatorname{Re} \mu > 0$; *for* (r_1, r_2) *in the right-hand side we have* $\operatorname{Re} \mu < 0$, *if* $\chi(\mu; r_1, r_2) = 0$ *and* $(\operatorname{Im} \mu, r_1, r_2) \in V$.

Proof. We use the variables s_1, s_2 and σ, τ introduced in (2) and (5).

Implicitly via $\chi(iy; r_1, r_2) = 0$ we define first $\tau(\sigma)$ and $y(\sigma)$, then $r_j(\sigma)$, $j = 1, 2$, in a neighborhood V of (ρ_1, ρ_2, y_0). To prove the statements regarding V_r, we do the same, only allowing arbitrary complex roots $\mu = x + iy$, thus getting $y(\sigma, x)$, $r_j(\sigma, x)$. Then we show that the partial derivative with respect to x points to the left of the curve $(r_1(\sigma, 0), r_2(\sigma, 0))$, from which the conclusions follow easily.

First, note that $y_0 \neq 0$ and $(\rho_1, \rho_2) \neq (0, 0)$, since $a + b + c > 0$.

$\chi(iy; r_1, r_2) = 0$ is equivalent to $g_1(y, \sigma, \tau) = g_2(y, \sigma, \tau) = 0$, where g_1, g_2 are defined by

$$\begin{aligned} g_1(y, \sigma, \tau) &= a + b\cos(\sigma + \tau) + c\cos(\tau - \sigma) = \operatorname{Re} \chi(iy; r_1, r_2) \\ g_2(y, \sigma, \tau) &= y - b\sin(\sigma + \tau) - c\sin(\tau - \sigma) = \operatorname{Im} \chi(iy; r_1, r_2). \end{aligned} \tag{6}$$

The given solution (ρ_1, ρ_2, y_0) corresponds to (σ_0, τ_0, y_0), where $\sigma_0 := \frac{1}{2}(\rho_1 - \rho_2)y_0$, $\tau_0 := \frac{1}{2}(\rho_1 + \rho_2)y_0$. At this point we have

$$\det \begin{pmatrix} \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_1}{\partial \tau} \\ \dfrac{\partial g_2}{\partial y} & \dfrac{\partial g_2}{\partial \tau} \end{pmatrix} = y_0 \neq 0,$$

hence using the implicit-function theorem, there is an interval I and maps $y, \tau\colon I \to \mathbb{R}_>$, such that $\sigma_0 \in I$, and

$$(\tau(\sigma_0), y(\sigma_0)) = (\tau_0, y_0), \qquad g_j(y(\sigma), \sigma, \tau(\sigma)) \equiv 0,\ \sigma \in I,\ j = 1, 2.$$

Returning to the $r_1 r_2$-plane by setting

$$r_1(\sigma) := \frac{\sigma + \tau(\sigma)}{y(\sigma)}, \qquad r_2(\sigma) := \frac{\tau(\sigma) - \sigma}{y(\sigma)}$$

the map $(r_1, r_2, y)(\sigma) = (r_1(\sigma), r_2(\sigma), y(\sigma))$ satisfies $\chi(iy(\sigma);\ r_1(\sigma),$

$r_2(\sigma)) \equiv 0$, and for $V \subset \mathbb{R}^3_>$ a neighborhood of (ρ_1, ρ_2, y_0) small enough, $(r_1, r_2, y)(\sigma)$, $\sigma \in I$ (eventually making I smaller), are the only roots of χ in V.

Now we show that $(r_1, r_2)(\sigma) \not\equiv const$, which we shall need later on.

Assume for a moment the opposite, i.e., $r_1(\sigma) \equiv \rho_1$, $r_2(\sigma) \equiv \rho_2$.

We cannot have $\rho_1 = \rho_2$, since then

$$\begin{aligned} 0 &\equiv \chi(iy(\sigma); \rho_1, \rho_2) \\ &= a + (b + c)\cos \rho_1 y(\sigma) + i(y(\sigma) - (b + c)\sin \rho_1 y(\sigma)) \end{aligned}$$

would imply $y(\sigma) \equiv const$, and thus $\chi(iy; r_1, r_2) \neq 0$ for all $(r_1, r_2, y) \in V \setminus \{(\rho_1, \rho_2, y_0)\}$, which clearly is false.

So assume $\rho_1 \neq \rho_2$.

$r_1(\sigma)$ and $r_2(\sigma)$ being constant, (5) implies τ is a fixed multiple of σ, and $g_1(y(\sigma), \sigma, \tau(\sigma))$ can be written explicitly as a function in σ. This we shall use to get a contradiction.

Set $\alpha := (\rho_1 + \rho_2)/(\rho_1 - \rho_2) \neq 0$. Then $\tau = \alpha\sigma$, and

$$\begin{aligned} 0 &\equiv \frac{d}{d\sigma} g_1(y(\sigma), \sigma, \tau(\sigma)) \\ &\equiv -b(1 + \alpha)\sin((1 + \alpha)\sigma) - c(\alpha - 1)\sin((\alpha - 1)\sigma). \end{aligned}$$

Differentiating this expression twice with respect to σ and solving with respect to $\sin((\alpha - 1)\sigma)$, we find either $\alpha = -1$, which yields $\sin(-2\sigma) \equiv 0$, or $(1 + \alpha)^2 - (\alpha - 1)^2 = 0$, which in turn gives $\alpha = 0$. Both cases cannot be, so indeed $(r_1, r_2)(\sigma) \not\equiv const$.

Taking V_r to be the projection of V onto the last two coordinates, eventually making V smaller, we can assume $(r_1, r_2)(\sigma)$, $\sigma \in I$, to divide V_r into two parts, the left-hand side respectively the right-hand side as σ increases.

To be able to prove the statements about the roots of $\chi(.; r_1, r_2)$ for (r_1, r_2) in either one of these parts of V_r, we have to look at how the root $\mu = iy$ of $\chi(.; r_1, r_2)$ evolves as (r_1, r_2) moves off the curve $(r_1(\sigma), r_2(\sigma))$. More specifically we need to know how the real part x of μ changes. So we make the same change of variables as before, only incorporating the real part x of μ: (r_1, r_2, x, y) becomes (σ, τ, x, y).

Define $g_1(x, y, \sigma, \tau) = \operatorname{Re} \chi(x + iy; r_1, r_2)$, $g_2(x, y, \sigma, \tau) = \operatorname{Im} \chi(x + iy; r_1, r_2)$, i.e.,

$$\begin{aligned} g_1(x, y, \sigma, \tau) &= x + a + be^{-\frac{x}{y}(\sigma + \tau)}\cos(\sigma + \tau) + ce^{-\frac{x}{y}(\tau - \sigma)}\cos(\tau - \sigma) \\ g_2(x, y, \sigma, \tau) &= y - be^{-\frac{x}{y}(\sigma + \tau)}\sin(\sigma + \tau) - ce^{-\frac{x}{y}(\tau - \sigma)}\sin(\tau - \sigma). \end{aligned}$$

Then, with the same reasoning as above, we get maps y, r_1, r_2: $I \times U(0) \to$

$\mathbb{R}_>$, $U(0) \subset \mathbb{R}$ an open interval, such that $(r_1, r_2, y)(\sigma, 0) = (r_1, r_2, y)(\sigma)$ as defined above, and $\chi(x + iy(\sigma, x), r_1(\sigma, x), r_2(\sigma, x)) \equiv 0$.

We claim that the scalar product between the normal of the curve $(r_1, r_2)(\sigma)$ pointing into the left-hand side of V_r, and the partial derivative $\partial(r_1, r_2)/\partial x$ is positive for at least one point on the dividing curve $(r_1, r_2)(\sigma) = (r_1(\sigma, 0), r_2(\sigma, 0))$.

This means there is at least one point $(\tilde{r}_1, \tilde{r}_2)$ in the left-hand side of V_r, for which the root μ of $\chi(.; \tilde{r}_1, \tilde{r}_2)$ with $(\tilde{r}_1, \tilde{r}_2, \operatorname{Im} \mu) \in V$ satisfies $\operatorname{Re} \mu < 0$. Since we know by construction that for $(r_1, r_2, y) \in V$ not on the dividing curve $(r_1, r_2)(\sigma)$, there is no root $\mu = iy$ of $\chi(.; r_1, r_2)$, the conclusion follows for the left-hand side, and analogously also for the other one.

To prove the claim, we need the partial derivatives of r_1, r_2.

