ELSEVIER
Topology and its Applications 120 (2002) 105–156
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

On maps with unstable singularities

Sergey A. Melikhov [1]

Department of Differential Geometry, Faculty of Mechanics and Mathematics, Moscow State University, 119899 Moscow, Russia

Received 16 September 1999; received in revised form 21 December 2000

Abstract

If a continuous map $f:X \to Q$ is approximable arbitrary closely by embeddings $X \hookrightarrow Q$, can some embedding be taken onto f by a pseudo-isotopy? This question, called Isotopic Realization Problem, was raised by Ščepin and Akhmet'ev. We consider the case where X is a compact n-polyhedron, Q a PL m-manifold and show that the answer is 'generally no' for $(n,m)=(3,6)$; $(1,3)$, and 'yes' when:

(1) $m>2n$, $(n,m)\neq(1,3)$;

(2) $m>3(n+1)/2$ and $\Delta(f)=\{(x,y)\mid f(x)=f(y)\}$ has an equivariant (with respect to the factor exchanging involution) mapping cylinder neighborhood in $X\times X$;

(3) $m>n+2$ and f is the composition of a PL map and a TOP embedding.

In doing this, we answer affirmatively (with a minor preservation) a question of Kirby: does small smooth isotopy imply small smooth ambient isotopy in the metastable range, verify a conjecture of Kearton–Lickorish: small PL concordance implies small PL ambient isotopy in codimension $\geqslant 3$, and a conjecture set of Repovs–Skopenkov. © 2002 Elsevier Science B.V. All rights reserved.

AMS classification: Primary 57N37, Secondary 18G10; 54C05; 55N07; 55P91; 55S91; 57M30; 57N45; 57N75; 57Q25; 57Q37; 57Q55; 57Q60; 57R45; 58D10

Keywords: Approximability by embeddings; Pseudo-isotopy; Decomposition into prime knots; Hauptvermutung; Pseudo-arc; p-adic solenoid; Bing sling; Wilder arc; Continuity versus exactness; derived limit functor; Isovariant map; ε-concordance; Mapping cylinder

1. Introduction

A general mathematical problem is to decide whether a singular state of some system is stable or unstable. In terms of geometric topology it can be expressed as follows: given

E-mail address: sergey@melikhov.mccme.ru, melikhov@math.ufl.edu (S.A. Melikhov).

[1] Current address: University of Florida, Department of Mathematics, 358 Little Hall, P.O. Box 118105, Gainesville, FL 32611-8105, USA. Partially supported by the Russian Foundation for Basic Research Grant No. 99-01-00009.

0166-8641/02/$ – see front matter
PII: S0166-8641(01)00012-8

a continuous map f of a compactum X into a manifold Q, can it be ε-approximated by an embedding $f_\varepsilon : X \hookrightarrow Q$ for each $\varepsilon > 0$? If this is the case, the map f is called *realizable* [80] or *discretely realizable* [2]. If f is a constant map, its realizability evidently coincides with embeddability of X into $\mathbb{R}^{\dim Q}$, meanwhile embeddability of a compactum into $\mathbb{R}^m$ can be reduced to realizability of certain PL maps, cf. [80,43,75,2].

As far as in some cases the ε-approximation of f can be made in infinite number of inequivalent ways (i.e., by embeddings, not joined by sufficiently small ambient isotopies — e.g., a map $S^n \sqcup S^n \to S^n \vee S^n \hookrightarrow \mathbb{R}^{2n+1}$, which embeds each sphere, can be approximated by links with arbitrary linking number), it is natural to ask, whether an approximation of f can be viewed as a *continuous process*, parametrized by real numbers? The map f is *isotopically realizable*, if there exists a homotopy $H_t : Q \to Q$, $t \in I = [0, 1]$, such that H_t is a homeomorphism for $t < 1$ (such homotopy is called a *pseudo-isotopy*, cf. [46]), $H_0 = \mathrm{id}_Q$ and $H_1 \circ g = f$ for some embedding $g : X \hookrightarrow Q$.

Isotopic Realization Problem (E.V. Ščepin, 1993; P.M. Akhmet'ev [2]). When does discrete realizability imply isotopic realizability?

To the best of the author's knowledge, the concept of isotopic realizability was first considered by Blass and Holsztyński in 1971 [39]. It was independently introduced under the present name in a paper by Ščepin and Štan'ko [75] (subsequent to the work of Ščepin on uncountable inverse spectra and the earlier work of Štan'ko on embedding dimension). Although both papers [39,75] also dealt with discrete realizability, the relationship was not discussed there, and it appears that until recently the IR Problem above has been virtually untouched.

Actually it traces back to the Keldyš Problem (1966) on realizability of wildly embedded polyhedra by pseudo-isotopy of subpolyhedra [44,45] (see also [26], compare [65]). That is, in the above X should be replaced by a polyhedron, Q by a PL manifold, f by an embedding and g by a PL embedding, and, strictly speaking, the pointwise equality $H_1 \circ g = f$ by the setwise $H_1(g(X)) = f(X)$. In a few succeeding years the Keldyš Problem was solved positively for wild surfaces in 3-manifolds [45] and for wild n-polyhedra in PL m-manifolds, $m - n \geqslant 3$ [25] (cf. Theorem 3.5(a) below), and negatively for certain wild knots in $\mathbb{R}^3$ [47,81] (see Example 1.2 below). On the other hand, it should be noticed that isotopic realizability as a property of maps in the closure of the space of embeddings is similar to tameness as a property of TOP embeddings in the closure of the space of PL embeddings, moreover, in codimension $\geqslant 3$, the fact that all TOP embeddings lie in the latter closure was used in proofs of equivalence of tameness and the 1-LCC property [14,18,15] (see also [25, 8.2], [68, 2.5.1]).

The concept of discrete realizability was studied widely (see brief surveys in [69] and [4]). For example, each self-map of the pseudo-arc is realizable (see [54]), meanwhile for locally connected continua X, Y, $\dim X \leqslant 1$, all maps $X \to Y \hookrightarrow \mathbb{R}^2$ are realizable iff either X is contained in triod and Y is in S^1, or X is contained in the 'letter q' and Y is in I [80]. Realizability of a given map $X \to \mathbb{R}^2$ seems to be a harder question (see [67] for the PL case). Any map of an n-dimensional compactum X into $\mathbb{R}^m$ is realizable for $m \geqslant 2n+1$ (cf. [79]) and even for $m = 2n$ if $\dim X \times X < 2n$ [24,84]. All maps $T^n \to T^n \hookrightarrow_{\text{standard}} \mathbb{R}^{2n}$ are realizable if $n > 1$ [43], meanwhile the maps $S^n \to S^n \hookrightarrow_{\text{standard}} \mathbb{R}^{2n}$ are realizable whenever $n \neq 1, 2, 3, 7$, and are not, generally speaking, if $n = 1,\ 3$ or 7 [2,4]. Furthermore, for each k one can find an n such that all maps $S^n \to S^n \hookrightarrow_{\text{st.}} \mathbb{R}^{2n-k}$ are realizable [3,4]. Surprisingly, in the space of maps $S^2 \to \mathbb{R}^3 \hookrightarrow_{\text{st.}} \mathbb{R}^4$ the subset of non-realizable maps is dense [6].

As for isotopic realizability, two principal results had been previously known.

From Černavskij's Theorem on local contractibility of the homeomorphism group of a closed manifold M [19,27] it follows that discrete realizability implies isotopic for self-maps of M (thus for $\dim M \neq 3$ both are equivalent to the property of being cell-like [78,68]). Secondly, Akhmet'ev showed in 1996 that all maps $S^n \to S^n \hookrightarrow_{\text{st.}} \mathbb{R}^{2n}$ are isotopically realizable for $n = 4k+1 \geqslant 9$ [2].

The real question, implicit in the above universal statement of the IR Problem and originally motivating this deep result of Akhmet'ev as well as the present paper was, does discrete realizability imply isotopic for maps of nice spaces (say, of a manifold into Euclidean space) in high codimensions (say, greater than 2, in order to kill the fundamental group)? Our main result is that it does not. The counter-example (Example 1.9) is an explicit geometric construction with a self-contained verification, but it was not until the rest of the paper had been written when it naturally appeared. The major part of the paper is devoted to the reduction of the IR Problem for maps of nice spaces in the metastable range to a homotopy-theoretic question, which, in turn, admits an answer in terms of vanishing of certain cohomological obstructions.

Despite such an algebraization, it is still unknown whether the above condition $n = 4k+1$ is really necessary, and even whether[2] discrete realizability implies isotopic for all maps $S^n \to \mathbb{R}^{2n-1} \hookrightarrow_{\text{st.}} \mathbb{R}^{2n}$ (see [59] for partial results). A new technique seems to be necessary here, which may be also useful in attacking the following problem [6]: Suppose $f : M \to \mathbb{R}^n$ is a generic smooth map and $i : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ the standard inclusion, does discrete realizability of $i \circ f$ imply that f can be factored into the composition of an embedding $M \hookrightarrow \mathbb{R}^{n+k}$ and the projection $\mathbb{R}^{n+k} \to \mathbb{R}^n$? (The latter clearly implies isotopic realizability of $i \circ f$.)

1.1. Low-codimensional examples

In the general setting it is easy to construct discretely realizable maps which are not isotopically realizable.

Example 1.1. Let S be the countable union of n-spheres $S^n_1, S^n_2, \ldots,$ compactified by a point p, and let $f : S \sqcup q \to \mathbb{R}^{n+1}$ be a map, throwing the points p, q onto the origin and each S^n_k homeomorphically onto the standard sphere of radius $1/k$ centered at the origin. Clearly, f is realizable but not isotopically.

[2] The answer here appears to be negative. The example for each $n = 2k \geqslant 4$ is included in [59]. It is also shown in [59] that if $n \neq 5, 6$ and neither n nor $n+1$ is a power of 2, then any map $S^n \to S^n \subset \mathbb{R}^{2n}$ is isotopically realizable.

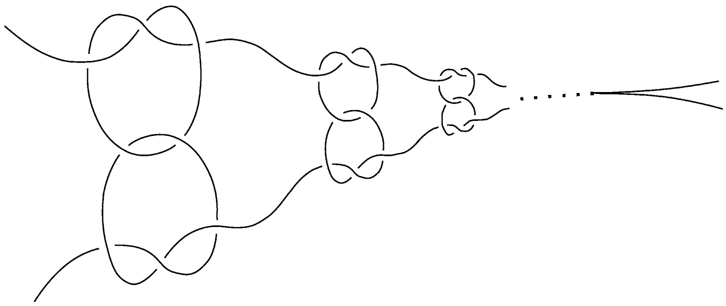

Fig. 1.

There are also somewhat less straightforward examples.

Example 1.1′. Let P denote the pseudo-arc, $p : P \sqcup P \to P$ the trivial double cover and $i : P \hookrightarrow \mathbb{R}^2$ any embedding yielded by the Bing definition of the pseudo-arc [8,54] where all links are round disks in the plane. Clearly, the composition $i \circ p$ is discretely realizable, however in Section 2 we show that it is not isotopically realizable.

Perhaps it is worth determining, which compacta admit such natural maps into Euclidean space, realizable discretely but not isotopically, in particular, whether the standard embedding of the p-adic solenoid into $\mathbb{R}^3$ (cf. Example 1.9) precomposed with the trivial double cover is isotopically realizable.

However, in this paper we treat such cases as pathological, and to eliminate them we restrict the spaces under consideration in the IR Problem.

From now *we assume the domain X to be a compact n-polyhedron and the target space Q a PL m-manifold* (*without boundary*). In this setting, the following example was known.

Example 1.2. Let $f : I \hookrightarrow \mathbb{R}^3$ be the Wilder arc (i.e., one of the two wild arcs shown on Fig. 1) or, more generally, a non-trivial Wilder arc in the sense of [31]. Up to an ambient isotopy, we can assume that f consists of infinitely many tame knots $f|_{[a_i,a_{i+1}]} : [a_i, a_{i+1}] \hookrightarrow \mathbb{R}^2 \times [a_i, a_{i+1}]$ (each of them can be chosen of arbitrary non-trivial isotopy class), where $a_i = 1/2 - 1/2^i$, $i = 1, 2, \dots,$ and of a straight line segment $f|_{[1/2,1]}$. It was noticed by Keldyš [47] and Sikkema [81] that f cannot be obtained by a pseudo-isotopy of a tame arc.

For convenience of the reader (since in [81] the reduction of Theorem 1 to Theorem 2 was omitted, while the argument in [47] seems to be too complicated to prove this particular statement), we outline a proof. Indeed, suppose on the contrary that $g : I \hookrightarrow \mathbb{R}^3$ is a PL arc and $H_t : \mathbb{R}^3 \to \mathbb{R}^3$ a pseudo-isotopy such that $H_0 = \mathrm{id}$ and $H_1 \circ g = f$. The arc g is the restriction of a PL knot $\bar{g} : S^1 \hookrightarrow \mathbb{R}^3$ (it is supposed that $I \subset S^1$), and without loss of generality $H_t \circ \bar{g}(S^1 \setminus I)$ is sufficiently far from $f([\frac{1}{4}, \frac{3}{4}])$ for all $t \in I$ (for H_t and g can be assumed as close as desired to the identity and to f, respectively). But then for each n

there exists an $\varepsilon > 0$ such that $H_{1-\varepsilon} \circ \bar{g}$ can be decomposed into at least n knots, which contradicts the uniqueness of decomposition of $\bar{g}$ into prime knots (cf. [30]).

We call a discretely realizable map $f : X \to Q$ *continuously realizable*, if $\forall\varepsilon > 0\ \exists\delta > 0$ such that each embedding $g_\delta : X \hookrightarrow Q$, δ-close to f, can be taken onto f by an ε-pseudo-isotopy. Of course, every continuously realizable map is isotopically realizable, but not vice versa, as Example 1.2 shows. We will see below that maps, realizable discretely but not continuously, are often easier to find and to classify than ones realizable discretely but not isotopically. That is why in what follows we keep in mind, along with the IR Problem, the following *Pre-limit IR Problem*: When does realizability imply continuous realizability?

Example 1.3. The map $f : I \sqcup I \to I \vee I \hookrightarrow \mathbb{R}^3$, whose image is shown on Fig. 1, is not isotopically realizable. As in Example 1.2, the proof rests on the Schubert Theorem of uniqueness of decomposition into prime knots.

The following argument was inspired by an idea due to Akhmet'ev. First let us define an invariant of PL links. Given a PL link $l : S^1_1 \sqcup S^1_2 \to \mathbb{R}^3$ with vanishing linking number, we consider a decomposition of $l|_{S^1_1}$ into the connected sum of prime knots $k_1, k_2, \dots, k_p : S^1_1 \to \mathbb{R}^3$. We call k_i *inessential*, if the homotopy class of $l|_{S^1_2}$ in $\mathbb{R}^3 \setminus l(S^1_1)$, regarded as a conjugate class in $\pi_1(\mathbb{R}^3 \setminus l(S^1_1))$, lies in the kernel of the homomorphism

$$\pi_1\bigl(\mathbb{R}^3 \setminus l\bigl(S^1_1\bigr)\bigr) \to \pi_1\bigl(\mathbb{R}^3 \setminus k_i\bigl(S^1_1\bigr)\bigr),$$

yielded by introduction of the commutativity relations killing the rest prime knots $k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_p$. We define $\alpha(l)$ to be the number of essential prime knots among $k_1, \dots, k_p$.

Now suppose that there exists a (possibly wild) embedding $g : I_1 \sqcup I_2 \hookrightarrow \mathbb{R}^3$ and a pseudo-isotopy $H_t : \mathbb{R}^3 \to \mathbb{R}^3$ such that $H_0 = \mathrm{id}$ and $H_1 \circ g = f$. Extend g by adding two arcs to obtain a link of two (possibly wild) knots $\bar{g} : S^1_1 \sqcup S^1_2 \hookrightarrow \mathbb{R}^3$ with zero linking number. It can be assumed that $H_t \circ \bar{g}(S^1_i \setminus I_i)$, $i = 1, 2$, is sufficiently far from $f([\frac{1}{4}, \frac{3}{4}] \sqcup [\frac{1}{4}, \frac{3}{4}])$ for all $t \in I$. It is easy to see that for each positive integer n there exists an $\varepsilon > 0$ such that for every PL link l, sufficiently close to $H_{1-\varepsilon} \circ \bar{g}$, the invariant $\alpha(l)$ is greater than n. On the other hand, for reasons of compactness (see Section 2 for details) $\alpha(l)$ is bounded for PL links l, sufficiently close to $\bar{g}$.

Remark. It is not clear whether $\alpha(l)$ necessarily stabilizes as $l \to \bar{g}$. This is evidently true if some polyhedral neighborhood of $\bar{g}(S^1_1)$ in $\mathbb{R}^3 \setminus \bar{g}(S^1_2)$ is homeomorphic to the solid torus. In general it seems (cf. Example 1.5) that, whatever happens to the quantity of the essential prime knots, their isotopy types need not stabilize.

Example 1.3′. If we replace the Wilder arcs in the previous example by the wild arcs from [66, p. 303], we will obtain an isotopically realizable map (compare to [47, Example 1]). Initial steps of a pseudo-isotopy are indicated on Fig. 2.

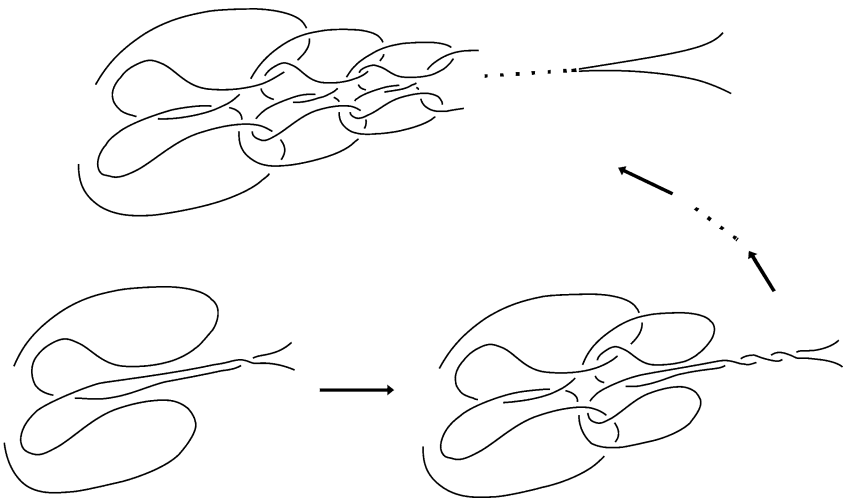

Fig. 2.

The above argument in Example 1.3 works equally well for plenty of maps, similar to the one on Fig. 1, in particular when one of the two wild arcs on Fig. 1 is replaced by a straight line segment. One may focus his attention on a class of maps for which it does not work, namely, the maps $I \sqcup I \to I \vee I \hookrightarrow \mathbb{R}^3$ whose restriction on each component is a tame arc. (For example, the map $I \sqcup I \to \mathbb{R}^3$, obtained from Fig. 1 by replacing each elementary link of two trefoils with the Whitehead link.) The question, whether these maps are isotopically realizable, seems to be important, because the contrary would show that, in the range, the phenomenon of a map, realizable but not isotopically, is not just a ramification of the phenomenon of a wild embedding, but is somewhat completely different. More generally:

Question I. Does there exist a discretely realizable but not isotopically realizable map which is a locally flat topological immersion? [3]

The positive answer would follow from the positive answer to a general problem in the link theory [60]. Speaking informally, is there a natural theory of 'links modulo knots' with a well-defined operation of connected sum admitting accumulation of complexity (that is, for some 'link modulo knot' λ and any λ' and any positive integer n there exists a positive integer N such that for any λ'', the connected sum $(\sharp_N\lambda)\sharp\lambda''$ is not equivalent to $(\sharp_n\lambda)\sharp\lambda')$? See [60] and [61] for precise statement and some partial results concerning the latter question, which turns out to be somewhat related to the long-standing problem of equivalence of the bounded Engel condition and nilpotence in the class of finitely generated groups.

[3] Recently such maps were found, namely a series of maps $S^n \to \mathbb{R}^{2n}$, $n \geqslant 3$ [59]. However, for maps of a 1-manifold into $\mathbb{R}^3$ the question remains open.

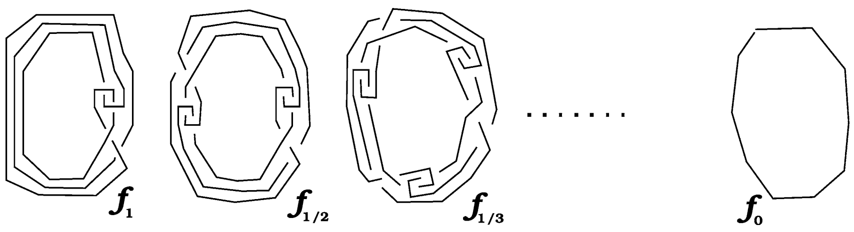

Fig. 3.

Another question, arising from the above examples: should a map, realizable discretely but not isotopically, necessarily be of infinite complexity (in some sense), or does there exist, say, a PL map such that the better we try to approximate it, the more 'knotted' embedding we should use? We make this more precise as follows:

Question II. Does there exist a PL map which is PL discretely realizable but not PL isotopically realizable?

The definitions of *PL* (*discrete*, *isotopic*, *continuous*) *realizability* can be obtained by stating the definitions above in the PL category. Although the answer is unknown in general (see Section 1.2 for the codimension $\geqslant 3$ case), we suggest the following negative answer to the pre-limit version of the latter problem:

Example 1.4. The PL unknot $f_0 : S^1 \hookrightarrow \mathbb{R}^3$ is not PL continuously realizable, for there exist PL knots $f_{1/k} : S^1 \hookrightarrow \mathbb{R}^3$ (see Fig. 3), arbitrarily close to f_0, which cannot be taken onto f_0 by a small PL pseudo-isotopy. (One can drop 'small' by the price of replacing the knots $f_{1/k}$ with the links $f'_{1/k}$, obtained in the similar way from the Hopf link f'_0.) We prove this in Section 2 by showing that f_0 is not equivalent to $f_{1/k}$'s by a small PL (possibly not locally flat) isotopy. One cannot obtain such example by tying small knots on the image of f_0, since they can be untied by a small PL pseudo-isotopy pushing them to points.

Example 1.4′. Alternatively, recall the Hsiang–Shaneson–Wall–Casson–Kirby–Siebenmann example of PL homeomorphisms of an n-torus, $n \geqslant 5$, arbitrarily close to the identity (therefore small isotopic to the identity) but not PL isotopic to it [49, proof of Theorem C], [50, Appendix 2 to Essay IV]. (This example was the key ingredient in the elementary disproof of the Hauptvermutung for manifolds [77, §2], [49, §0], [50].) Perhaps the knots $f_{1/k}$ from the previous example can lead to a similar construction, cf. [23, §12]. See also [25, end of §7] and [16].

Finally we remark that the straightforward way to disprove isotopic realizability of a continuous map $f : S^1 \sqcup S^1 \to \mathbb{R}^3$ is to measure the way of linking of two simple closed curves by a positive integer N, tending to infinity as they get closer to f. In general, the problem of finding such invariants of a wild link seems to be intricate and poorly studied (however, see [51, Part II] and [61] for possible sources of such invariants, leaving alone $\alpha(l)$ from Example 1.3). One of the reasons for this difficulty is existence of the Bing sling [9], closely related to the knots $f_{1/k}$ from Example 1.4.

Example 1.5. Given a link $l: S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$, we define $N(l)$ to be the minimal number of intersections of distinct components under a null-homotopy of the second component. Let h_0 be the Hopf link and $h_i : S^1_1 \sqcup S^1_2 \hookrightarrow \mathbb{R}^3$ be obtained from h_{i-1} by replacing a small regular neighborhood of $h_{i-1}(S^1_1)$ by the solid torus containing the knot $f_{1/k(i)}$, where k is some sufficiently fast growing function, providing $\operatorname{dist}(h_{i-1}, h_i) \leqslant 1/2^i$. In other words, $h_{i-1}|_{S^1_1}$ is the axis of $h_i|_{S^1_1}$ in the sense of [30], meanwhile $h_i|_{S^1_2} = h_{i-1}|_{S^1_2}$. The limit of h_i's is a wild link $h: S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$ (compare to [9, Fig. 1]). In the verification of Example 1.4 in Section 2 we show that $N(f'_{1/i}) = 3$ for each i. Hence $N(h_i) = 3^i$ and $N(h) = \infty$.

1.2. Isotopic realization in higher codimensions

Whereas every map $f: X^n \to Q^m$ is discretely realizable (even approximable by PL embeddings) whenever $m \geqslant 2n+1$, the 'stable range' for isotopic and continuous realizability is, generally speaking, $m \geqslant 2n+2$ (this restriction is sharp by the above examples). Indeed, sufficiently close PL embeddings $X^n \hookrightarrow Q^m$, $m \geqslant 2n+2$, are joined by a small PL ambient isotopy (cf. [11, 5.5]), and the statement follows (cf. [46, Lemma 1]). We shall see that often the restriction $m \geqslant 2n+2$ can be weakened, especially for maps satisfying some additional assumptions of 'niceness'.