Differentiating $g_j(x, y(\sigma, x), \sigma, \tau(\sigma, x))$, $j = 1, 2$, with respect to σ and x and solving with respect to the various partial derivatives, a tedious but straightforward calculation yields

$$\begin{aligned}
\frac{\partial y}{\partial \sigma}(\sigma_0, 0) &= -\frac{2bc}{y_0}\sin(2\sigma_0) \\
\frac{\partial \tau}{\partial \sigma}(\sigma_0, 0) &= \frac{1}{y_0}(-b\sin(\rho_1 y_0) + c\sin(\rho_2 y_0)) \\
\frac{\partial y}{\partial x}(\sigma_0, 0) &= \frac{-1}{y_0}\big(a + \rho_1 b^2 + \rho_2 c^2 + (\rho_1 + \rho_2) bc\cos(2\sigma_0)\big) \\
\frac{\partial \tau}{\partial x}(\sigma_0, 0) &= \frac{1}{y_0}(1 - \rho_1 b\cos(\rho_1 y_0) - \rho_2 c\cos(\rho_2 y_0)).
\end{aligned} \tag{7}$$

Using these equations in the definitions of $r_j(\sigma)$, $j = 1, 2$, and keeping in mind $g_1(0, y_0, \sigma_0, \tau_0) = g_2(0, y_0, \sigma_0, \tau_0) = 0$, a simple calculation shows

$$\begin{aligned}
\frac{\partial r_1}{\partial \sigma}(\sigma_0, 0) &= \frac{2c}{y_0^2}(\sin \rho_2 y_0 + \rho_1 b\sin(\rho_1 - \rho_2) y_0) \\
\frac{\partial r_2}{\partial \sigma}(\sigma_0, 0) &= \frac{2b}{y_0^2}(-\sin \rho_1 y_0 + \rho_2 c\sin(\rho_1 - \rho_2) y_0) \\
\frac{\partial r_1}{\partial x}(\sigma_0, 0) &= \frac{1}{y_0^2}\big(1 - \rho_1 b\cos \rho_1 y_0 - \rho_2 c\cos \rho_2 y_0 \\
&\qquad + \rho_1\big(a + \rho_1 b^2 + \rho_2 c^2 + (\rho_1 + \rho_2) bc\cos(\rho_1 - \rho_2) y_0\big)\big)
\end{aligned}$$

$$\frac{\partial r_2}{\partial x}(\sigma_0, 0) = \frac{1}{y_0^2}\Big(1 - \rho_1 b \cos \rho_1 y_0 - \rho_2 c \cos \rho_2 y_0$$
$$+\rho_2\big(a + \rho_1 b^2 + \rho_2 c^2 + (\rho_1 + \rho_2) bc \cos(\rho_1 - \rho_2) y_0\big)\Big).$$

Keeping in mind $\chi(iy_0; \rho_1, \rho_2) = 0$, another tedious but straightforward calculation shows

$$\left(-\frac{\partial r_2(\sigma_0, 0)}{\partial \sigma}, \frac{\partial r_1(\sigma_0, 0)}{\partial \sigma}\right)\left(\begin{array}{c}\frac{\partial r_1(\sigma_0, 0)}{\partial x} \\ \frac{\partial r_2(\sigma_0, 0)}{\partial x}\end{array}\right)$$
$$= \frac{2}{y_0^3}\left(\left(\frac{\partial g_1(0, y_0, \sigma_0, \tau_0)}{\partial x}\right)^2 + \left(\frac{\partial g_2(0, y_0, \sigma_0, \tau_0)}{\partial x}\right)^2\right) \geq 0. \quad (8)$$

In other words, the scalar product between the normal to the curve $(r_1, r_2)(\sigma, 0)$ pointing to the left-hand side of V_r and $\nabla_x(r_1, r_2)$ at the point $(r_1, r_2) = (\rho_1, \rho_2)$ is never negative.

We have only to show that in (8) we have a strict $>$ for at least one point $(r_1, r_2)(\sigma)$ on the dividing curve.

Assume this to be false. Then for all $\sigma \in I$ we have

$$\frac{\partial g_1}{\partial x}(0, y(\sigma), \sigma, \tau(\sigma)) = \frac{\partial g_2}{\partial x}(0, y(\sigma), \sigma, \tau(\sigma)) = 0.$$

But by definition of g_1, g_2, this means

$$\frac{\partial \chi}{\partial \mu}(iy(\sigma); r_1(\sigma), r_2(\sigma)) = \chi'(iy(\sigma)) = 0.$$

So we have for $(r_1, r_2, y) = (r_1, r_2, y)(\sigma)$

$$0 \equiv y - b \sin r_1 y - c \sin r_2 y \qquad (\operatorname{Im} \chi \equiv 0)$$
$$0 \equiv r_1 b \sin r_1 y + r_2 c \sin r_2 y \qquad (\operatorname{Im} \chi' \equiv 0)$$
$$0 \equiv 1 - r_1 b \cos r_1 y - r_2 c \cos r_2 y \qquad (\operatorname{Re} \chi' \equiv 0).$$

Setting $s_1 = r_1 y$ and $s_2 = r_2 y$, we get the following equivalent set of equations:

$$\begin{aligned} y &\equiv b \sin s_1 + c \sin s_2 \Leftrightarrow \\ 0 &\equiv s_1 b \sin s_1 + s_2 c \sin s_2 \\ 0 &\equiv b(\sin s_1 - s_1 \cos s_1) + c(\sin s_2 - s_2 \cos s_2). \end{aligned} \quad (9)$$

s_1 and s_2 are functions of σ. Differentiating the last equation in (9), we have

$$\begin{aligned} 0 &\equiv b(\sin s_1 + s_1 \cos s_1) s_1' + c(\sin s_2 + s_2 \cos s_2) s_2' \\ 0 &\equiv b s_1 \sin s_1 s_1' + c s_2 \sin s_2 s_2'. \end{aligned} \tag{10}$$

Using the second equation of (9) and the last of (10), we get $0 \equiv c s_2 \sin s_2 (s_2' - s_1')$. $\sin s_2 = 0$ implies $\sin s_1 = 0$, and thus $y = 0$, which we know not to be. So $s_1' \equiv s_2'$ follows. If $s_1' \equiv s_2' = 0$, then $y' \equiv 0$ too, hence $r_1' \equiv (s_1/y)' \equiv 0 \equiv r_2'$, but we know already $(r_1, r_2) \not\equiv \mathit{const}$. So we arrive at

$$0 \equiv b(\sin s_1 + s_1 \cos s_1) + c(\sin s_2 + s_2 \cos s_2),$$

and with the last equation of (9), $0 \equiv 2b \sin s_1 + 2c \sin s_2$, hence $y \equiv 0$, a contradiction.

This proves the claim and completes the proof of the lemma.

Given the problem of joining two given points in U_r, with Lemma 2 it is sufficient to find a connecting curve γ which consists partially of parts of solution curves $(r_1, r_2)(\sigma)$. This is so because where such a γ coincides with a solution curve, one can shift it a little bit onto the left-hand side of the solution curve, and the resulting $\tilde{\gamma}$ lies indeed totally within U_r.

This is even possible if γ is locally one solution curve glued to a second one, if both cross (= intersect non-tangentially) at this point, as we have already mentioned at the beginning of this section.

The set of (s_1, s_2) which solves Eqs. (3) and (4) for a suitable $y > 0$ is a curve $(s_1, s_2)(t)$, $t \in I$. The following lemma lists all possible cases with respect to the behaviour of this curve $(s_1, s_2)(t)$.

Roughly speaking, in its case (i), $(s_1, s_2)(t)$ is one-half of a closed simple curve, where the corresponding $y(t)$ becomes 0 at the endpoints. See also Fig. 1 for a typical example of $(s_1, s_2)(t)$. Figure 2 depicts corresponding curves $(r_1, r_2)(t)$; the notation is as defined in (17). In case (ii) one variable is a function of the other one, e.g., $s_2 = s_2(s_1)$, $y = y(s_1)$, and both functions are 2π-periodic. A typical example is shown in Fig. 3, and in Fig. 4 one finds corresponding curves $(r_1, r_2)(t)$. Case (iii) is an intermediate one. One can still write, e.g., $s_2(s_1)$, but where in (ii) always $y(s_1) > 0$, now $y(s_1) = 0$ occurs. Examples of $s_2(s_1)$ and the corresponding curve $(r_1(s_1), r_2(s_1))$ are shown in Figs. 5 and 6. Case (iv) is the very special case where the solutions of (3) form squares in the $s_1 s_2$-plane, and on two sides of a square the corresponding y is equal to 0.

In Lemma 3 we shall speak of the first, ..., fourth quadrant with respect to a center point p. With this we mean the first, ..., fourth quadrant (counting as usual) of the (local) Cartesian coordinate system with origin in p.

LEMMA 3. *Assume* $a + b + c > 0$.

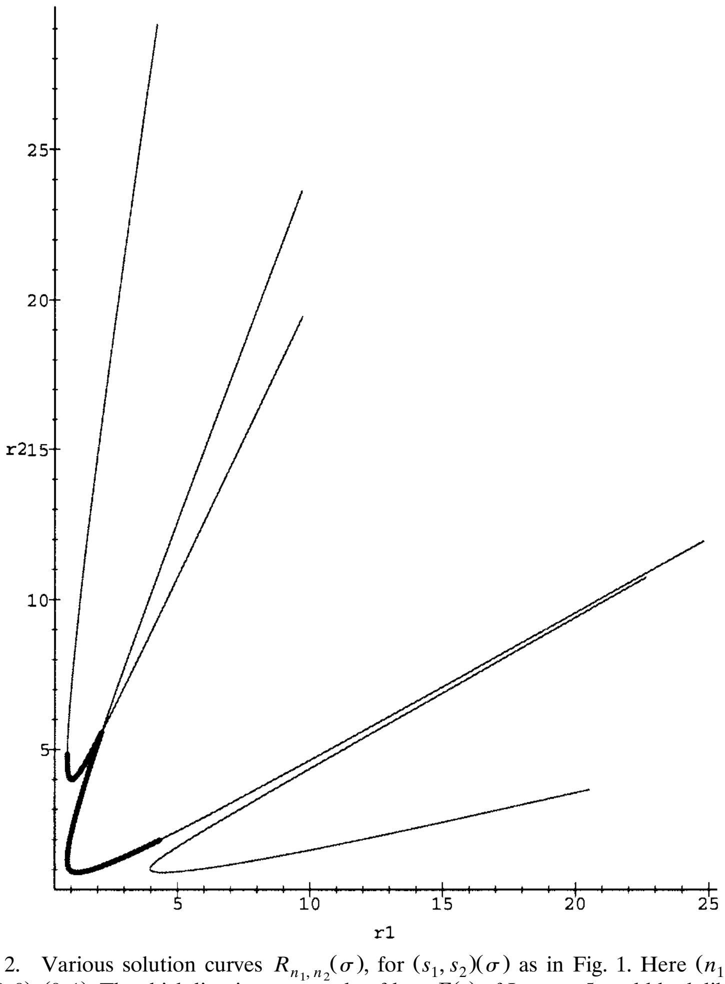

FIG. 2. Various solution curves $R_{n_1, n_2}(\sigma)$, for $(s_1, s_2)(\sigma)$ as in Fig. 1. Here $(n_1, n_2) = (1, 0), (0, 0), (0, 1)$. The thick line is an example of how $\Gamma(t)$ of Lemma 5 could look like. U_r is above it.