Theorem 1.6. *Let X^n be a compact polyhedron, Q^m a PL manifold, $m-n \geqslant 3$.*

(a) *Any PL realizable PL map $f: X \to Q$ is PL continuously realizable.*

(b) *If $h: X \to Y$ is a PL map into a polyhedron Y, $i: Y \hookrightarrow Q$ is an embedding and $i \circ h$ is realizable, then $i \circ h$ is continuously realizable.*

In part (a), under a stronger assumption $m > 3(n+1)/2$, $Q = \mathbb{R}^m$, a weaker conclusion of PL isotopic realizability was conjectured in [70, 1.9d]. A special case of (b) was proved in [6]: if X is a closed smooth manifold, $f: X^n \to \mathbb{R}^{2n-1}$ a generic smooth map, $i: \mathbb{R}^{2n-1} \hookrightarrow \mathbb{R}^{2n}$ the standard inclusion, $i \circ f$ is realizable, then $i \circ f$ is isotopically realizable. The proof of Theorem 1.6 is based on the results of [25] (see Section 3). We reduce (b) to (a) and prove the latter using slicing techniques (see Section 4). For $(n,m) = (1,3)$ both statements of Theorem 1.6 fail by the above examples, and so does isotopic realizability in (b), but using the proof of (a), it is easy to verify that PL isotopic realizability in (a) holds in this case.

To approach the general case, where f is an arbitrary continuous mapping, we introduce ε's into the Haefliger–Harris theory of isovariant maps. In the metastable range there is a certain correspondence between embeddings and isovariant maps, and in Section 7 we extend it for discrete and isotopic realizability:

Criterion 1.7. Let X^n be a compact polyhedron, Q^m a PL manifold, $f : X \to Q$ a (PL) map, $m \geqslant 3(n+1)/2$ in the (a)'s and $m > 3(n+1)/2$ in the (b)'s.

- (a−) [37] f is (PL) homotopic to a (PL) embedding iff $f^2 : X \times X \to Q \times Q$ is equivariantly homotopic to an isovariant map.
- (a) f is (PL) realizable iff f^2 is ε-approximable by isovariant maps for each $\varepsilon > 0$.
- (a+) Moreover, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if f^2 is δ-close to an isovariant map, then f is ε-close to a PL embedding.
- (b−) [37,25] (PL) embeddings $g, h : X \hookrightarrow Q$ are (PL ambient) isotopic iff g^2, h^2 are isovariantly homotopic. (In the TOP case, no restrictions of local flatness are imposed on isotopy, in the spirit of [66].)
- (b) f is (PL) isotopically realizable iff there is a homotopy $\Phi_t : X \times X \to Q \times Q$ such that $\Phi_1 = f^2$ and Φ_t is isovariant for $t < 1$.
- (b+) Moreover, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $g : X \hookrightarrow Q$ is a (PL) embedding and g^2 is δ-homotopic to f^2 by a homotopy Φ_t, isovariant for $t < 1$, then g is taken onto f by a (PL) ε-pseudo-isotopy.

A map $\Phi : X \times X \to Q \times Q$ is *equivariant* if it commutes with the involutions $(x, y) \leftrightarrow (y, x)$ on $X \times X$ and $Q \times Q$, and *isovariant* (cf. [35]), if in addition $\Phi^{-1}(\Delta_Q) = \Delta_X$, where Δ_X means the diagonal of the product $X \times X$. The 'only if' parts are evidently true without any dimensional restrictions. The TOP case of (b−) follows from its PL case, proved in [37], and an easy corollary (see Theorem 3.5(a)) of [25, 6.1+8.1]. See [35] for smooth and [82,83] for various deleted product versions of (a−) and (b−).

For $Q = \mathbb{R}^m$ the PL case of (a) was proved in [70], and seemingly its methods suffice to prove for $Q = \mathbb{R}^m$ the statement of (a+), cf. [70, pre-limit formulation of 1.2]. On the other hand, the deleted product theory of [70] does not work in an arbitrary Q, and, which seems to be more important, its natural generalizations beyond the metastable range, the deleted nth power obstructions, turn out to be incomplete even in Euclidean space [83]. That is why we reestablish the result of [70] in the more reliable setting of isovariant maps. To prove (b+), whose special case was conjectured in [70, 1.9c], we need, besides the straightforward boundary version of (a+), the controlled version of the classical Concordance Implies Isotopy Theorem, which turns out to be non-trivial and of independent interest (see Section 1.3).

Since a constant map $X \to Q$ realizes discretely (or isotopically) iff X embeds into $\mathbb{R}^m$, (a) and (b) generalize the case $Q = \mathbb{R}^m$ of (a−). Consequently by [76] (a), (b) are untrue for each (n, m) such that $3 < m < 3(n+1)/2$. Counterexamples directly to (a), (b) for $(n, m) = (2, 4)$ can be deduced from [6].

Corollary 1.8. *Let X^n be a compact polyhedron and Q^m a PL manifold.*

(a) *If $m = 2n + 1 > 3$, every continuous map $f : X \to Q$ is continuously realizable.*

(b) *If $m > 3(n+1)/2$, discrete realizability implies continuous for a map $f : X \to Q$ such that $\Delta(f) = \{(x, y) \in X \times X \mid f(x) = f(y)\}$ has an equivariant, with respect to the factor exchanging involution, mapping cylinder neighborhood in $X \times X$.*

The proof is given in Section 8. Under *equivariant mapping cylinder neighborhood* of an invariant subspace A of a space B we mean a closed invariant neighborhood of A in B which is equivariantly homeomorphic to the mapping cylinder $A \cup_{n\times 0\mapsto g(n)} N \times I$ of some equivariant map $g : N \to A$. In particular, the hypothesis of (b) is satisfied if $\Delta(f)$ is an invariant subpolyhedron of $X \times X$. Thus we obtain an alternative proof of Theorem 1.6(b) in the metastable range. Analogously for Theorem 1.6(a) (the required PL version of Corollary 1.8(b) follows from the PL part of Criterion 1.7 analogously to the proof of Corollary 1.8(b)); this yields a different proof of [70, Conjecture 1.9d].

Corollary [5]. *Let X^n be a compact polyhedron, Q^m a PL manifold, $m \geqslant 3(n+1)/2$, and $f : X \to Q$ a discretely realizable map. The composition of f and the inclusion $Q = Q \times 0 \hookrightarrow Q \times \mathbb{R}$ is isotopically realizable.*

Example 1.9. Let us construct a map $S^1 \times B^2 \sqcup B^3 \to \mathbb{R}^6$ (in codimension 3), realizable discretely but not isotopically.

Let $T = T_0 \supset T_1 \supset T_2 \supset \cdots$ be a sequence of solid tori $T_i \cong S^1 \times B^2$ such that each inclusion $T_i \subset T_{i-1}$ induces multiplication by 3 in 1-dimensional homology. The intersection $S = \bigcap T_i$ is the triadic solenoid (cf. [28]). We define a sequence of maps $f_i : T \to \mathbb{R}^3 \setminus 0$ as follows. For each $i > 0$ let B_i^3 be the (2^{-i})-neighborhood of the origin 0 in $\mathbb{R}^3$, and x_i be a point in $B_i^3 \setminus 0$. Let f_0 map T onto x_1, and for $i > 0$ put $f_i = f_{i-1}$ on $T \setminus T_i$ and let $f_i|_{T_i} : (T_i, \partial T_i) \to (B^2, \partial B^2) \to (B_i^3 \setminus 0, x_i)$ be any map taking T_{i+1} onto x_{i+1} and inducing isomorphism in relative 2-cohomology.

Then the limit map $f : T \to \mathbb{R}^3$ meets 0 in $f(S)$ and has the following property: the absolute magnitude of the difference $d(\varphi, f_0) \in H^2(T, \partial T; \pi_2(\mathbb{R}^3 \setminus 0))$ is arbitrarily great for every map $\varphi : (T, \partial T) \to (\mathbb{R}^3 \setminus 0, x_1)$, sufficiently close to f. Indeed, from the fact that each homomorphism in the sequence

$$\cdots \to H^2(T, T \setminus T_2) \to H^2(T, T \setminus T_1) \to H^2(T, \partial T)$$

is multiplication by 3 of the group $\mathbb{Z}$ of integers, it follows, firstly, that $d(f_i, f_0) = 1 + 3 + \cdots + 3^{i-1} = (3^i - 1)/2$ for $i > 0$ and, secondly, that $d(\varphi, \psi) \in 3^i\mathbb{Z}$ for each two maps $\varphi, \psi : (T, \partial T) \to (\mathbb{R}^3 \setminus 0, x_1)$ agreeing with f on $T \setminus T_i$. Given $i > 0$, we take φ so close to f that it can be homotoped, keeping image in $\mathbb{R}^3 \setminus 0$, to agree with f, hence with f_i, on $T \setminus T_i$. Then

$$d(\varphi, f_0) \in \frac{3^i - 1}{2} + 3^i\mathbb{Z}$$

and, consequently,

$$\left|d(\varphi, f_0)\right| \geqslant \frac{3^i - 1}{2}.$$

It follows that there does not exist a homotopy $h_t : T \to \mathbb{R}^3$ such that $h_1 = f$ and $\operatorname{im} h_t \subset \mathbb{R}^3 \setminus 0$ for $t < 1$.

Now let us use f and the standard inclusion $B^3 \hookrightarrow \mathbb{R}^3$ to obtain a new map $F : T \sqcup B^3 \to \mathbb{R}^3 \times 0 \cup 0 \times \mathbb{R}^3 \hookrightarrow \mathbb{R}^6$. It is discretely realizable, for there are embeddings $F_i : T \sqcup B^3 \to \mathbb{R}^6$, $i = 1, 2, \ldots,$ defined by $F_i|_T(p) = (f_i(p), g_i(p))$, where $g_i : T \hookrightarrow B_i^3 \subset \mathbb{R}^3$ are some embeddings, and by $F_i|_{B^3} = F|_{B^3}$. On the other hand, F is not isotopically realizable, for otherwise it could be assumed (see Remark 6.1) that the image of B^3 was fixed under the pseudo-isotopy, and hence there would exist a homotopy h_t as above.

Alternatively, isotopic realizability of F would imply existence of a homotopy $H_t : T \times B^3 \to \mathbb{R}^6$ such that $H_1(p, q) = F(p) - F(q)$ for each $(p, q) \in T \times B^3$ and $\operatorname{im} H_t \subset \mathbb{R}^6 \setminus 0$ for $t < 1$, which can be shown to be impossible analogously to the above argument. (H_t is $(\text{pseudo-isotopy}|_{\text{embedded } T}) \times (\text{pseudo-isotopy}|_{\text{embedded } B^3})$, composed with the projection $\mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}^6$ given by $(x, y) \mapsto x - y$.)

Example 1.9′. It is easy to see that, if in the above construction the map $f_i|_{T_i}$ was replaced with one inducing multiplication by k for each i, where $k \not\equiv 1 \bmod 3$, or, instead, the triadic solenoid was replaced with the dyadic one, then the resulting map, although isotopically realizable, would still be not continuously realizable.

Actually Example 1.9 can be improved to yield a series of maps $S^n \to \mathbb{R}^{2n}$, $n \geqslant 3$, realizable discretely but not isotopically [5]. On the other hand, this example, in view of Criterion 1.7, opens up the way to a complete algebraic description of isotopically and continuously realizable maps among discretely realizable maps in the metastable range. Such a description was obtained recently, and for completeness we state it briefly (for the case $X = S^n$, $Q = \mathbb{R}^m$; the general case is conceptually the same but involves additional technicalities).