Depending on the parameters a, b, c, the solutions (s_1, s_2, y) to Eqs. (3) and (4), with $y > 0$, have the following form:

(i) $||b| - |c|| < |a| < |b| + |c|$. Let $n_b, n_c \in \{0, 1\}$ be defined by (12) below.

There exist an interval $I = [\alpha, \beta]$, functions y: $I \to \mathbb{R}_{\geq}$, s_1: $I \to](n_b - 1)\pi, (n_b + 1)\pi[$, s_2: $I \to](n_c - 1)\pi, (n_c + 1)\pi[$, such that every solution

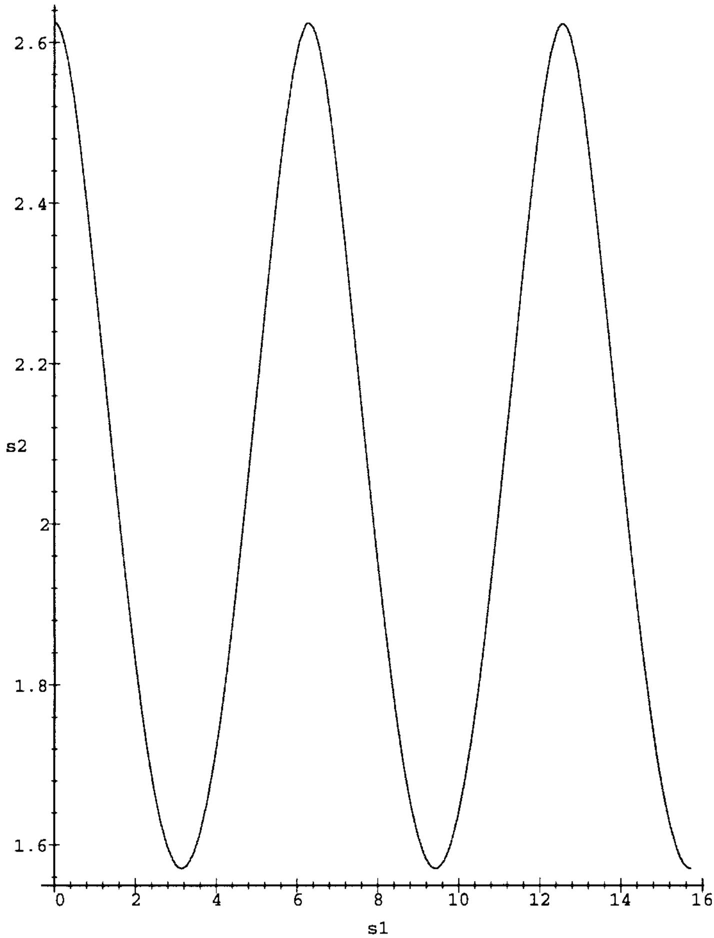

FIG. 3. Example for case (ii) of Lemma 3. The curve is $s_2(s_1)$, where $a = b = 1$, $c = 2.3$, $n_c = 1$.

(s_1, s_2, y) can be written

$$(s_1, s_2, y) = \big(s_1(\sigma) + 2n_1\pi, s_2(\sigma) + 2n_2\pi, y(\sigma)\big),$$

$a\sigma \in]\alpha, \beta[$ and $n_1, n_2 \in \mathbb{Z}$.

Moreover, $y(\alpha) = y(\beta) = 0$, $y(\sigma) > 0$ for $\sigma \in]\alpha, \beta[$. $(s_1(\sigma), s_2(\sigma))$ starts in the second and ends in the fourth quadrant with respect to the center $(n_b\pi, n_c\pi)$, but $(n_b\pi, n_c\pi) \neq (s_1(\sigma), s_2(\sigma))$, for all $\sigma \in I$. $(s_1(I), s_2(I))$ intersects each local axis $s_1 \equiv n_b\pi$ and $s_2 \equiv n_c\pi$ once.

Note that here σ is that variable defined in (5).

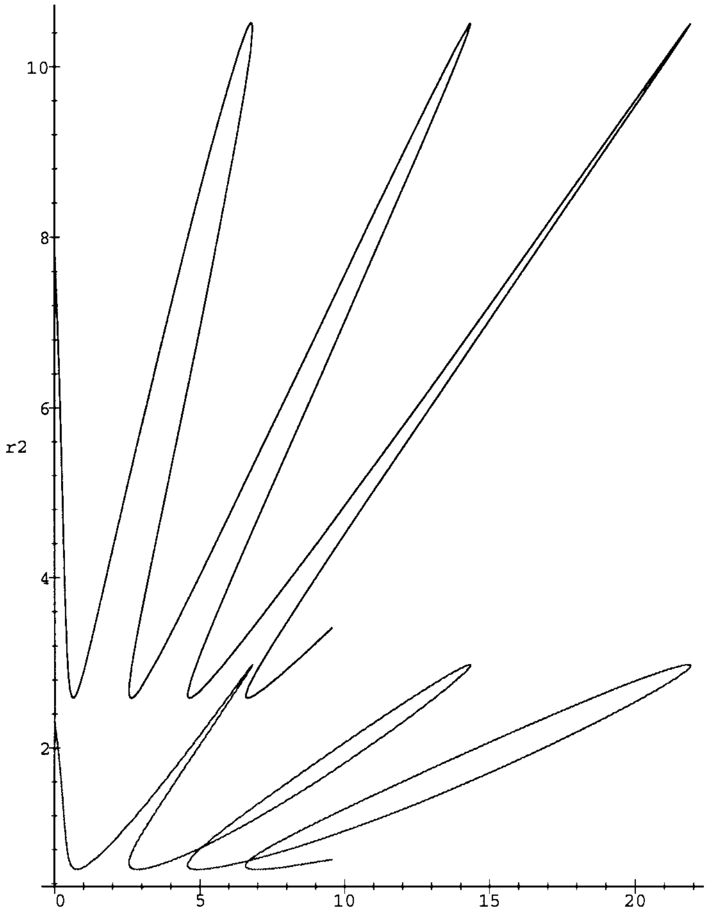

FIG. 4. Various solution curves $(r_1(s), r_2(s)) = \frac{1}{y(s)}(s_1, s_2(s_1) + 2n_2\pi)$, where $s_2(s_1)$ is as in fig. 3. Here $n_2 = 0, 1$.

(ii) $|a| < ||b| - |c||$. If $|a| < |c| - |b|$, then there are 2π-periodic functions $s_2, y\colon \mathbb{R} \to \mathbb{R}$, such that every solution (s_1, s_2, y) can be written

$$(s_1, s_2, y) = (s_1, s_2(s_1) + 2n_2\pi, y(s_1)),$$

an $n_2 \in \mathbb{Z}$. $y(s_1) > 0$, and $s_2(s_1) \notin \mathbb{Z}\pi$, for all $s_1 \in \mathbb{R}$.

An analogous statement holds for the case $|a| < |b| - |c|$ exchanging s_1 and s_2.

(iii) $0 \neq |a| = ||b| - |c||$. Let $n_b, n_c \in \{0, 1\}$ be defined as in Eq. (12) below.

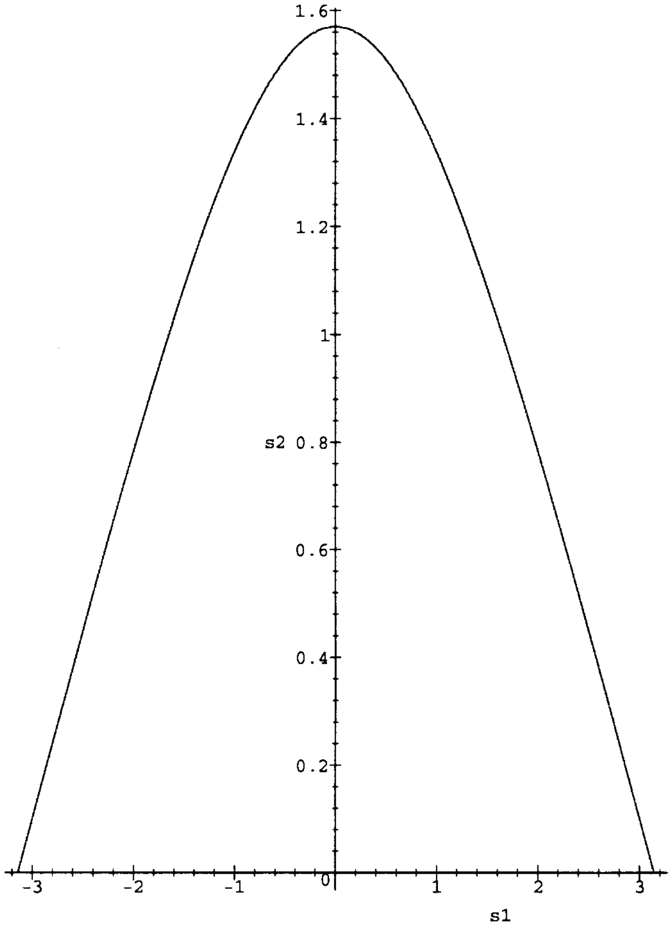

FIG. 5. Example for case (iii) of Lemma 3. The curve is $(s_1, s_2)(s)$, where $a = -1$, $b = 1$, $c = 2$, $n_b = n_c = 0$.