Given a continuous map $f : S^n \to \mathbb{R}^m$, let us consider open sets

$$U = S^n \times S^n \setminus \Delta_{S^n} \supset U^f = S^n \times S^n \setminus \Delta(f) \supset U^f_\varepsilon = S^n \times S^n \setminus P_\varepsilon,$$

where P_ε is some fixed closed polyhedral neighborhood of $\Delta(f)$, containing $N_\varepsilon = \{(x, y) \colon \|f(x) - f(y)\| < \varepsilon\}$ and contained in $N_{2\varepsilon}$. The composition $\tilde{f} : U^f \to S^{m-1}$ of the restriction $f^2|_{U^f}$ and the obvious canonical homotopy equivalence

$$\mathbb{R}^m \times \mathbb{R}^m \setminus \Delta_{\mathbb{R}^m} \to \mathbb{R}^m \setminus 0 \to S^{m-1}$$

is equivariant with respect to the factor exchanging involution t on $U^f \subset S^n \times S^n$ and the antipodal involution s on S^{m-1}. On the quotient space U/t, let us consider the locally constant sheaf $\mathcal{Z}_m$ with each stalk isomorphic to $\mathbb{Z}$ and the action of $\pi_1(U/t)$ on the stalks defined by

$$\alpha \mapsto \begin{cases} s_*, & \delta_*(\alpha) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\delta_* : \pi_1(U/t) \to \mathbb{Z}_2$ denotes the connecting homomorphism from the exact sequence of the bundle $U \to U/t$ and 1 denotes the non-trivial element of $\mathbb{Z}_2$, while 0 denotes the trivial automorphism and s_* the automorphism $1 \mapsto (-1)^m$ (induced by the involution s)

of the group $\mathbb{Z} = \pi_{m-1}(S^{m-1})$. For each $X \subset U$ we write $H^{m-1}_{\text{eq}}(X)$ for the cohomology group $H^{m-1}(X/t; \mathcal{Z}_m|_{X/t})$. Let $(f) \in H^{m-1}_{\text{eq}}(U^f)$ denote the first obstruction for equivariant homotopy (cf. [22, §§2,4]) of the maps $\tilde{f}$ and $\tilde{i}|_{U^f}$, where $i : S^n \hookrightarrow \mathbb{R}^m$ is the standard (or any, in view of [87]) inclusion. Similarly, we denote by $o_\varepsilon(f) \in H^{m-1}_{\text{eq}}(U^f_\varepsilon)$ the first obstruction for equivariant homotopy of the restrictions $\tilde{f}|_{U^f_\varepsilon}$, $\tilde{i}|_{U^f_\varepsilon}$. The latter obstruction can be equivalently defined in the spirit of van Kampen (cf. [5], compare to [70, 1.4]). Finally, for embeddings $g_1, g_2 : S^n \hookrightarrow \mathbb{R}^m$, ε-close to f, the first obstruction $d(g_1, g_2)$ for equivariant ε-homotopy of $(\tilde{g}_i|_{U^f})$'s is an element of $H^{m-1}_{\text{eq}}(U, U^f_\varepsilon)$. Let us write $G_k = H^{m-1}_{\text{eq}}(U, U^f_{2^{-k}})$ and let $j^l_k : G_l \to G_k$, $l > k$, be the forgetful homomorphism.

The following result, whose proof is based on Criterion 1.7, shows in particular that from the algebraic viewpoint, maps yielding negative solution to the metastable case of the IR Problem look quite similar to phantom maps [32].

Theorem[4] [5]. *Let $f : S^n \to \mathbb{R}^m$ be a continuous map, $m > 3(n+1)/2$.*

(a) *f is discretely realizable iff $o_\varepsilon(f) = 0$ for $\varepsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$.*

(b) *f is isotopically realizable iff $o(f) = 0$.*

(c) *Suppose that f is discretely realizable. f is continuously realizable iff the inverse spectrum $\{G_k; j^l_k\}$ satisfies the Mittag–Leffler condition or, equivalently* [32],

$$\varprojlim{}^1 \{G_k; j^l_k\} = 0.$$

(d) *Suppose that f is discretely realizable. f is isotopically realizable iff*

$$0 = O(f) \in \varprojlim{}^1 \{G_k; j^l_k\}.$$

The obstruction $O(f)$ can be defined (cf. [59]) as the class of the sequence $d(g_1, g_2)$, $d(g_2, g_3), \ldots,$ where $g_k : S^n \hookrightarrow \mathbb{R}^m$ is an embedding, (2^{-k})-close to f. See [57] for definition and basic properties of the derived limit functor. The fact that no obstructions arise in dimensions other than $m-1$ is due to the Serre Theorem on finiteness of homotopy groups of spheres. Using the fact that the forgetful homomorphism $H^*_{\text{eq}}(\cdot) \to H^*(\cdot)$ factors

[4] In the case $m < 2n$, the proof of this theorem in [5] contains a mistake (on page 81, line 8). The argument in [5] works to prove the theorem only under the additional assumption that the given map f is discretely k-realizable for each $k = m+1, m+2, \ldots, 2n$ (see definition below). Without this assumption, the parts (a) and (b) are incorrect already for $m = 2n-1$, while (d) fails for $m = 2n-5 \geqslant 9$ [59]. Fortunately, the part (c) (and hence its corollary) is correct as stated, but its proof in [5] is insufficient without the additional assumption; the correct proof rests on higher cohomology operations and appears in [59] (see also Erratum to [5]). Let us state the required definition of discrete k-realizability. Assume $2n > m > 3(n+1)/2$ and $m \leqslant k \leqslant 2n$. Let us call a PL map $f : S^n \to \mathbb{R}^m$ a *k-embedding* if there is a triangulation T of S^n such that f is simplicial in some subdivision of T and embeds each simplex of T, and $f(\sigma) \cap f(\tau) = f(\sigma \cap \tau)$ for any two simplices σ^s, τ^t of T such that $s + t \leqslant k$. Let us call a map $f : S^n \to \mathbb{R}^m$ *discretely k-realizable*, if $\forall \varepsilon > 0\ \exists \delta > 0$ such that any $(k-1)$-embedding, δ-close to f, is PL ε-homotopic in the class of $(k-2)$-embeddings to some k-embedding. It is easy to see that f is discretely m-realizable iff $o_\varepsilon(f) = 0$ for all $\varepsilon > 0$.

through the multiplication by 2 in $H^*(\cdot)$, and the Alexander duality, one immediately obtains the following

Corollary [5]. *Let $f : S^n \to \mathbb{R}^m$ be a discretely realizable map, $m > 3(n+1)/2$. If the canonical epimorphism $H_{2n-m}(\Delta(f)) \to \check{H}_{2n-m}(\Delta(f))$ between the reduced Steenrod* (*exact*; *cf.* [57]) *and the reduced Čech* (*continuous*; *cf.* [28]) *homology has trivial kernel, then f is continuously realizable.*

This puts the IR Problem in the metastable range in the context of the discussion 'continuity versus exactness' in Eilenberg and Steenrod [28, p. 265] (see [29] for a modern version). It follows, e.g., that if a map $f : S^n \to \mathbb{R}^{2n}$, $n \geqslant 4$, is realizable discretely but not continuously, then the compactum $\Delta(f)$ cannot be zero-dimensional or have countable Steenrod 0-homology (cf. [36]).

Remark. In proving Criterion 1.7 we obtain a number of interesting results in the PL category in the metastable range, among which are: sufficient conditions for existence of an embedding in the ε-homotopy class of a map (Theorem 7.1), of an embedding in the ε-regular homotopy class of an immersion (Theorem 7.2), of an immersion in the ε-homotopy class of a map (Theorem 7.4), of an ε-ambient isotopy between close embeddings (Corollary 7.9(a)). These all are controlled versions of Harris' criteria [37], however we use some ideas, additional (proof of Theorem 7.2) and alternative (proof of Theorem 7.4) to that of [37]. In fact, our proof of Theorem 7.4 is a new geometric proof of [37, Theorem 2] (roughly a half of Criterion 1.7(a−)) and a good candidate for generalization for k-tuples of points.

Conjecture 1.10. In each range $m \geqslant (k+1)(n+1)/k$ $(m > (k+1)(n+1)/k)$ the analogue of Criterion 1.7 holds for isovariant maps $(X^n)^k \to (Q^m)^k$, provided $m - n \geqslant 3$.

We call a map $\Phi : (X^n)^k \to (Q^m)^k$ *isovariant* if it commutes with the actions of the symmetric group S_k on $(X^n)^k$ and $(Q^m)^k$, and if $\Phi^{-1}(\Delta^S_Q) = \Delta^S_X$ for each $S \subset \{1, 2, \dots, k\}$, where $\Delta^S_X = \{(x_1, \dots, x_k) \in (X^n)^k \mid i, j \in S \Rightarrow x_i = x_j\}$. Such a result cannot be expected in codimension 2: the reader may wish to verify that non-triviality of the link $f'_{1/k}$ from Example 1.5 (as well as that of the kth Milnor's link [66] and of the link Whitehead$_k$, cf. [51]) is not detected by the isovariant homotopy class of the $(k+2)$th power mapping. It is worth observing that, in contrast to the metastable range (where smooth embeddability and quasi-embeddability are equivalent to PL embeddability), the smooth and deleted versions of Conjecture 1.10 are untrue [34,83].

1.3. Other definitions of realizability and relations on close embeddings

A (PL) map $F : X \times I \to Q \times I$ will be called a (PL) *pseudo-concordance* if $F^{-1}(Q \times 1) = X \times 1$ and $F|_{X \times [0,1)}$ is an embedding. We call a (PL) map $f : X \to Q$ (PL) *concordantly realizable* if $f \times \mathrm{id}_1$ extends to a (PL) pseudo-concordance $F : X \times I \to Q \times I$. If, in

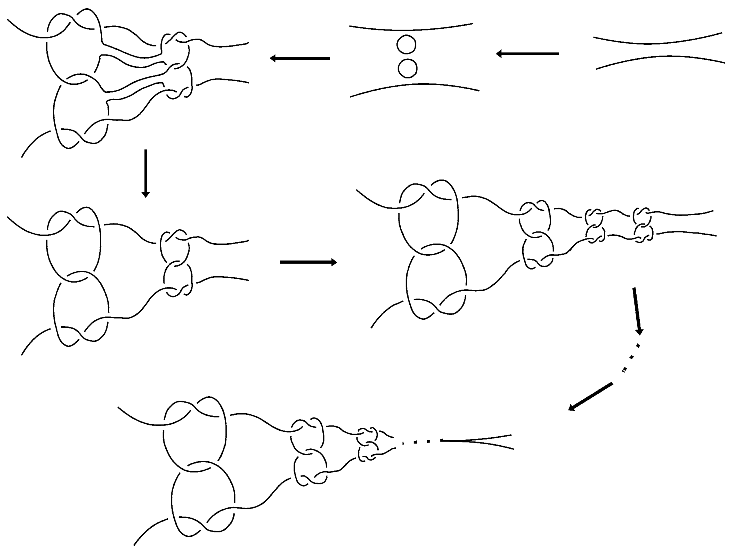

Fig. 4.

addition, $F^{-1}(Q \times 0) = X \times 0$, then f is (PL) *pseudo-concordant to* the unique (PL) embedding $g : X \hookrightarrow Q$ such that $g \times \mathrm{id}_0 = F|_{X\times 0}$.

Example 1.11. The map f from Example 1.3 is concordantly realizable. Several slices $(\operatorname{im} F) \cap \mathbb{R}^3 \times t$ of a pseudo-concordance F are shown on Fig. 4.

In higher codimensions the situation is again quite different:

Theorem 1.12. *Let X^n be a compact polyhedron, Q^m a PL manifold, $m - n \geqslant 3$. Then each* (*PL*) *concordantly realizable* (*PL*) *map $f : X \to Q$ is* (*PL*) *isotopically realizable. Moreover, $\forall \varepsilon > 0\ \exists \delta > 0$ such that every* (*PL*) *embedding $g : X \hookrightarrow Q$,* (*PL*) *δ-pseudo-concordant to f, can be taken onto f by a* (*PL*) *ε-pseudo-isotopy.*

The proof is given in Section 6; the TOP case is based on the following controlled version Theorem 1.13(a) of the classical PL Concordance Implies Isotopy Theorem (CIIT):

Theorem 1.13. *For each $\varepsilon > 0$ and a positive integer n there exists $\delta = \delta(n, \varepsilon) > 0$ such that the following holds.*

(a) *Let X^n be a compact polyhedron and Q^m a PL manifold, $m - n \geqslant 3$. Then each two PL δ-concordant embeddings $f, g : X \hookrightarrow Q$ are PL ε-ambient isotopic.*

(b) *Let X^n be a compact smooth manifold, Q^m a smooth manifold, $m > 3(n + 1)/2$. Then each two smoothly δ-concordant embeddings $f, g : X \hookrightarrow Q$ are smoothly ε-ambient isotopic.*

It seems that Hudson's original proof of CIIT [40] (as well as Lickorish's proof of the case $Q = S^m$ [55, Theorem 6]) does not work to prove 1.13(a) (compare to remarks in [62, Introduction], [70, §2]). In [73] Rourke sketched a new proof of CIIT, and in [42, last paragraph] it was 'expected that, when the details of Rourke's proof are published, they will apply' to prove 1.13(a). A special case of 1.13(a) was conjectured in [70, 1.9a]. Perhaps 1.13(a) can be also proved by the methods of [25, proof of 7.1], but hardly by that of [21, proof of Lemma 1].