If $|a| = |c| - |b|$, let $I = [(n_b - 1)\pi, (n_b + 1)\pi]$. There are functions y: $I \to \mathbb{R}_{\geq}$, s_2: $I \to [(n_c - 1)\pi, (n_c + 1)\pi]$, such that every solution (s_1, s_2, y) can be written

$$(s_1, s_2, y) = \big(s + 2n_1\pi, s_2(s) + 2n_2\pi, y(s)\big),$$

an $s \in](n_b - 1)\pi, (n_b + 1)\pi[$, $n_1, n_2 \in \mathbb{Z}$.

Moreover, $y((n_b - 1)\pi) = y((n_b + 1)\pi) = 0$, $s_2((n_b - 1)\pi) = s_2((n_b + 1)\pi) = n_c\pi$, and for $s \in](n_b - 1)\pi, (n_b + 1)\pi[$ we have $y(s) > 0$ and $s_2(s) \in](n_c - 1)\pi, (n_c + 1)\pi[\setminus\{n_c\pi\}$.

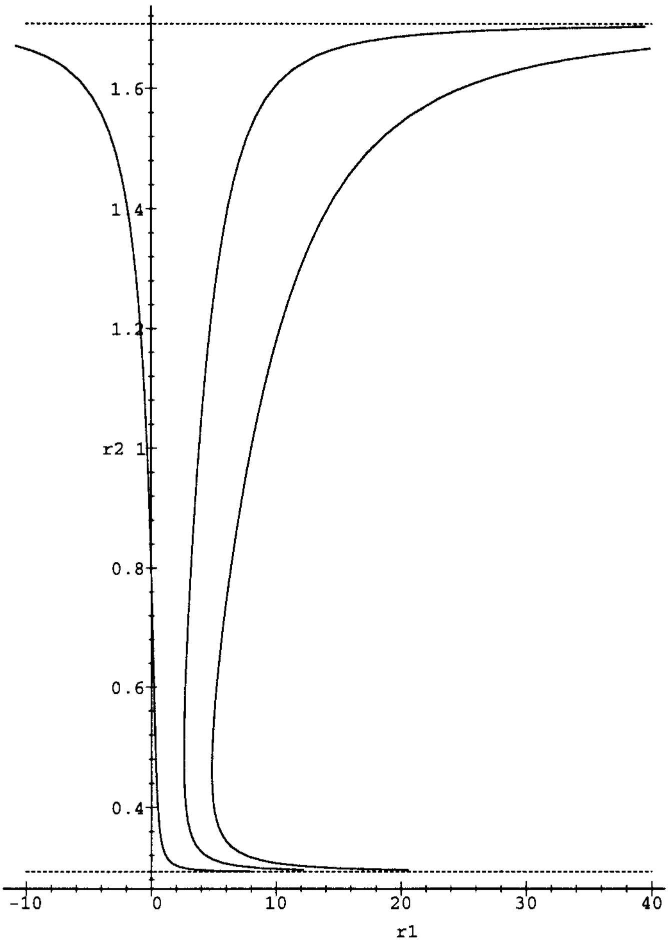

FIG. 6. Various solution curves $R_{n_1, n_2}(s)$ for $(s_1, s_2)(s)$ as in Fig. 5. The dotted lines are the asymptotics for these solution curves. Here $(n_1, n_2) = (0, 0), (1, 0), (2, 0)$.

An analogous statement holds for the case $|a| = |b| - |c|$ exchanging s_1 and s_2.

(iv) $a = 0$, $b = (-1)^{n_0} c$. Let $n_b = \mathit{sign}\, b$.

There are functions s_1, s_2, y: $[n_b \pi, (n_b + 1)\pi] \to \mathbb{R}$, such that every solution (s_1, s_2, y) can be written

$$(s_1, s_2, y) = \left(s + 2n_1\pi, -s + (n_0 + 1 + 2n_2)\pi, y(s)\right),$$

an $s \in]n_b \pi, (n_b + 1)\pi[$, $n_1, n_2 \in \mathbb{Z}$. $y(n_b \pi) = y((n_b + 1)\pi) = 0$.

For $|a| \geq |b| + |c|$ there is no solution of (3) and (4) with positive y.

Note that if in case (iv) we define $n_c = \mathit{sign}\, c$, then as in case (i), we can also say in cases (iii) and (iv) that $(s_1(\sigma), s_2(\sigma))$ starts in the second and ends in the fourth quadrant with respect to the center $(n_b\pi, n_c\pi)$, without passing through $(n_b\pi, n_c\pi)$.

Defining $r_j(t) := s_j(t)/y(t)$, $j = 1, 2$, using the functions $s_j(t)$, $y(t)$ given in Lemma 3, we have a parametrization of a solution curve which is different from that in Lemma 2 (with the exception of case (i) above). However, both parametrizations have the same orientation. Hence we can speak of the left-(right-)hand side of a solution curve, given either by Lemma 2, or defined via Lemma 3.

Proof. After a short calculation, Eqs. (3), (4) yield

$$y^2 = b^2 + c^2 - a^2 + 2bc\cos(s_1 - s_2) = b^2 + c^2 - a^2 + 2bc\cos(2\sigma). \tag{11}$$

It is obvious that (s_1, s_2, y) solves Eqs. (3) and (4) iff $(s_1 + 2n_1\pi, s_2 + 2n_2\pi, y)$ solves these equations, for all $n_1, n_2 \in \mathbb{Z}$.

If $|a| \geq |b| + |c|$, then there are no solutions to Eqs. (3) and (4) with $y > 0$. The remaining cases can be divided as in the conclusion of the lemma.

Case i. $||b| - |c|| < |a| < |b| + |c|$.
Define

$$n_b = \begin{cases} 0 & \text{if } ab < 0 \\ 1 & \text{if } ab > 0 \end{cases} \qquad n_c = \begin{cases} 0 & \text{if } ac < 0 \\ 1 & \text{if } ac > 0. \end{cases} \tag{12}$$

For the whole proof of case (i) we shall always assume

$$(s_1, s_2) \in [(n_b - 1)\pi, (n_b + 1)\pi] \times [(n_c - 1)\pi, (n_c + 1)\pi].$$

The conclusions of the lemma will be shown by combining the two representations we have for solutions s_1, s_2, y of (3), (4), namely s_1, s_2, and σ, τ as defined in (5), and the explicit formulas

$$\cos s_2 = -\frac{a + b\cos s_1}{c} \in \left[-\frac{a}{c} - \left|\frac{b}{c}\right|, -\frac{a}{c} + \left|\frac{b}{c}\right|\right] =: [\alpha_1, \beta_1], \tag{13}$$

$$y = b\sin s_1 + c\sin s_2 \tag{14}$$

we get from these equations.

First we show that the solutions (s_1, s_2) of Eq. (3) form a closed simple curve in the interior of the above mentioned square.

Equation (3) is equivalent to (13). A straightforward calculation shows $\alpha_1 < -1 < \beta_1 < 1$, if $ac > 0$, and $-1 < \alpha_1 < 1 < \beta_1$, if $ac < 0$.

Note that this proves $s_2 \in](n_c - 1)\pi, (n_c + 1)\pi[$, and analogously $s_1 \in](n_b - 1)\pi, (n_b + 1)\pi[$.

So for suitably chosen $(n_b - 1)\pi < \tilde{a} < \tilde{b} < (n_b + 1)\pi$, we get a function $s_2(s_1)$, $s_1 \in [\tilde{a}, \tilde{b}]$, defined by (13), such that (s_1, s_2) solves (3) iff (modulo multiples of 2π) $s_2 = \pm s_2(s_1)$, and $s_2(\tilde{a}) = s_2(\tilde{b}) = n_c\pi$. That is, the set of solutions of (3) is a closed simple curve S.

Note that S intersects the axis $s_1 = n_b\pi$ (resp. $s_2 = n_c\pi$) exactly twice, once with $s_2 > n_c\pi$ ($s_1 > n_b\pi$), and once with $s_2 < n_c\pi$ ($s_1 < n_b\pi$), and for exactly one of these intersections $y > 0$ is true (see (4)).

We are interested only in those parts of S for which the corresponding y in (14) is positive.

As in the proof of Lemma 2, whenever $y = y(s_1, s_2) > 0$, y as in (14), we can locally define τ (and y) as a function of σ, thus expressing s_1 and s_2 as functions of σ too: $s_j = s_j(\sigma)$, $y = y(\sigma)$, $j = 1, 2$.

So S can be covered by curves $(s_1^{(j)}(\sigma), s_2^{(j)}(\sigma)$, $\sigma \in I_j$, $j \in J$, leaving only those s_1, s_2 for which $y = 0$. The corresponding $y^{(j)}(\sigma)$ for each of these curves satisfies either $y^{(j)}(\sigma) > 0$ or $y^{(j)}(\sigma) < 0$, and $y^{(j)}(\sigma) \to 0$, as σ approaches an endpoint of I_j.

We shall prove

(1) $y^{(j)'}(\sigma) \neq 0$, if the corresponding $(s_1(\sigma), s_2(\sigma))$ lies in the second or fourth quadrant.

(2) $y(s_1, s_2) \neq 0$, if (s_1, s_2) lies in the first or third quadrant.

(3) If $y(s_1, s_2) > 0$, for (s_1, s_2) in the first (resp. third) quadrant, then $y(s_1, s_2) < 0$, for (s_1, s_2) in the third (resp. first) quadrant.

The claims (1) and (2), S crossing each (local) axis twice, and that the $\sigma\tau$-plane is the $s_1 s_2$-plane rotated by $\frac{\pi}{4}$ (plus a contraction), prove that each curve $(s_1^{(j)}(\sigma), s_2^{(j)}(\sigma))$, $\sigma \in I_j$, starts in the second quadrant and ends in the fourth. That is, there are only two such curves: $J = \{1, 2\}$. By claim (3) exactly one of these gives positive $y(\sigma)$.

Before proving the claims, let us show how the remaining conclusion follows.