Be that as it may, in Section 5 we present an explicit proof of 1.13(a). It is far from being a trivial extension of either known proof of CIIT (this is clear at once from the statement of Lemma 5.8), and it is also a new proof of CIIT (since Theorem 1.13(a) generalizes CIIT, by taking a metric on Q with all distances $< \delta$). Theorem 1.13(a), along with CIIT, is untrue in codimension 2 because of slice knots and links. It is worth observing that in our proof of 1.13(a) the main efforts are applied to obtain ε-ambient isotopy, rather than ambient ε-isotopy. In the proof of 1.13(a) we use Theorem 3.3(a), which includes Miller's controlled version [62, Theorem 9] of Zeeman's Unknotting Balls [87]. In turn, the $\partial Q \neq \emptyset$ version of 1.13(a), which is proved analogously, immediately implies [62, Theorem 9] (this was pointed out in [42, last paragraph]) and [21, Lemma 1]. In [14, Idea of proof of Theorem 3] a statement, similar to 1.13(a) was used with 'reference' to Hudson's CIIT (see [40]); the misquotation disappears in the revised proof [15].

Next we convert 1.13(a) to the smooth category and obtain Theorem 1.13(b). It answers, at least to some extent, a question of Kirby [48, discussion preceding 2.1]: 'suppose $m > 3(n+1)/2$, is there a function ε of δ such that any smoothly δ-isotopic smooth embeddings are smoothly ε-ambient isotopic?' (From the proof of 1.13(b) it follows that ε can be taken as $c(n) * \delta$, where $c(n)$ is a constant depending on $n = \dim X$.) It should be mentioned that our proof of 1.13(b) uses Kirby's partial answer to his question (see Theorem 3.3(b)). We conjecture that 1.13(b) holds in codimension $\geqslant 3$.

A corollary of Theorem 1.13(b) is the smooth version 3.2(b+) of Edwards' Theorem 3.2(a) on ε-equivalence of PL embeddings, close to a TOP embedding (see also Theorem 3.5(b+)). Using 1.13(a) we also obtain an alternative controlled version 3.7+ of CIIT.

Example 1.14. In general, small (smooth or TOP/PL locally flat) isotopy, in particular small concordance, does not imply small ambient isotopy. (Of course, it implies a great smooth or TOP/PL ambient isotopy [38,27,74].) Indeed, take the standard circle $S^1 \subset \mathbb{R}^3$ and tie, near a point $x \in S^1$, a small (e.g., trefoil) knot on it to obtain an embedding $f_0 : S^1 \hookrightarrow \mathbb{R}^3$. One can shift this small knot along S^1 by a (smooth or TOP/PL locally flat) isotopy f_t, which, at each moment $t \in I$, has support in a small neighborhood of the current position of the small knot on the circle. (Such an isotopy cannot be obtained by means of rotation of the whole circle.) But it is clear (see Section 2 for details) that f_0 and f_1 can not be joined by a small ambient isotopy.

Finally, we relate the Ščepin–Štan'ko definition of isotopic realizability to the two Akhmet'ev's definitions [2]. A map $f : X \to Q$ of a compact smooth manifold into a smooth manifold is A_1-(A_2-)*isotopically realizable*, if there is a homotopy $f_t : X \to Q$

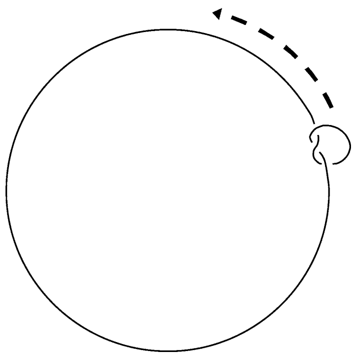

Fig. 5.

(respectively $H_t : Q \to Q$), called an A_1-(A_2-)*pseudo-isotopy*, such that $f_1 = f$ (respectively $H_0 = \mathrm{id}$ and $H_1 \circ g = f$ for some smooth embedding $g : X \hookrightarrow Q$), and which is a smooth isotopy (respectively smooth ambient isotopy) for $t < 1$. Certainly, A_i-isotopic realizability is not equivalent to isotopic realizability, see Example 1.2. Evidently, A_2-isotopic realizability implies A_1-. But the author does not see why the reverse implication holds, as claimed in [2].

Example 1.15. The standard embedding $f : S^1 \hookrightarrow \mathbb{R}^3$ is, of course, isotopically realizable in either sense. However, there is an A_1-pseudo-isotopy f_t from f to an embedding, which cannot be covered by a pseudo-isotopy (in particular, by an A_2-pseudo-isotopy).

Indeed, rotate a small knot around the circle, as in Example 1.14, so that its size tends to zero (hence $f_t \to f$) as $t \to 1$ and so that the speed of its rotation, along with the number of turns, tends to infinity as $t \to 1$. If f_t was covered by a pseudo-isotopy $H_t : Q \to Q$, we would obtain a contradiction with Example 1.14.

In order to avoid too restrictive assumptions of smoothness, let us say that a map $f : X \to Q$ is *M-isotopically realizable*, if there exists a homotopy $f_t : X \to Q$, called an *M-pseudo-isotopy*, such that $f_1 = f$ and for each $t < 1$ the map f_t is a topological embedding. The letter 'M' accounts for the fact that f_t for $t \in [0, 1)$ is an isotopy in the sense of Milnor [66].

Question III. Does there exist an M-isotopically realizable map which is not isotopically realizable?

By Theorem 1.12 such a map cannot be found in the codimension $\geqslant 3$ range. Furthermore, the DIFF case of the following theorem, proved in Section 6, implies (in view of Theorem 1.12) that in the metastable range all four definitions of pseudo-isotopy, as well as all four definitions of isotopic realizability (A_1 and A_2 of Akhmet'ev, M in the spirit of Milnor, and the classical one of Ščepin–Štan'ko) are equivalent. For another application of Theorems 1.12 and 1.16, see Remark 6.1.

Theorem 1.16. *Let X^n be a compact polyhedron (compact smooth manifold), Q^m a PL (smooth) manifold, $f : X \to Q$ a continuous map, $m - n \geqslant 3$ (respectively $m \geqslant 3(n+1)/2$, $n > 1$ in (a), $m > 3(n+1)/2$ in (b)).*

(a) *If f is isotopically realizable, then there exists a pseudo-isotopy, taking a PL (smooth) embedding $g : X \hookrightarrow Q$ onto f.*

(b) *If a PL (smooth) embedding $g : X \hookrightarrow Q$ is taken onto f by a pseudo-isotopy, then g can be taken onto f by a pseudo-isotopy $H_t : Q \to Q$ such that whenever $t \in [0, 1)$, H_t is a PL (smooth) isotopy.*

A continuous map $f : X_1 \sqcup \cdots \sqcup X_k \to Q$ (where the components $X_1, \ldots, X_k$ are fixed, but not necessarily connected) is called *disjoinable*, if it is approximable by link maps (cf. [85,24,84]); a map $g : X_1 \sqcup \cdots \sqcup X_k \to Q$ is called a (*generalized*) *link map* if $g(X_i) \cap g(X_j) = \emptyset$ whenever $i \neq j$ (cf. [58]). We call f *homotopically disjoinable* if there is a homotopy f_t such that $f_1 = f$ and f_t is a link map for $t < 1$.

Example 1.17.

(i) The proof of Example 1.1′ allows to replace '(isotopically) realizable' with '(homotopically) disjoinable' in its statement.

(ii) The construction of Example 1.9 yields a map $S^1 \times B^2 \sqcup pt \to \mathbb{R}^3$, which is disjoinable but not homotopically disjoinable.

(iii) The map f from Example 1.3 turns out to be homotopically disjoinable. Indeed, we start from two disjoint arcs. In the spirit of Example 1.11 we generate linking trefoils, keeping ends of arcs fixed, by the price of self-intersections of components. In the spirit of Example 1.3′ we compensate the increase of the linking number by small loops tending to the singular point as the time approaches 1.

We conjecture that the analogues of 1.6–1.8 for homotopic disjoinability hold and can be proved analogously. Moreover, the remark 1.17(iii) in conjuction with the facts that for classical links singular link concordance implies link homotopy and that κ-invariant has trivial kernel up to link homotopy motivate a conjecture that every map $S^1 \sqcup \cdots \sqcup S^1 \to \mathbb{R}^3$ is homotopically disjoinable.

Remark. For completeness let us consider a concept, approximately dual to homotopic disjoinability, that is, admitting distant self-intersections and prohibiting close ones. We call a map $f : X \to Q$ *locally isotopically realizable* if there is a homotopy $f_t : X \to Q$ such that $f_1 = f$ and f_t is a topological immersion for $t < 1$.

From the C^0-dense h-principle for smooth immersions it follows [2, proof of Lemma 2] that if a compact smooth manifold X^n smoothly immerses into a smooth manifold Q^m, $m - n \geqslant 1$, then each map $f : X \to Q$ is locally isotopically realizable.

2. Verification of examples

Verification of 1.1′. The argument below was inspired by an idea of Skopenkov (compare [72] and [70]). Suppose that $f_t : P \times \{0,1\} \to \mathbb{R}^2$ is a homotopy such that $f_1 = i \circ p$ and $f_t(P \times 0) \cap f_t(P \times 1) = \emptyset$ whenever $t < 1$. Let $F_t : P \times P \to \mathbb{R}^2 \times \mathbb{R}^2$ be the map defined by $(p,q) \mapsto (f(p \times 0), f(q \times 1))$. (In other words, $F_t = f_t^2|_{(P\times 0)\times(P\times 1)}$.)

Then $F_t^{-1}(\Delta_{\mathbb{R}^2})$ is empty for $t < 1$ and equals $\Delta_P = \{(p,p) \in P \times P\}$ for $t = 1$.

Since P is acyclic, $P \times P$ is acyclic and the map $F_0 : P \times P \to \mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta_{\mathbb{R}^2}$ is null-homotopic in $\mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta_{\mathbb{R}^2}$. Therefore the map

$$F_1|_{P\times P\setminus\Delta_P} : P \times P \setminus \Delta_P \to \mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta_{\mathbb{R}^2}$$

is also null-homotopic in $\mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta_{\mathbb{R}^2}$. The latter map is equivariant with respect to the involutions $(p,q) \leftrightarrow (q,p)$ on $P \times P \setminus \Delta_P$ and $\mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta_{\mathbb{R}^2}$, and the latter space is equivariant homotopy equivalent to S^1 equipped with the antipodal involution. Thus we obtain an inessential equivariant map $P \times P \setminus \Delta_P \to S^1$.

By [71], existence of such a map implies that $P \times P \setminus \Delta_P$ is not connected. Suppose that (p_1,p_2) and (q_1,q_2) lie in distinct connected components of $P \times P \setminus \Delta_P$; without loss of generality $p_2 \neq q_1$. Then either (p_1,p_2), (q_1,p_2) or (q_1,p_2), (q_1,q_2) lie in distinct components, say the first ones. Consequently, p_1 and q_1 lie in distinct connected components of $P \setminus p_2$. But P has no separating points, and we arrive at a contradiction.

Verification of 1.3. We are to prove that for every wild link $\bar{g} : S^1_1 \sqcup S^1_2 \hookrightarrow \mathbb{R}^3$

$$\limsup_{\substack{l\to\bar{g},\\ l\in\mathcal{L}_{\mathrm{PL}}}} \alpha(l) < \infty,$$

where $\mathcal{L}_{\mathrm{PL}}$ denotes the subspace of PL embeddings in the space of all continuous maps $S^1 \sqcup S^1 \to \mathbb{R}^3$, equipped with the topology of uniform convergence.

Assume on the contrary that there is a sequence of PL links $l_1, l_2, \ldots,$ converging to $\bar{g}$ and such that $\alpha(l_i) \to \infty$. Let N be a polyhedral neighborhood of $\bar{g}(S^1_1)$ in $\mathbb{R}^3 \setminus \bar{g}(S^1_2)$. We can assume that $l_i(S^1_1) \subset N$ for each i. We fix a decomposition of N into handles: $N = B^3 \cup H_1 \cup \cdots \cup H_q$, where $H_j \cong D^2 \times I$ via a homeomorphism $h_j : D^2 \times I \to H_j$ and $H_j \cap B^3 = h_j(D^2 \times \partial I)$ (without loss of generality there are no 2-handles). By the definition of a prime knot, there is a collection of disjoint 3-balls $B_{i,1}, B_{i,2}, \ldots, B_{i,\alpha(l_i)}$ such that the boundary of each B_{ij} meets $l_i(S^1_1)$ precisely in two points, separating one essential prime knot $k_{i,e_j} : (I, \partial I) \to (B_{ij}, \partial B_{ij})$ from all other prime knots in the decomposition of $l_i|_{S^1_1}$. Since k_{i,e_j} is essential, its image meets each D^2-fiber of some handle $H_{m_{ij}}$. By our assumption $\alpha(l_i) \to \infty$ as $i \to \infty$, hence for some handle H_l the number n_i of the knots k_{i,e_j} such that $m_{ij} = l$ tends to infinity as $i \to \infty$. Let us fix the handle $H_l = h_l(D^2 \times I)$ and denote the images of the latter knots by $\kappa_{i,1}, \kappa_{i,2}, \ldots, \kappa_{i,n_i}$.