If $s_1(\alpha) = n\pi$, an $n \in \mathbb{Z}$, then $y(\alpha) = 0$ implies $s_2(\alpha) = m\pi$, $m \in \mathbb{Z}$. But then Eq. (3) shows $0 = a \pm b \pm c$, for suitably chosen signs, which cannot be. Thus $(s_1(\sigma), s_2(\sigma))$ does not pass through the center $(n_b, n_c)\pi$.

There remains only to prove the claims (1), (2), and (3).

Let $s_i^{(j)} = s_i^{(j)}(\sigma)$, $y^{(j)} = y^{(j)}(\sigma)$, $\sigma \in I_j$, $i = 1, 2$, $j \in J$, be given.

By Eq. (11), $y^{(j)'}(\sigma) = 0$ only if $\sin(s_1^{(j)} - s_2^{(j)}) = 0$, i.e., $s_2^{(j)} = s_1^{(j)} + n\pi$, an $n \in \mathbb{Z}$. For $(s_1^{(j)}, s_2^{(j)})$ in the second or fourth quadrant, this means

$s_2^{(j)} = s_1^{(j)} + (n_c - n_b \pm 1)\pi$. Inserting this into (3), we get

$$0 = a + \left(b + (-1)^{n_c - n_b \pm 1} c\right) \cos s_1^{(j)} = a \pm (|b| - |c|) \cos s_1^{(j)},$$

which can only happen for $|a| \le \||b| - |c|\|\#$. This proves the first claim.

Now for (s_1, s_2) in the first or third quadrant

$$b \sin s_1 + c \sin s_2 = \pm \mathit{sign}\, a(|b| + |c|),$$

where the sign in the right-hand side depends on the quadrant, and is different for both. $y \neq 0$ follows, and claims (2) and (3) have been proven.

Case ii. $|a| < | \, |b| - |c| \, |$. We assume $|a| < |c| - |b|$, the other case being symmetrical.

For any solution of $\chi(iy) = 0$ we have

$$\left| \frac{a + b \cos r_1 y}{c} \right| \le \frac{|a| + |b|}{|c|} < 1,$$

so for $s_1 \in \mathbb{R}$, we can define s_{+2} and s_{-2} by

$$\begin{aligned} s_{\pm 2}(s_1, n) &:= 2n\pi \pm \arccos \frac{-a - b \cos s_1}{c} \\ &\in](2n - 1)\pi, (2n + 1)\pi[\setminus \{2n\pi\} \end{aligned} \tag{15}$$

for $n \in \mathbb{Z}$, and Eqs. (3), (4) are equivalent to $s_2 = s_{\pm 2}(s_1, n)$, and $y = y_-$ or $y = y_+$ defined by

$$y_{\pm}(s_1) := b \sin s_1 + c \sin s_{\pm 2}(s_1, n).$$

Note that $y_{\pm}(s_1)$ is indeed independent of n, and $s_{\pm 2}(., n)$ and $y_{\pm}(.)$ are 2π periodic functions. Given the conditions on a, b, c in this case, Eq. (11) shows $y_{\pm}(s_1) \neq 0$ for all $s_1 \in \mathbb{R}$. Choose the sign in (15), so that y becomes positive. Fix n and call the resulting functions $s_2(s_1)$ and $y(s_1)$. They satisfy the conclusions.

Case iii. $0 \neq |a| = \||b| - |c|\|$. We assume without loss of generality $|a| = |c| - |b|$.

As in Case 2, we can define $s_{\pm 2}(s_1, n)$, $y_{\pm}(s_1)$, the only difference being that now $s_{\pm 2}((n_b + 1)\pi) \in \mathbb{Z}\pi$, and $y_{\pm}((n_b + 1)\pi) = 0$, n_b as in (12). Letting n_c as in (12), choosing sign and $n \in \mathbb{Z}$ in (15) suitably, and calling the resulting functions $s_2(s_1)$ resp. $y(s_1)$, we get

$$s_2((n_b \pm 1)\pi) = n_c \pi, \qquad y((n_b \pm 1)\pi) = 0$$

and for $s_1 \in](n_b - 1)\pi, (n_b + 1)\pi[$

$$s_2(s_1) \in](n_c - 1)\pi, (n_c + 1)\pi[\setminus \{n_c \pi\}, \qquad y(s_1) > 0.$$

Case iv. $a = 0$, $b = (-1)^{n_0} c$. Equations (3) and (4) become

$$0 = \cos s_1 + (-1)^{n_0} \cos s_2$$

$$y = b\big(\sin s_1 + (-1)^{n_0} \sin s_2\big).$$

Thus (3) is equivalent to

$$s_2 = (-1)^m s_1 + (n_0 + 1 + 2n)\pi,$$

an $m \in \{0, 1\}$, $n \in \mathbb{N}$, and (4) gives

$$y = b\big(1 - (-1)^m\big) \sin s_1.$$

Only $m = 1$ gives positive y. The conclusions follows immediately.

For s_1, s_2, y: $I = [\alpha, \beta] \to \mathbb{R}$ as in Lemma 3, and $n_1, n_2 \in \mathbb{Z}$, define the translated curve γ_{n_1, n_2}: $I \to \mathbb{R}^2$ by

$$\gamma_{n_1, n_2}(\sigma) = \big(s_1(\sigma) + 2n_1\pi, s_2(\sigma) + 2n_2\pi\big), \tag{16}$$

and the corresponding solution curve $R_{n_1, n_2}:]\alpha, \beta[\to \mathbb{R}^2$ by

$$R_{n_1, n_2}(\sigma) = \big(r_1^{(n_1)}(\sigma), r_2^{(n_2)}(\sigma)\big) = \left(\frac{s_1(\sigma) + 2n_1\pi}{y(\sigma)}, \frac{s_2(\sigma) + 2n_2\pi}{y(\sigma)}\right). \tag{17}$$

LEMMA 4. *Let* $p = (p_1, p_2) \in \mathbb{R}^2_{\geq} \cap \mathbb{Q}^2$, $n_0, n_1 \in \mathbb{N} \setminus \{0\}$, *and* $\alpha \in]0, \frac{\pi}{2}[$. *Then there are* $n_2, m_2 \in \mathbb{N}$, $n_2 \geq n_1$ *or* $m_2 \geq n_1$, *and the half-line* $g_\alpha(t) = t(\cos\alpha, \sin\alpha)$, $t > 0$, *intersects the disc* $\{(x_1, x_2) \in \mathbb{R}^2: |(x_1, x_2) - ((p_1 + 2n_2)\pi, (p_2 + 2m_2)\pi)| < \frac{1}{n_0}\}$.

Proof. Is it sufficient to show that for arbitrary $\alpha, p_1, p_2, n_0, n_1$ as above, there are a $t > 0$, $n, m \in \mathbb{N}$, $n \geq n_1$, or $m \geq n_1$, and

$$(t\alpha - p_1 - n)^2 + (t - p_2 - m)^2 < \frac{1}{n_0^2}. \tag{18}$$

Let $N \in \mathbb{N}$, such that $Np_1, Np_2 \in \mathbb{N}$, and set $t = p_2 + m$. Then (18) follows from

$$\left|\alpha - \frac{p_1 + n}{p_2 + m}\right| = \left|\alpha - \frac{Np_1 + Nn}{Np_2 + Nm}\right| < \frac{1}{N(p_2 + m)n_0}.$$

This inequality has infinitely many solutions (see, e.g., Hurwitz Theorem [7, p. 133]), which proves the lemma.

LEMMA 5. *Let* $a + b + c \geq 0$, $||b| - |c|| \leq |a| < |b| + |c|$, *and* $k_0 > 0$. *Let* $g_\varphi(t)$ *denote the half-line* $t \mapsto t(\cos\varphi, \sin\varphi)$, $t > 0$, $\varphi \in]0, \frac{1}{2}\pi[$. *Then there is a continuous curve* $\Gamma(t)$: $\tilde{I} =]\tilde{\alpha}, \tilde{\beta}[\to \mathbb{R}^2_>$, *such that* $\Gamma(t)$ *intersects* g_φ, *for all* $\varphi \in]0, \frac{1}{2}\pi[$.

Moreover, locally either $\Gamma(t)$ *is part of a solution curve* $(r_1(\sigma), r_2(\sigma))$ *from Lemma* 2, *or two solution curves which cross each other.*

Proof. If we say in this proof a point $p \in \mathbb{R}^2_>$ lies on the left (right) of the straight line g_φ, we mean p lies in that component of $\mathbb{R}^2_> \backslash g_\varphi(\mathbb{R}_>)$ which contains $(0,1)$ ($(1,0)$ resp.).

The curve $\Gamma(t)$ of the conclusion will be constructed by glueing together parts of solution curves $R_{n_1,n_2}(\sigma)$ (see Fig. 2 for a simple example). There are some restrictions as to which (n_1, n_2) can be used for this, so we define an admissible set of indices

$$N := \Big\{(n_1, n_2) \in \mathbb{Z}^2 \colon \gamma_{n_1,n_2}(\sigma) \in \mathbb{R}^2_> ,$$
$$|\gamma_{n_1,n_2}(\sigma)| > k_0 \max(y(s) \colon s \in I),\ \sigma \in I\Big\}$$

which assures $R_{n_1,n_2}(\sigma)$—and hence $\Gamma(t)$ too—to lie in $\mathbb{R}^2_>$ and we have absolute values $|R_{n_1,n_2}(\sigma)| > k_0$.

The fundamental idea is to define $\Gamma(t)$ by setting it equal to the "first" intersection of $g_{(1/2)\pi - t}$ with a solution curve $R_{n_1,n_2}(\sigma)$, $(n_1,n_2) \in N$. Unfortunately it does not work that simply, the main problem being that tangential intersections between different solution curves might occur.

The way to avoid this difficulty is to restrict ourselves to intersections between solution curves $R_{n_1,n_2}(\sigma)$, $(n_1,n_2) \in N$, with straight lines g_φ, for which $R_{n_1,n_2}(\sigma)$ starts left of and ends right of g_φ. With this restriction, loosely stated, $\Gamma(t)$ consists of those parts of solution curves, which are "first intersections" for the appropriate g_φ plus the adjoining parts of every such solution curve until its crossing with the next one.

The proof is organized in a series of claims. But before we state these, we give two definitions we shall need further on.

For given $(n_1,n_2) \in N$ define the interval $]\varphi_1(n_1,n_2), \varphi_2(n_1,n_2)[$ of angles φ, for which $\gamma_{n_1,n_2}(\sigma)$ (or $R_{n_1,n_2}(\sigma)$) starts left of and ends right of g_φ, by letting $\varphi_j(n_1,n_2)$, $j = 1,2$, be such that $\gamma_{n_1,n_2}(\alpha)$ ($\gamma_{n_1,n_2}(\beta)$) lies on $g_{\varphi_2(n_1,n_2)}$($g_{\varphi_1(n_1,n_2)}$), respectively.

Note that by Lemma 3 and the choice of N we have indeed $0 < \varphi_1(n_1,n_2) < \varphi_2(n_1,n_2) < \frac{1}{2}\pi$.

For each $\varphi \in]0, \frac{1}{2}\pi[$ define the index $(n_1,n_2) = (n_1,n_2)(\varphi) \in N$ of the "first intersection" of an $R_{n_1,n_2}(\sigma)$ with g_φ, as the unique (n_1,n_2)

satisfying

(i) $\varphi_1(n_1, n_2) < \varphi < \varphi_2(n_1, n_2)$.

(ii) If for each $(\tilde{n}_1, \tilde{n}_2) \subset N$ we define $t_j = t_j(\tilde{n}_1, \tilde{n}_2)$, $j = 1, \ldots, p = p(\tilde{n}_1, \tilde{n}_2)$, by

$$R_{\tilde{n}_1, \tilde{n}_2}(]\alpha, \beta[) \cap g_\varphi(\mathbb{R}_>) = \{g_\varphi(t_1), \ldots, g_\varphi(t_p)\},$$

then

$$\min\bigl(|g_\varphi(t_j(n_1, n_2))|:\ j = 1, \ldots, p(n_1, n_2)\bigr) = \inf\bigl(|g_\varphi(t_j(\tilde{n}_1, \tilde{n}_2))|:$$
$$(\tilde{n}_1, \tilde{n}_2) \in N,\ \varphi_1(n_1, n_2) < \varphi < \varphi_2(n_1, n_2),\ j = 1, \ldots, p(\tilde{n}_1, \tilde{n}_2)\bigr).$$

If more than one (n_1, n_2) satisfies (i) and (ii), let $(n_1, n_2)(\varphi)$ be the minimal one of these (using any order on $\mathbb{Z}^2$).

By the following claim (1), there is always an intersection between a given line g_φ and a solution curve, so $(n_1, n_2)(\varphi)$ is well defined.

Claim 1. For each $\varphi \in]0, \frac{1}{2}\pi[$ there is an infinite set $N_\varphi \subset \{(n_1, n_2) \in N:\ \varphi_1(n_1, n_2) < \varphi < \varphi_2(n_1, n_2)\}$, such that for all $(n_1, n_2) \in N_\varphi$, g_φ intersects $\gamma_{n_1, n_2}(\sigma)$.

Claim 2. For all $\varphi_0 \in]0, \frac{1}{2}\pi[$ there exists an $\varepsilon > 0$, such that

$$\#\{(n_1, n_2)(\varphi):\ \varphi_0 - \varepsilon < \varphi < \varphi_0\} = 1,$$
$$\#\{(n_1, n_2)(\varphi):\ \varphi_0 < \varphi < \varphi_0 + \varepsilon\} = 1.$$

Claim 3. For each $\varepsilon > 0$ the interval $[\varepsilon, \frac{1}{2}\pi - \varepsilon]$ can be covered by finitely many intervals $[a_i, b_i]$, $a_i < b_i$, $i = 1, \ldots, p = p(\varepsilon)$, such that $]a_i, b_i[\cap]a_j, b_j[= \emptyset$, and $(n_1, n_2)(\varphi) \equiv (n_1^{(i)}, n_2^{(i)})$, for all $\varphi \in]a_i, b_i[$, for all possible $i \neq j$.

Before we prove these claims, let us construct $\Gamma(t)$ using them.

We construct first a $\Gamma_\varepsilon(t)$, $\varepsilon > 0$ small, which only intersects g_φ for certain φ, then extend it by letting $\varepsilon \to 0$ to get $\Gamma(t)$.

Let $\varepsilon > 0$ be small. By Claim 3 there are $[a_i, b_i]$, $i = 1, \ldots, p$, such that

$$\bigcup_{i=1}^{p} J_i = \left[\varepsilon, \tfrac{1}{2}\pi - \varepsilon\right], \qquad]a_i, b_i[\ \cap\ ]a_j, b_j[\ = \emptyset,\ i \neq j,\ i, j = 1, \ldots, p,$$

and $(n_1, n_2)(\varphi) = (n_1^{(i)}, n_2^{(i)})$, for $\varphi \in]a_i, b_i[$.

Without loss of generality assume $b_i = a_{i+1}$, $i = 1, \ldots, p-1$. By construction (definition of $(n_1, n_2)(\varphi)$), $R_{n_1^{(i)}, n_2^{(i)}}(\sigma)$ is asymptotic (as $\sigma \uparrow \beta$) to a line g_φ, where $\varphi > b_i$, and $R_{n_1^{(i+1)}, n_2^{(i+1)}}(\sigma)$ asymptotic (as $\sigma \downarrow \alpha$) to a line g_φ, where $\varphi < a_{i+1} = b_i$. This means these two solution curves cross each other. Denote this intersection by $R^{(i)} \in \mathbb{R}^2_>$.

Now let $\Gamma(t)$ just consist of those parts of the curves $R_{n_1^{(i)}, n_2^{(i)}}(\sigma)$, which lie between two consecutive intersection points $R^{(i)}$ and $R^{(i+1)}$, $i = 1, \dots, p-2$.

The resulting $\Gamma_\varepsilon(t)$ is continuous, $|\Gamma_\varepsilon(t)| > k_0$, and it intersects g_φ at least for $\varphi \in [\varphi_1(n_1^{(2)}, n_2^{(2)}),\ \varphi_2(n_1^{(p-1)}, n_1^{(p-1)})]$.

Letting $\varepsilon \to 0$, and starting with a countable covering of $]0, \frac{1}{2}\pi[$ from which we choose the finite ones for $[\varepsilon, \frac{1}{2}\pi - \varepsilon]$, we get an extension $\Gamma(t)$ of the above constructed $\Gamma_\varepsilon(t)$ which satisfies the conclusions of the lemma.

There remain the proofs of the three claims.

Proof of Claim 1. This claim follows from the particular structure of the curve $(s_1, s_2)(\sigma)$ from Lemma 3, which allows us to reduce the problem to that of intersections of straight lines with a certain disk, and Lemma 4 which proves there are infinitely many solutions to the latter problem.

First we show that there is an $n_0 = n_0(\varphi) \in \mathbb{N}$, such that if $(n_1, n_2) \in N$, and g_φ intersects the disk $B((n_b + 2n_1, n_c + 2n_2)\pi; \frac{1}{n_0})$ with center $(n_b + 2n_1, n_c + 2n_2)\pi$ and radius $\frac{1}{n_0}$, then g_φ intersects $\gamma_{n_1, n_2}(\sigma)$ too.

By Lemma 3(i), (iii), and (iv) (resp. the comment after the lemma, and the notation thereof) $(s_1, s_2)(\sigma)$, $\sigma \in I$, starts in the second and ends in the fourth quadrant, but it does not pass through its center $(n_b, n_c)\pi$.

So for a small enough $(\hat{s}_1, \hat{s}_2)$, the straight line $t \mapsto (\hat{s}_1, \hat{s}_2) + (n_b, n_c)\pi + t(\cos\varphi, \sin\varphi)$ intersects $(s_1, s_2)(\sigma)$. In other words, there is an $n_0 \in \mathbb{N}$, such that all straight lines intersecting the disk $B((n_b, n_c)\pi; 1/n_0)$ and being parallel to g_φ intersect $(s_1, s_2)(\sigma)$ too. $\gamma_{n_1, n_2}(\sigma)$ is just the translated $(s_1, s_2)(\sigma)$, so if g_φ intersects $B((n_b + 2n_1, n_c + 2n_2)\pi\colon \frac{1}{n_0})$ it intersects $\gamma_{n_1, n_2}(\sigma)$ too.

Since $(s_1, s_2)(\sigma)$ starts in the second and ends in the fourth quadrant, $\gamma_{n_1, n_2}(\sigma)$ starts left and ends right of g_φ.

Now an easy application of Lemma 4 proves Claim 1.

Proof of Claim 2. We only prove the first statement, the second follows by symmetry.

Assume the first statement does not hold. Then there is a sequence $\varphi_j \uparrow \varphi_0$ such that for all $j_0 \in \mathbb{N}$

$$\#\left\{(n_1, n_2)(\varphi_j)\colon j \geq j_0\right\} \geq 2. \tag{19}$$

By definition, for j big enough, $\varphi_1((n_1, n_2)(\varphi_0)) < \varphi_j < \varphi_2((n_1, n_2)(\varphi_0))$, hence

$$\min\left\{|g_j\big(t_l\big((n_1, n_2)(\varphi_j)\big)\big)|\colon l = 1, \dots, p\big((n_1, n_2)(\varphi_j)\big)\right\}$$

$$\leq \min\{|g_0(t_l((n_1, n_2)(\varphi_0))|: l = 1, \dots, p((n_1, n_2)(\varphi_0))\}.$$

Since only finitely many $R_{n_1, n_2}(\sigma)$ intersect any given finite disk, for every $j_0 \in \mathbb{N}$ the set in (19) is finite. This in turn implies there are at least two distinct $(n_1, n_2) \in N$, such that the corresponding solution curves $R_{n_1, n_2}(\sigma)$ intersect each other infinitely often within a finite region, which cannot be, because there are analytic curves.