Now for each i, j the intersection $\kappa_{ij} \cap H_l$ is the union of some PL arcs. Each of these arcs meets either $h_l(D^2 \times 0)$ or $h_l(D^2 \times 1)$, and at least one of these arcs, denoted by a_{ij}, meets $h_l(D^2 \times \frac{1}{2})$. The diameter of a_{ij} is therefore at least $d = \mathrm{dist}(h_l(D^2 \times \frac{1}{2}), h_l(D^2 \times \{0,1\})$. Since the arcs $a_{i,1}, \ldots, a_{i,n_i}$ are contained in disjoint curves $\kappa_{i,1}, \ldots, \kappa_{i,n_i} \subset l_i(S^1_1)$, the PL curve $l_i(S^1_1)$ contains at least n_i disjoint subarcs, each of diameter at least d. Since n_i tends to infinity as $i \to \infty$, this is in contradiction with the assumption of convergence of l_i's to $\bar{g}$.

Verification of 1.5. Let T be a small regular neighborhood of $f_0(S^1)$ and l be a circle in the complement of T, linked with $f_0(S^1)$ with linking number one. If $H_t:\mathbb{R}^3 \to \mathbb{R}^3$ is a PL pseudo-isotopy taking a PL embedding f_ε onto f_0, then $h_t = H_t \circ f_\varepsilon$ is a PL (possibly not locally flat) isotopy (with all points of failure of local flatness occurring for $t = 1$), and moreover if H_t is sufficiently small, the image of h_t lies in T. The statement of Example 1.5 follows from Claims 2.1, 2.2 below.

Claim 2.1. *If $g:S^1 \hookrightarrow T$ is a PL embedding, the minimal number $I(g)$ of transversal intersections of a singular disk, spanned by l, with $g(S^1)$ is invariant under PL (generally not locally flat) isotopy in T.*

Proof. Let $h_t:S^1 \hookrightarrow T$ be a PL (possibly not locally flat) isotopy and D be a disk, spanned by l and meeting $h_0(S^1)$ in $I(h_0)$ points. It suffices to show that there is a disk D', spanned by l and meeting $h_1(l)$ in $I(h_0)$ points. Without loss of generality we can assume that h_t is either locally flat, or locally knotted at a unique point $a \in S^1$ in the moment $t = 1/2$, so that $h_t = h_0$ outside a small neighborhood U of a and $h_t(U) \subset W$, where W is a regular neighborhood of $h_{1/2}(S^1)$ relative $h_{1/2}(S^1 \setminus U)$.

In the first case h_t can be covered by an ambient isotopy H_t [74], which carries the disk D so that the number of intersections of $D_t = H_t(D)$ with $h_t(S^1) = H_t(h_0(S^1))$ remains constant. In the second case we modify the disk D as follows. First we shift any intersections with $h_0(S^1)$ along $h_0(S^1)$ out of $h_0(U)$. Then the shifted D, denoted by $\widetilde{D}$, does not meet $h_0(S^1)$ in W. Next we push $\widetilde{D}$ out of W. It is possible since the kernel of

$$incl_*:\pi_1\big(\mathbb{R}^3 \setminus (h_0(S^1) \cup W)\big) \to \pi_1\big(\mathbb{R}^3 \setminus h_0(S^1)\big)$$

is trivial. Indeed, introduction of commutativity relations into the subgroup of $\pi_1(\mathbb{R}^3 \setminus (h_0(S^1)))$, consisting of conjugates to the loops lying in W, yields

$$comm_*:\pi_1\big(\mathbb{R}^3 \setminus (h_0(S^1))\big) \to \pi_1\big(\mathbb{R}^3 \setminus (h_0(S^1) \cup W)\big)$$

such that

$$incl_* \circ comm_* = \mathrm{id}.$$

Hence $incl_*$ has trivial kernel, consequently we can replace $\widetilde{D}$ by a disk D' avoiding W. Since h_t has its support in W, the disk D' meets $h_0(S^1)$ and $h_1(S^1)$ in the same points. □

Now let us recall the knots $f_{1/k}$ from Example 1.5.

Claim 2.2. $I(f_{1/k}) = 3$ *for each* $k = 1, 2, \ldots$ (*while* $I(f_0)$ *is clearly* 1).

Proof. It is clear that $I(f_{1/k}) \leqslant 3$ for each $k = 1, 2\ldots.$ There is a k-fold cover $p:T \to T$ such that $p^{-1}(f_1(S^1)) = f_k(S^1)$, hence $I(f_{1/k}) \geqslant I(f_1)$. It remains to show that $I(f_1)$

is not less than 3. Actually f_1 is the trefoil knot, and l represents $a^{-1}b^2$ in its group $G = \langle a, b \mid aba = bab\rangle$. A disk, spanned by l, cannot meet $f_1(S^1)$ in 2 points, for this would imply an even linking number of l and $f^1(S^1)$.

Suppose that there is a disk, spanned by l and meeting $f_1(S^1)$ transversely in one point. Then l is homotopic to a loop representing an element of G of type $g^{-1}b^{\varepsilon}g$, where $g \in G$, $\varepsilon = 1$ or -1. Since $[[l]] = [a^{-1}b^2] = [b]$ in $G/[G, G] = H_1(\mathbb{R}^3 \setminus f(S^1))$, necessarily $\varepsilon = 1$, hence for some $g \in G$ the equality $a^{-1}b^2 = b^g$ holds in G (here b^g denotes $g^{-1}bg$). We will show this to be impossible by considering a representation of G.

It is easy to see that the formulae $a \mapsto (123)$, $b \mapsto (432)$ yield a representation $\varphi : G \to A_4 \subset S_4$ in the symmetric group (the well-known representation ψ in S_3 is insufficient, since $\psi(a^{-1}b^2) = \psi(a^{-1}) = \psi(b^{ba})$). We have $\varphi(a^{-1}b^2) = (412)$ and $\varphi(b^g) = (432)^{\varphi(g)}$. But (432) and $(412) = (432)^{(13)}$ are not conjugate in A_4, which is a contradiction. □

Verification of 1.14. Suppose that the small isotopy f_t can be covered by a small ambient isotopy $H_t : \mathbb{R}^3 \to \mathbb{R}^3$, $H_0 = \mathrm{id}$, $H_1 \circ f_0 = f_1$ (we omit the epsilonics). Denote by π the fundamental group $\pi_1(\mathbb{R}^3 \setminus f_0(S^1))$. Let $a \in \pi$ be the class of a small circle around S^1 far from x. Let b be any element of π which is not a power of a, and represent b by a small loop l (which necessarily lies near x). Then $H_1(l)$ lies possibly little farther from x, but still near it. Now f_1 has its small knot far from x, hence far from l. This means that l should represent a power of a in $\pi = \pi_1(\mathbb{R}^3 \setminus f_1(S^1))$, which is a contradiction.

3. Some facts on close PL, DIFF and TOP embeddings

In this section we recall some approximation theorems to be heavily used in the rest of the paper. Exceptions are Theorems 3.2(b+), 3.5(b+), 3.7+, which are not used in the sequel; on the contrary, their proofs require Theorem 1.13, proved in Section 5.

Theorem 3.1.

(a) [20], [63], [12], [25, 8.1], [13]. *Let (X^n, Y^{n-1}) be a polyhedral pair, $(Q^m, \partial Q)$ a PL manifold, $m - n \geqslant 3$. Then any TOP embedding $f : (X, Y) \hookrightarrow (Q, \partial Q)$ is ε-approximable, for each $\varepsilon : X \to (0, \infty)$, by a PL embedding $g : (X, Y) \hookrightarrow (Q, \partial Q)$. Moreover if Z is a subpolyhedron of X and $f|_Z$ is PL, then it can be assumed that $g|_Z = f|_Z$.*

(b) [33], [48, 2.2]. *Let X^n be a compact smooth manifold, Q^m a smooth manifold and $m \geqslant 3(n + 1)/2$. Any TOP embedding $f : X \hookrightarrow Q$ is ε-approximable, for each $\varepsilon > 0$, by a smooth embedding $g : X \hookrightarrow Q$. Moreover if Z is a closed subset of X and f is smooth on the δ-neighborhood of Z, then it can be assumed that $g|_Z = f|_Z$.*

Theorem 3.2.

(a) [7], [15], [63], [25, 6.1]. *Let (X^n, Y^{n-1}) be a polyhedral pair, $(Q^m, \partial Q)$ a PL manifold, $m - n \geqslant 3$, Z a subpolyhedron of X, and $f : (X, Y) \hookrightarrow (Q, \partial Q)$ a TOP embedding. For each $\varepsilon : Q \to (0, \infty)$ there exists $\delta : X \to (0, \infty)$ such that any PL embeddings $g, h : (X, Y) \hookrightarrow (Q, \partial Q)$, δ-close to f, are PL ε-ambient* *isotopic. Moreover if* $g|_Z = h|_Z$*, then the isotopy can be chosen fixing* $g(Z) = h(Z)$.

(b) *Let* X^n *be a compact smooth manifold,* Q^m *a smooth manifold,* $m > 3(n+1)/2$*,* Z *a closed subset of* X*, and* $f : X \hookrightarrow Q$ *a TOP embedding. For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any smooth embeddings* $g, h : X \hookrightarrow Q$*,* δ*-close to* f*, are smoothly* ε*-isotopic. Moreover if* $g = h$ *on the* δ*-neighborhood of* Z*, then the isotopy can be chosen fixing* $g(Z) = h(Z)$.

(b+) *In the* (b) *part, '*ε*-isotopic' can be replaced with '*ε*-ambient isotopic'.*

The (b) and (b+) parts are proved later in this section. We point out the following special case of Theorem 3.2.

Theorem 3.3.

(a) [17,62,21]. *Let* X^n *be a compact polyhedron,* Q^m *a PL manifold,* $m - n \geqslant 3$*, and* $f : X \hookrightarrow Q$ *a PL embedding. For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any PL embedding* $f' : X \hookrightarrow Q$*,* δ*-close to* f*, is PL* ε*-ambient isotopic to* f.

(b) [48, 2.1]. *Let* X^n *be a compact smooth manifold,* Q^m *a smooth manifold,* $m > 3(n+1)/2$*, and* $f : X \hookrightarrow Q$ *a smooth embedding. For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any smooth embedding* $f' : X \hookrightarrow Q$*,* δ*-close to* f*, is smoothly* ε*-ambient isotopic to* f.

Theorem 3.4.

(a) *Let* X^n *be a compact polyhedron,* Q^m *a PL manifold,* $m - n \geqslant 3$*. Any TOP isotopy* f_t *between PL embeddings* $f_0, f_1 : X \hookrightarrow Q$ *is* ε*-approximable, for each* $\varepsilon > 0$*, by a PL isotopy* g_t *between* f_0 *and* f_1*. Moreover if* f_t *fixes a subpolyhedron* Z *of* X*, then* g_t *can be chosen fixing* Z.

(b) [33], [48, 2.3]. *Let* X^n *be a compact smooth manifold,* Q^m *a smooth manifold and* $m > 3(n+1)/2$*. Any TOP isotopy* f_t *between smooth embeddings* $f_0, f_1 : X \hookrightarrow Q$ *is* ε*-approximable, for each* $\varepsilon > 0$*, by a smooth isotopy* g_t *between* f_0 *and* f_1*. Moreover if* f_t *fixes the* δ*-neighborhood of a closed subset* Z *of* X*, then* g_t *can be chosen fixing* Z.

Proof of 3.4(a). (Compare to [56], [64, proof of Theorem 3]; see Remark 3.8 for an alternative proof.) For each $t \in I$ let $U(t)$ denote an open neighborhood of t in I such that for each $s \in U(t)$ the embedding f_s is $\beta(t)$-close to f_t, where $2\beta(t) = \delta_{3.2(\mathrm{a})}$ is given by 3.2(a) for $\varepsilon_{3.2(\mathrm{a})} = \varepsilon/2$ and $f_{3.2(\mathrm{a})} = f_t$; we can assume $\beta(t) < \varepsilon/4$.