Proof of Claim 3. By Claim 2, for every $\varphi_0 \in]0, \frac{1}{2}\pi[$ there is a neighborhood $U(\varphi_0) \subset]0, \frac{1}{2}\pi[$ such that

$$\#\{(n_1, n_2)(\varphi): \varphi \in U(\varphi_0)\} \leq 3.$$

$[\varepsilon, \frac{1}{2}\pi - \varepsilon]$ can be covered by finitely many of these. But since each $U(\varphi_0)$ itself can be divided into at most two intervals (plus eventually the point $\{\varphi_0\}$), on each of which $(n_1, n_2)(\varphi)$ is constant, the claim follows easily.

This closes the proof of Lemma 5.

LEMMA 6. *Let $a + b + c \geq 0$, $||b| - |c|| \leq |a| < |b| + |c|$, s_1, s_2, y: $I = [\alpha, \beta] \to \mathbb{R}$ be the functions from Lemma* 3, *and, for $k_0 > 0$, $\Gamma(t)$ the curve from Lemma* 5. *Let $\gamma_{n_1, n_2}(\sigma)$ and $R_{n_1, n_2}(\sigma)$ be defined by* (16) *and* (17).

Let $(n_1^{(0)}, n_2^{(0)})$ be given such that $\gamma_{n_1^{(0)}, n_2^{(0)}}(I) \cap \mathbb{R}^2_> \neq \emptyset$. Then for $\sigma \in]\alpha, \beta[$

$$R_{n_1^{(0)}, n_2^{(0)}}(\sigma) = \left(r_1^{(n_1^{(0)})}(\sigma), r_2^{(n_2^{(0)})}(\sigma)\right)$$
$$= \left(\frac{s_1(\sigma) + 2n_1^{(0)}\pi}{y(\sigma)}, \frac{s_2(\sigma) + 2n_2^{(0)}\pi}{y(\sigma)}\right),$$

crosses $\Gamma(t)$, or it can be connected by a curve $\tilde{\Gamma}(t) \subset U_r$ to $\Gamma(t)$, for a k_0 big enough.

Proof. If $\gamma_{n_1^{(0)}, n_2^{(0)}}(\alpha) \subset \mathbb{R}^2_>$ or $\gamma_{n_1^{(0)}, n_2^{(0)}}(\beta) \subset \mathbb{R}^2_>$, then $R_{n_1^{(0)}, n_2^{(0)}}(\sigma)$ asymptotically approaches a straight line g_φ (namely $\varphi = \varphi_j(n_1^{(0)}, n_2^{(0)})$, an $j \in \{1, 2\}$, as defined in the proof of Lemma 5), where $0 < \varphi < \frac{1}{2}\pi$. Hence it crosses $\Gamma(t)$, if $k_0 > \inf(|R_{n_1^{(0)}, n_2^{(0)}}(\sigma)|: \sigma \in]\alpha, \beta[)$.

There remains the case that $\gamma_{n_1^{(0)}, n_2^{(0)}}(\sigma)$ starts and ends on an axis, or crosses it, that is,

$$\gamma_{n_1^{(0)}, n_2^{(0)}}(\alpha), \gamma_{n_1^{(0)}, n_2^{(0)}}(\beta) \notin \mathbb{R}^2_> . \tag{20}$$

Define $\alpha \leq \tilde{\alpha} < \tilde{\beta} \leq \beta$ by

$$\gamma_{n_1^{(0)}, n_2^{(0)}}(]\tilde{\alpha}, \tilde{\beta}[) \subset \mathbb{R}^2_> ,$$

and $]\tilde{\alpha}, \tilde{\beta}[$ is maximal. (Note that by Lemma 3, $\gamma_{n_1, n_2}(\sigma)$ can intersect each axis at most once.)

Lemma 3 and (20) imply that either $\gamma_{n_1^{(0)}, n_2^{(0)}}([\tilde{\alpha}, \tilde{\beta}])$ intersects both the r_1- and r_2-axis, in which case by Lemma 2 it goes from the latter to the former or it intersects one of them twice.

We shall first treat the former case of intersection of both axes outside the origin.

$\gamma_{n_1^{(0)}, n_2^{(0)}}(]\tilde{\alpha}, \tilde{\beta}[)$ intersects $g_{\pi/4}$, in say $g_{\pi/4}(t_0)$. By Lemma 2, $g_{\pi/4}$ crosses $R_{n_1^{(0)}, n_2^{(0)}}(\sigma)$ from the right to the left, i.e., $g_{\pi/4}(t) \subset U_r$ for $t > t_0$ small enough. But for $t > t_0$, $g_{\pi/4}(t)$ eventually has to cross a solution curve $R_{n_1, n_2}(\sigma)$ (for example, part of $\Gamma(t)$ from Lemma 5 for k_0 big enough). So there are $(n_1^{(1)}, n_2^{(1)}) \in N$, such that $g_{\pi/4}$ crosses $R_{n_1^{(1)}, n_2^{(1)}}(\sigma)$, in say $g_{\pi/4}(t_1)$, and $g_{\pi/4}(]t_0, t_1]) \subset U_r$.

Now $\gamma_{n_1^{(1)}, n_2^{(1)}}(\sigma)$ does not satisfy (20), because for at least one $j \in \{1, 2\}$, $n_j^{(1)} > n_j^{(0)}$, hence the corresponding solution curve crosses $\Gamma(t)$, for k_0 big enough.

We have just shown $R_{n_1^{(0)}, n_2^{(0)}}(\sigma)$ is connected in U_r to a solution curve which in turn intersects $\Gamma(t)$, but with Lemma 2 this suffices to prove the existence of $\tilde{\Gamma}(t)$ as required.

There remains only the case of $\gamma_{n_1^{(0)}, n_2^{(0)}}([\tilde{\alpha}, \tilde{\beta}])$ intersecting one axis twice. Without loss of generality assume it to be the r_1-axis, the other case being symmetrical.

This is a very special case, which by Lemma 3 can only happen if $0 \neq |a| = |c| - |b|$, $n_c = 0$, and $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$.

If $n_b = 0$, then $\alpha = -\pi$, $\beta = \pi$, $ab, ac < 0$, and b and c have the same sign. Also, $s_2(s) \in]0, \pi[$, $s \in]-\pi, \pi[$, and $0 < y(0)$ implies $c > 0$. All together, for $n_b = 0$ we have $n_1^{(0)} \geq 1$, $n_2^{(0)} = 0$, and

$$a = b - c, \qquad 0 < b < c.$$

If $n_b = 1$, then $\alpha = 0$, $\beta = 2\pi$, $ac < 0 < ab$, and b and c have different sign. $s_2(s) \in]0, \pi[$, $s \in]0, 2\pi[$, and $y(\pi) > 0$ implies again $c > 0$. That is, for $n_b = 1$ we have $n_1^{(0)} \geq 0$, $n_2^{(0)} = 0$, and

$$a = -b - c, \qquad 0 < -b < c.$$

The proof that in this case $R_{n_1^{(0)}, n_2^{(0)}}(s)$ can also be connected to $\Gamma(t)$ proceeds through three claims:

(1) For $n_1 > 1$, $R_{n_1, 0}(s)$ can be connected to $R_{1,0}(s)$.

(2) $R_{1,0}(s)$ can be connected to $R_{0,0}(s)$.

(3) $R_{0,0}(s)$ can be connected to $\Gamma(t)$, for k_0 big enough.

Crucial for proving these claims is

$$r_2^{(0)}(s) \to \begin{cases} \dfrac{1}{c - \sqrt{|b|c}}, & s \downarrow \alpha, \\ \dfrac{1}{c + \sqrt{|b|c}}, & s \uparrow \beta. \end{cases} \tag{21}$$

We show (21) only for $n_b = 0$ and $s \downarrow \alpha = -\pi$, the other cases can be proved in much the same way.

With Eq. (3)

$$s_2(s) = \arccos \frac{-a - b\cos s}{c} = \arccos \frac{c - b(1 + \cos s)}{c}, \qquad s \in [-\pi, \pi]. \tag{22}$$

For $s \downarrow -\pi$ we get

$$\begin{aligned} s_2'(s) &= \frac{b}{\sqrt{2bc - b^2(1 + \cos s)}} \frac{-\sin s}{\sqrt{1 + \cos s}} \\ &= \frac{b}{\sqrt{2bc - b^2(1 + \cos s)}} \sqrt{1 - \cos s} \to \sqrt{\frac{b}{c}}, \\ y'(s) &= b\cos s + s_2'(s) c \cos s_2(s) \to -b + \sqrt{bc}, \end{aligned}$$

and thus

$$\lim_{s \downarrow -\pi} r_2^{(0)} = \lim_{s \downarrow -\pi} \frac{s_2(s)}{y(s)} = \lim_{s \downarrow -\pi} \frac{s_2'(s)}{y'(s)} = \frac{1}{c - \sqrt{bc}}.$$

For later use, note that if $n_b = 1$ and $s \downarrow \alpha = 0$,

$$\begin{aligned} s_2'(s) &\to \sqrt{\frac{|b|}{c}}, \\ y'(s) &\to -|b| + \sqrt{|b|c}, \\ r_1^{(0)}(s) &= \frac{s}{y(s)} \to \frac{1}{\sqrt{|b|c} - |b|}. \end{aligned} \tag{23}$$

Relation (21) implies that $\underline{R_{n_1,0}(s)}$, $n_1 \geq 1$, approaches asymptotically the straight lines $r_2 \equiv 1/(c - \sqrt{|b|c})$ and $r_2 \equiv 1/(c + \sqrt{|b|c})$, respectively. Thus $R_{n_1,0}(s)$ cuts $\mathbb{R}^2_>$ into two unbounded connected components (plus maybe a finite number of bounded ones), say $C_0(n_1)$ containing the origin, and $C_1(n_1)$ containing the unbounded part of $r_2 \equiv (1/2)(1/(c - \sqrt{|b|c}) + 1/(c + \sqrt{|b|c}))$.