Since I is compact, it can be covered by a finite number k of open intervals $U_1, U_2, \ldots, U_k$, where $U_i = U(s_i)$ for some $s_i \in I$, $s_1 = 0$, $s_k = 1$ and $U_i \cap U_{i+1} \neq \emptyset$ for each $i = 1, \ldots, k-1$; let t_i be a point in $U_i \cap U_{i+1}$. By 3.1(a) for each $i = 1, \ldots, k-1$ there is a PL embedding $g_i : X \hookrightarrow Q$, agreeing with f_{t_i} on Z and such that $\mathrm{dist}(g_i, f_{t_i}) < \min(\beta(s_i), \beta(s_{i+1}))$. We put $g_0 = f_0$ and $g_k = f_1$. Then g_i is $2\beta(s_i)$-close to f_{s_i} and $2\beta(s_{i+1})$-close to $f_{s_{i+1}}$ for each $i = 0, \ldots, k$. By 3.2(a) for each $i = 0, \ldots, k-1$ the embeddings g_i and g_{i+1} are PL $\frac{1}{2}\varepsilon$-isotopic fixing Z. The stacked composition of these isotopies is the required isotopy, ε-close to f_t. □

Theorem 3.5.

(a) *Suppose that X^n is a compact polyhedron, Q^m a PL manifold, $m-n \geqslant 3$, Z a subpolyhedron of X, and $f: X \hookrightarrow Q$ a TOP embedding. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for any PL embedding $g: X \hookrightarrow Q$, δ-close to f, there is an ε-isotopy $f_t: X \to Q$ such that $f_0 = g$, $f_1 = f$ and such that f_t for $t < 1$ is a PL isotopy. Moreover, if $g|_Z = f|_Z$, then f_t fixes $g(Z) = f(Z)$.*
Furthermore, f_t is covered by an ε-homotopy $H_t: Q \to Q$ such that:

$H|_{Q\times[0,1)}$ *is a PL homeomorphism*;

$H_0 = \mathrm{id}_Q$ *and* $H_1 \circ g = f$;

if $g|_Z = f|_Z$, *then* H_t *fixes* $g(Z) = f(Z)$.

(b) *Suppose that X^n is a compact smooth manifold, Q^m a smooth manifold, $m > 3(n+1)/2$, Z a closed subset of X, and $f: X \hookrightarrow Q$ a TOP embedding. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for any smooth embedding $g: X \hookrightarrow Q$, δ-close to f, there is an ε-isotopy $f_t: X \hookrightarrow Q$ such that $f_0 = g$, $f_1 = f$ and such that f_t for $t < 1$ is a smooth isotopy. Moreover if $f = g$ on the δ-neighborhood of Z, then f_t can be chosen fixing Z.*

(b+) *In the* (b) *part it can be assumed that f_t is covered by an ε-homotopy $H_t: Q \to Q$ such that $H|_{Q\times[0,1)}$ is a diffeomorphism, $H_0 = \mathrm{id}_Q$, and $H_1 \circ g = f$.*

Theorem 3.5(a) is an immediate corollary of 3.1(a) and 3.2(a).

Proof of 3.2(b). Triangulate X and ambient isotop g onto a PL embedding $H \circ g$. Let Z' be a subpolyhedron of X such that $g|_{Z'} = h|_{Z'}$ and $Z \subset Z'$.

By 3.1(a) and 3.5(a), $H \circ h$ is TOP isotopic, by an arbitrarily small isotopy, fixing Z', to a PL embedding h'. Hence h and $H^{-1} \circ h'$ can be assumed TOP $\frac{1}{3}\varepsilon$-isotopic fixing Z'.

By 3.2(a), $H \circ g$ and h' can be assumed topologically (even PL) isotopic, by a sufficiently small isotopy, fixing Z'. Hence g and $H^{-1} \circ h'$ can be assumed TOP $\frac{\varepsilon}{3}$-isotopic fixing Z'.

Finally, by 3.4(b), the obtained TOP $\frac{2}{3}\varepsilon$-isotopy between g and h can be approximated by a smooth ε-isotopy fixing Z. □

Theorem 3.2(b+) follows immediately from 3.2(b) and 1.13(b).
Theorem 3.5(b) follows immediately from 3.1(b) and 3.2(b).
Theorem 3.5(b+) follows immediately from 3.1(b) and 3.2(b+).

Theorem 3.6. *Let X^n be a compact polyhedron, Q^m a PL manifold, $m-n \geqslant 3$.*

(a) *Suppose that $f: X \to Q$ is an embedding. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if an embedding $g: X \hookrightarrow Q$ is δ-close to f, then for any $\gamma > 0$ there is an ε-ambient isotopy, taking g onto an embedding, γ-close to f.*

(a′) *In addition, there is an ε-ambient isotopy, taking f onto an embedding, γ-close to g.*

(b) *Suppose that* $f : X \to Q$ *is a map and* $g : X \to Q$ *an embedding. For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if an embedding* $g' : X \hookrightarrow Q$, δ*-close to* g*, is taken onto* f *by a pseudo-isotopy* H'_t*, then* g *is taken onto* f *by a pseudo-isotopy* H_t, ε*-close to* H'_t.

Proof. (a) Let $2\delta = \delta_{3.5(\mathrm{a})}$ be given by 3.5(a) for $f_{3.5(\mathrm{a})} = f$ and $\varepsilon_{3.5(\mathrm{a})} = \frac{1}{2}\varepsilon$. In addition let $\delta' = \delta_{3.5(\mathrm{a})}$ be given by 3.5(a) for $f_{3.5(\mathrm{a})} = g$ and $\varepsilon_{3.5(\mathrm{a})} = \frac{1}{2}\varepsilon$. We can assume that $\delta' < \delta$. By 3.1(a) g is δ'-close to a PL embedding $h : X \hookrightarrow Q$. By 3.5(a) h can be taken by an $\frac{1}{2}\varepsilon$-pseudo-isotopy G_t onto g and by an $\frac{1}{2}\varepsilon$-pseudo-isotopy F_t onto f.

Let $U \subset Q \times I$ be the closed neighborhood of $G_1 \circ h(X) \times [0,1)$ in $Q \times I$ such that $U \cap Q \times 1 = G_1 \circ h(X) \times 1$. Then $G|_U$ is injective, and since U is compact, the map $G^{-1}|_U : U \to G^{-1}(U)$ is uniformly continuous. Hence for each $\beta > 0$ there is a number $t_0 < 1$ such that the embedding $h' = G_{t_0}^{-1} \circ g$ is β-close to $h = G_1^{-1} \circ g$. The map $G'_t = G_{t_0}^{-1} \circ G_{t_0(1-t)}$, $t \in I$, yields an $\frac{1}{2}\varepsilon$-ambient isotopy taking g onto h'. Finally, since F is uniformly continuous, the number β can be chosen so that for each $t \in I$ the embeddings $F_t \circ h'$, $F_t \circ h$ are $\frac{1}{2}\gamma$-close, while h' and $F_t \circ h'$ are clearly $\frac{1}{2}\varepsilon$-ambient isotopic. If $t < 1$ is such that $F_t \circ h$ and $F_1 \circ h = f$ are $\frac{1}{2}\gamma$-close, then f and $F_t \circ h'$ are γ-close, while $F_t \circ h'$ and g are ε-ambient isotopic.

(a$'$) Proceed as in the proof of (a) until F_t, G_t are constructed, and after that exchange their roles.

(b) We can assume that Q is compact, hence H'_t is uniformly continuous. For any fixed $t_0 < 1$ the map $(H'_t)^{-1}$, $t \in [0, t_0]$, is uniformly continuous, and so is the map $H'_{st} = H'_t \circ (H'_s)^{-1}$, $s \in [0, t_0]$, $t \in I$. For $k = 0, 1, \ldots$ let $\lambda_k > 0$ be such number that $\operatorname{dist}(H'_{st}(p), H'_{st}(q)) < \lambda \operatorname{dist}(p, q)$ whenever $p, q \in Q$, $s \in [0, 1 - 2^k]$, $t \in I$. Let ε_k, $k = 0, 1, \ldots$ be a sequence of reals such that

$$\sum_{k=0}^{\infty} \varepsilon_k \lambda_k < \varepsilon. \tag{$*$}$$

Let $\delta = \delta_{3.6(\mathrm{a})}$ be given by the (a$'$) part for $f_{3.6(\mathrm{a})} = g$ and $\varepsilon_{3.6(\mathrm{a})} = \varepsilon_0$. Let $\delta_k = \delta_{3.6(\mathrm{a})}$ be given by the (a) part for $f_{3.6(\mathrm{a})} = g'_k = H_{1-2^{-k}} \circ g'$ and $\varepsilon_{3.6(\mathrm{a})} = \varepsilon_k$, $k = 1, 2, \ldots$.

For $k = 0, 1, \ldots$ put $\gamma_k = \delta_{k+1}/\lambda_{k+1}$. Then for any embedding g_k, γ_k-close to $g'_k = H_{1-2^{-k}} \circ g'$, and for each $t \in [1 - 2^{-k}, 1 - 2^{-k-1}]$ the embedding $H'_{1-2^{-k},t} \circ g_k$ is δ_{k+1}-close to $H'_t \circ g'$. In particular, the embedding $\bar{g}_{k+1} = H'_{1-2^{-k},1-2^{-k-1}} \circ g_k$ is δ_{k+1}-close to g'_{k+1}, $k = 0, 1, \ldots$. The statement '$\bar{g}_k$ is δ_k-close to g'_k' holds also for $k = 0$ if we put $\bar{g}_0 = g$, $\delta_0 = \delta$.

Now by (a) and (a$'$) for $k = 0, 1, \ldots$ one can take an embedding $\bar{g}_k$, which is δ_k-close to g'_k, onto an embedding g_k, which is γ_k-close to g'_k, by an ε_k-ambient isotopy $G^k_t : Q \to Q$. Then the stacked composition of isotopies

$$G^0_t;\ H'_t,\ t \in [0, \tfrac{1}{2}];\quad G^1_t;\ H'_t,\ t \in [\tfrac{1}{2}, \tfrac{3}{4}];\quad G^2_t;\ \ldots$$

yields a pseudo-isotopy H_t (the possibility of continuous extension as $t \to 1$ is guaranteed by ($*$), cf. [46, Lemma 1]). By ($*$) H_t is ε-close to H'_t. Since $\delta_k \to 0$ as $k \to \infty$ and

$H_t \circ g = H'_{1-2^{-k},t} \circ g_k$ is δ_{k+1}-close to $H'_t \circ g'$, where $k = [-\log_2(1-t)]$, we obtain that $H_1 \circ g = H'_1 \circ g' = f$. $\square$

Theorem 3.7 [25, 4.1]. *For each $\varepsilon > 0$ and a positive integer n there exists $\delta > 0$ such that the following holds.*

Let X^n be a compact polyhedron and Y^{n-1} its subpolyhedron, $(Q^m, \partial Q)$ a PL manifold, $m - n \geqslant 3$, and $f : (X, Y) \times [-1, 1] \to (Q, \partial Q) \times \mathbb{R}$ a PL embedding such that $\Pi \circ f$ is δ-close to π. (Here $\Pi : Q \times \mathbb{R} \to \mathbb{R}$, $\pi : X \times [-1, 1] \to [-1, 1] \subset \mathbb{R}$ denote the projections.) Then there is a PL ε-ambient isotopy H_t with support in $Q \times [-\varepsilon, \varepsilon]$, taking f onto a PL embedding g such that $g^{-1}(Q \times J) = X \times (J \cap [-1, 1])$ for each $J = (-\infty, 0], 0, [0, +\infty)$.

Furthermore, for each $\gamma > 0$ given in advance it can be assumed that $P \circ H_t$ moves points less than γ, where $P : Q \times \mathbb{R} \to Q$ denotes the projection. Moreover, if $f^{-1}(\partial Q \times J) = Y \times J$ for each J as above, H_t can be chosen to fix $\partial Q \times \mathbb{R}$.

Theorem 3.7, called Slicing Lemma in [25,64], was one of the key steps in the proof of the (a) parts of Theorems 3.1 and 3.2. (In the statement [25, 4.1(3)] one should read '$(h_1 \circ g)^{-1}$' instead of 'g^{-1}'.) In view of an analogy between Lemmas 4.1 and 7.6 below, one can regard Theorem 3.7 as a geometric version of the Freudental Suspension Theorem. The proof of Theorem 3.7 in [25] is somewhat similar to the proof of the Penrose–Whitehead–Zeeman–Irwin Embedding Theorem, meanwhile Miller proves a generalization of Theorem 3.7 in [64] using his controlled version (see [62]) of sunny collapsing (see Section 5).