Note that $C_1(n_1)$ lies on the left-hand side of $R_{n_1,0}(s)$.

Proof of Claim (1). We define equivalence classes $[n] \subset \mathbb{N}$ by setting $n \sim m$, if there are finitely many $n = n^{(0)}, n^{(1)}, \dots, n^{(p)} = m$, such that $R_{n^{(i)},0}(s)$ and $R_{n^{(i+1)},0}(s)$ cross each other, $i = 0, \dots, p-1$.

If n, m belong to the same class $[n]$, then they can be connected in U_r, so all we have to show is that $[1]$ and $[n_1]$, $n_1 > 1$, can be connected.

Order the equivalence classes by the minimal members, i.e., by setting $[n] > [m]$, if there is an $\tilde{m} \in [m]$ such that $\tilde{m} < \tilde{n}$, for all $\tilde{n} \in [n]$. Since for $n > m$, $R_{n,0}(s)$ has a bigger minimum than $R_{m,0}(s)$, and both solution curves approach asymptotically the same straight lines, $[n] > [m]$ implies $R_{\tilde{n},0}(s) \subset C_1(\tilde{m})$, for all $\tilde{n} \in [n]$, $\tilde{m} \in [m]$. In this sense the equivalence classes are nested within each other in increasing order, and one can connect $[n]$ with the next one by a curve which lies in $C_1(n)$, and at most intersects $R_{\tilde{n},0}(s)$ for $\tilde{n} \in [n] \cup [m]$. That is, two consecutive classes can be connected in U_r. There are only finitely many classes between $[1]$ and $[n_1]$, thus they—and $R_{1,0}(s)$ and $R_{n_1,0}(s)$ too—can indeed be joined.

Proof of Claim (2). Either $R_{1,0}(s)$ and $R_{0,0}(s)$ cross each other, or the former lies on the left of the latter (since $r_2^{(0)}(\alpha) - r_2^{(0)}(\beta) > 0, r_1^{(0)}(s) < r_1^{(1)}(s)$), and going in a straight line from 0 to the minimum of $R_{1,0}(s)$, $s \in]\alpha, \beta[$, one crosses $R_{0,0}(s)$ once (from the right to the left), without having intersections with curves $R_{0,n_2}(s)$, $n_2 > 0$ (because no g_φ intersects both $\gamma_{0,n_2}(s)$ and $\gamma_{1,0}(s)$), nor with $R_{n_1,n_2}(s)$, $n_1 \geq 1$, $n_2 > 0$, or $n_1 > 1$, $n_2 = 0$ (because they have a bigger minimum). That is, via this straight line one can joint $R_{1,0}(s)$ with $R_{0,0}(s)$.

Proof of Claim (3). If $n_b = 0$, then $\gamma_{0,0}(s)$ crosses the s_2-axis, that is, it intersects both axes. We already treated this case.

Assume now $n_b = 1$. By (23) and (21),

$$R_{0,0}(s) \to \left(\frac{1}{\sqrt{|b|c} - |b|}, \frac{1}{c - \sqrt{|b|c}} \right),$$

which lies on the line

$$1 - r_1 b - r_2 c = 0.$$

For (r_1, r_2) on this line,

$$\chi(0; r_1, r_2) = \chi'(0; r_1, r_2) = 0,$$

and $(r_1, r_2) \in U_r$ follows.

For (r_1, r_2) with

$$1 - r_1 b - r_2 c < 0,$$

$\chi(.; r_1, r_2)$ has a root in 0 with negative derivative, since χ tends to infinity as $x \to \infty$, there has to be a real root $x > 0$. So

$$\left\{(r_1, r_2) \in \mathbb{R}^2_> : 1 - r_1 b - r_2 c \le 0\right\} \subset U_r.$$

Now $R_{0,0}(s)$ ends in the set above, which in turn intersects $\Gamma(t)$, for $k >_0$ big enough.

This proves Claim (3), and completes the proof of the lemma.

Now comes the main proposition of this section, namely that the region of instability is connected and unbounded in the case of fixed parameters and variable delays:

PROPOSITION 2. *$U_r \subset \mathbb{R}^2_\ge$ is connected and unbounded.*

Proposition 2 implies in particular that there are no "islands" of instability within the stable region. With Corollary 1, there are not even "islands" of stability, but not asymptotic stability, within this region. This follows also from the description of the stability region given by Hale and Huang in [2], if one knows the stability region to be connected. This seems to be true, but it has—to our knowledge—as yet not been proven.

Proof. If $b = 0$ or $c = 0$, then we have only one delay, and the conclusion holds (see, e.g., [1]). If $a + b + c < 0$, then $\mathbb{R}^2_\ge = U_r$ (Lemma 1). For the rest of this proof we assume $b \ne 0 \ne c$ and $a + b + c \ge 0$.

We distinguish four cases:

Case 1. $|a| > |b| + |c|$. In this case $\chi(\mu)$ has no purely imaginary roots, and either $\mathbb{R}^2_\ge = S_r$ or $\mathbb{R}^2_\ge = U_r$.

Case 2. $|a| = |b| + |c|$. We have $|a| > 0$. $\operatorname{Re} \chi(x + iy) = 0$ implies $r_1 y, r_2 y \in \pi\mathbb{Z}$, which in turn gives $y = 0$. In other words, 0 is the only possible purely imaginary root of χ.

If $a + b + c \ne 0$, then as in Case 1 either $\mathbb{R}^2_\ge = S_r$ or $\mathbb{R}^2_\ge = U_r$.

If $a + b + c = 0$, then $\chi(0; r_1, r_2) = 0$ for all $r_1, r_2 \ge 0$, and 0 is the only root on the imaginary axis. If $\chi'(0) = 1 - r_1 b - r_2 c \le 0$, then $\chi(.)$ has a (second) non-negative real root, and $(r_1, r_2) \in U_r$.

If $\chi'(0) > 0$ and $\chi(x_0 + iy_0) = 0$ for an $x_0 \ge 0$, $x_0 + iy_0 \ne 0$, then $x_0 > 0$ and $|y_0| < |b| + |c|$. $\chi(\mu; r_1, r_2)$ is an analytic function; in this case it has (at least) two roots in $S_1 := \{x + iy\colon x \ge 0, |y| \le |b| + |c|\}$. For t going from 1 to 0, $\chi'(\cdot; tr_1, tr_2) > 0$, hence for (tr_1, tr_2) there is no purely imaginary root, and for all these t, the two roots have to remain in S_1. But $\chi(.; 0, 0)$ has only 0 as a simple root here. Hence there cannot have been a root $x_0 + iy_0$, $x_0 > 0$, and $\chi'(0; r_1, r_2) > 0$. So

$$U_r = \left\{(r_1, r_2) \in \mathbb{R}^2_{\geq} : 1 - r_1 b - r_2 c \leq 0\right\}$$

is connected and either empty or unbounded.

Case 3. $|a| < ||b| - |c||$. We shall only prove the case $|a| < |c| - |b|$, the other one being similar.

With Lemma 3 the set of r_1, r_2, such that $\chi(.; r_1, r_2)$ has purely imaginary roots is a (countable) union of curves $(r_1, r_2)(\sigma)$: $[\alpha, \infty[\to \mathbb{R}^2_{\geq}$, $r_1(\alpha) = 0$, $r_2(\sigma)$ periodic.

Each point of U_r is connected (within U_r) to such a curve, and with Lemma 2 to a point $(0, r_{2,0})$ on the r_2-axis. For $r_1 = 0$ we are in the one-dimensional case, and the set of r_2, such that Eq. (1) with $r_1 = 0$ is unstable and is an unbounded interval. This proves U_r to be connected and unbounded.

Case 4. $||b| - |c|| \leq |a| < |b| + |c|$. Let (ρ_1, ρ_2), $(\tilde{\rho}_1, \tilde{\rho}_2) \in U_r$ be given and assume $U_r \neq \mathbb{R}^2_{\geq}$.

If $a + b + c > 0$, then (ρ_1, ρ_2) is connected with the left-hand side of a curve $(r_1, r_2)(\sigma)$:$]\alpha, \beta[\to \mathbb{R}^2_{\geq}$, where $r_j(\sigma) = (s_j(\sigma) + 2n_j\pi)/y(\sigma)$, $j = 1, 2$, and $n_1, n_2 \in \mathbb{Z}$, s_1, s_2, y: $[\alpha, \beta] \to \mathbb{R}$ as in Lemma 3.

By Lemma 6 two such curves can be connected via $\Gamma(t)$ of Lemma 5. But then Lemma 2 shows (ρ_1, ρ_2) and $(\tilde{\rho}_1, \tilde{\rho}_2)$ to be connected in U_r.

If $a + b + c = 0$, then 0 is a root of χ, and U_r contains $S_2 := \{(r_1, r_2) \in \mathbb{R}^2_{\geq} : 1 - r_1 b - r_2 c \leq 0\}$ (see Case 2).

(ρ_1, ρ_2) can be connected to a solution curve $(r_1, r_2)(\sigma)$ or to S_2. Since S_2 intersects $\Gamma(t)$, for $k_0 > 0$ big enough, the conclusion follows as in the case $a + b + c > 0$.

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