The following curious statement, not required in the rest of this paper, can be regarded as an alternative controlled version of the Concordance Implies Isotopy Theorem. Call a concordance $F : X \times I \hookrightarrow Q \times I$ *ε-level-disturbing* if for each $t \in I$ there is a neighborhood $U(t)$ of t in I such that $\Pi \circ F(X \times t) \subset U(t)$, $\Pi : Q \times I \to I$ being the projection, and $F|_{X \times U(t)}$ moves points less than ε. Notice that a 0-level-disturbing concordance is an isotopy, and the property of being ε-level-disturbing is independent on the choice of metric in I.

Theorem 3.7+. *For each $\varepsilon > 0$ and a positive integer n there exists $\delta > 0$ such that the following holds. Consider a compact polyhedron X^n, a PL manifold Q^m, $m - n \geqslant 3$, and PL embeddings $f, g : X \hookrightarrow Q$. Then any PL δ-level-disturbing concordance between f, g is ε-close to a PL isotopy between f, g.*

Proof. Let $F : X \times I \hookrightarrow Q \times I$ be the given concordance. Without loss of generality $F(X \times I) \subset Q \times I$. Let $\gamma = \delta_{1.13(a)}$ be given by 1.13(a) for $\varepsilon_{1.13(a)} = \frac{1}{2}\varepsilon$ and suppose $\gamma < \frac{1}{2}\varepsilon$. If F moves points less than $\frac{1}{2}\gamma$, then by 1.13(a) F is $(\frac{1}{2}\gamma + \frac{1}{2}\varepsilon)$-close to an isotopy between f, g.

Otherwise we can split I into pieces J_i such that $F|_{X \times J_i}$ moves points less than $\frac{1}{2}\gamma$ but more than $\frac{1}{4}\gamma$ for each i. Then for each i there is a point $x_i \in X$ such that $\operatorname{diam} P \circ F(x_i \times J_i) > \frac{1}{4}\gamma$, $P : Q \times I \to Q$ being the projection. Hence for each i and any positive number $\delta < \frac{1}{4}\gamma$ one can find numbers $d_{i1} < \cdots < d_{im}$, where $m = [\gamma/4\delta]$, such that $J_i = (d_{i1}, d_{im})$ and $\operatorname{dist}(P \circ F(x_i \times d_{ij}), P \circ F(x_i \times d_{i,j+1})) \geqslant \delta$.

Now suppose that F is δ-level-disturbing, then for each $t \in I$ there exists $U(t)$ such that $\Pi \circ F(X \times t) \subset U(t)$, and $F|_{X \times U(t)}$ moves points less than δ. Then for each $t \in I$, $U(t)$ contains at most one point d_{ij}. Choose a metric on I such that $\operatorname{dist}(d_{ij}, d_{i,j+1}) = \frac{\delta}{2}$, then $\operatorname{diam} U(t) < \delta$, while $\operatorname{diam} J_i = \frac{1}{2}\delta m > \frac{1}{10}\gamma$, provided $\delta \leqslant \frac{4}{5}\gamma$. Finally let $\delta = \delta_{3.7}$, which is obtained from Lemma 3.7 for $\varepsilon_{3.7} = \frac{1}{20}\gamma$.

Then by 3.7, F is $\frac{1}{20}\gamma$-ambient isotopic (hence $\frac{1}{20}\gamma$-close) to a concordance G between f, g such that $G(X \times J_i) \subset Q \times J_i$ for each i. Thus G splits into concordances $G|_{X \times J_i}$, each moving points less than $\frac{1}{20}\gamma + \frac{1}{2}\gamma + \frac{1}{20}\gamma$. By 1.13(a) each of them is $(\frac{12}{20}\gamma + \frac{1}{2}\varepsilon)$-close to an isotopy. Together these isotopies yield an isotopy $\Phi : X \times I \hookrightarrow Q \times I$, $(\frac{12}{20}\gamma + \frac{1}{2}\varepsilon)$-close to G, hence $(\frac{13}{20}\gamma + \frac{1}{2}\varepsilon)$-close (thus ε-close) to F. $\square$

Remark 3.8. Given a homotopy $H : X \times I \to Q \times I$, for each $\varepsilon > 0$ it is easy to find $\delta > 0$ (depending on H) such that any concordance, δ-close to H, is ε-level-disturbing. Taking into account Theorems 3.1(a) and 3.7+, we thus obtain an alternative proof of Theorem 3.4(a).

4. Proof of Theorem 1.6

The following lemma is a corollary of Theorem 3.7:

Theorem 4.1. *Let X^n be a finite simplicial complex, Q^m a combinatorial manifold, $m - n \geqslant 3$, $f : X \to Q$ a simplicial map and C a union of some top-dimensional dual cells of Q. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. Suppose that $g : X \hookrightarrow Q$ is a PL embedding, δ-close to f and such that $g^{-1}(B) = f^{-1}(B)$ for each dual cell B of C. Then g is PL ε-ambient isotopic, keeping C fixed, to a PL embedding $h : X \hookrightarrow Q$ such that $h^{-1}(B) = f^{-1}(B)$ for each dual cell B of Q.*

Suppose that $\dim f(X) = k$, and $\overline{Q \setminus C}$ consists of l top-dimensional dual cells. Denote by $4.1(k,l)$ the statement of Theorem 4.1 for k and l. Then $4.1(i,0)$ and $4.1(0,j)$ are trivial for any i, j. Assuming that $4.1(i,j)$ is proved for $i < k$ and arbitrary j, and for $i = k$ and $j < l$, let us prove $4.1(k,l)$.

Proof of 4.1(k,l). Choose any vertex v of Q outside C. Let $D = \operatorname{st}(v, Q')$ be its dual cell, and write $E = \overline{\partial D \setminus \partial C}$. Notice that the pair $(E, \partial E)$ is bi-collared in $(\overline{Q \setminus C}, \partial C)$. By Theorem 3.7, for any $\gamma > 0$ the number δ can be chosen so that g is PL γ-ambient isotopic, keeping C fixed, to a PL embedding $\varphi : X \hookrightarrow Q$ such that $\varphi^{-1}(E) = f^{-1}(E)$. It follows that, in addition, $\varphi^{-1}(D) = f^{-1}(D)$.

Pseudo-radial projection [74] yields a PL homeomorphism $\partial D \to \partial \operatorname{st}(v, Q)$ (in general the latter complex does not coincide with $\operatorname{lk}(v, Q)$), which takes the intersection of D with a simplex of Q onto a simplex of $\partial \operatorname{st}(v, Q)$ and a dual cell of Q, lying in ∂D, onto a dual cell of $\partial \operatorname{st}(v, Q)$. We apply $4.1(k-1, l')$ in ∂D, equipped with triangulation, inherited from $\partial\,\mathrm{st}(v, Q)$, where l' is the number of dual cells of Q in $C \cap \partial D$. Using collaring, we obtain that for each $\beta > 0$ we can choose $\gamma + \delta$ so that φ (which is $(\gamma + \delta)$-close to f), is PL β-ambient isotopic, keeping C fixed, to a PL embedding $\psi : X \hookrightarrow Q$ such that $\psi^{-1}(B) = f^{-1}(B)$ for each dual cell B of $C \cup D$.

By 4.1$(k, l-1)$, for any $\alpha > 0$ the number $\beta+\gamma+\delta$ can be chosen so that ψ (which is $(\beta+\gamma+\delta)$-close to f) is PL α-ambient isotopic, keeping $C \cup D$ fixed, to a PL embedding $h : X \hookrightarrow Q$ such that $h^{-1}(B) = f^{-1}(B)$ for each dual cell B of Q. Thus g is ε-ambient isotopic to h, keeping C fixed, provided $\alpha+\beta+\gamma < \varepsilon$. □

Lemma 4.2. *Let X^n be a finite simplicial complex, Q^m a combinatorial manifold, $m - n \geqslant 3$, and $f : X \to Q$ a simplicial map. If $h : X \hookrightarrow Q$ is a PL embedding such that $h^{-1}(B) = f^{-1}(B)$ for each dual cell B of Q, then h is taken onto f by a PL pseudo-isotopy $H_t : Q \to Q$ such that $H_t(B) = B$ for each dual cell B of Q.*

Proof. Put $H_0 = \mathrm{id}_Q$ and $H_t = \mathrm{id}$ outside $N = \mathrm{N}(f(X), Q)$. Let $A_1, \dots, A_m$ be the dual cells of N, except for those in ∂N, arranged in an order of increasing dimension. Assuming that H is defined on $A_j \times I$ for all $j < i$ (hence on $(\partial A_i) \times I$), extend it to $A_i \times I$ as follows.

Denote the cone point of A_i by a_i. Let R be a relative regular neighborhood in A_i of $\partial A_i \cup h(f^{-1}(A_i))$ modulo $h(f^{-1}(a_i))$, and put $P = \overline{A_i \setminus R}$. Then $h^{-1}(P) = f^{-1}(a_i)$, and we define $H_t(p) = t * a_i + (1-t) * p$ for each $p \in P$ (we use here the cone structure $a_i * \partial A_i$ on A_i).

The quotient space A_i/P is PL homeomorphic keeping ∂A_i fixed to $A_i = a_i * \partial A_i$, and $f^{-1}(A_i)/f^{-1}(a_i)$ is PL homeomorphic keeping $f^{-1}(\partial A_i)$ fixed to the cone $a_i * f^{-1}(\partial A_i)$. Denote these homeomorphisms by φ and ψ, respectively, and let

$$i : A_i \setminus P \hookrightarrow A_i/P, \quad j : f^{-1}(A_i) \setminus f^{-1}(a_i) \hookrightarrow f^{-1}(A_i)/f^{-1}(a_i)$$

be the natural inclusions. Let $h' : a_i * f^{-1}(\partial A_i) \hookrightarrow a_i * \partial A_i$ be the embedding defined by the identity on a_i and by $\varphi \circ i \circ h \circ j^{-1} \circ \psi^{-1}$ elsewhere. By the Lickorish Cone Unknotting Theorem [55] there is a PL homeomorphism $\lambda : a_i * \partial A_i \to a_i * \partial A_i$ keeping ∂A_i fixed and such that $\lambda \circ h'$ is the conical map $\mathrm{id}_{a_i} * h'|_{f^{-1}(\partial A_i)}$. Define an isotopy $\Lambda : A_i \times I \to A_i \times I$ by

$$\Lambda = \big(\mathrm{id}_{a_i \times 1} * (\lambda^{-1} \times \mathrm{id}_0 \cup \mathrm{id}_{\partial A_i \times I})\big) \circ (\lambda \times \mathrm{id}_I),$$

then $\Lambda \circ (h' \times \mathrm{id}_I)$ is the conical map $\mathrm{id}_{a_i \times 1} * (h' \times \mathrm{id}_0 \cup h'|_{f^{-1}(\partial A_i)} \times \mathrm{id}_I)$.

Extend $H' = H|_{\partial A_i \times I \cup A_i \times 0}$ conewise to obtain the map

$$\mathrm{id}_{a_i \times 1} * H' : (a_i \times 1) * (\partial A_i \times I \cup A_i \times 0) \to (a_i \times 1) * (\partial A_i \times I \cup A_i \times 0).$$

Finally, define H on $(A_i \setminus P) \times I$ by

$$H = \big(\mathrm{id}_{a_i \times 1} * H'\big) \circ \Lambda \circ (\varphi \times \mathrm{id}_I) \circ (i \times \mathrm{id}_I).$$

Clearly, H is well-defined, is PL, and H_1 takes h onto f.

Assuming that $H|_{A_j \times I}$ is level-preserving for each $j < i$, we see from the construction above that so is $H|_{A_i \times I}$. Assuming that H is a homeomorphism on $A_j \times [0, 1)$ for each $j < i$ and recalling that it is a homeomorphism on $Q \times 0$, we see that H is a homeomorphism on $A_i \times [0,1)$. $\square$

Proof of 1.6. (a) Triangulate X and Q so that f is simplicial and each dual cell of Q is of diameter $< \frac{1}{2}\varepsilon$. Let δ be less than $\delta_{4.1}$, which is given by Lemma 4.1 for $\varepsilon_{4.1} = \frac{1}{2}\varepsilon$, and apply Lemma 4.1 to obtain a PL $\frac{1}{2}\varepsilon$-isotopy, taking g onto a PL embedding $h : X \hookrightarrow Q$ such that $h^{-1}(B) = f^{-1}(B)$ for each dual cell B of Q. Finally, apply Lemma 4.2 to obtain a PL pseudo-isotopy $H_t : Q \to Q$ taking h onto f. Then H moves no point as much as the maximal diameter of a dual cell of Q, which, in turn, is less than $\frac{1}{2}\varepsilon$.

(b) By 3.6(b) and 3.1(a), we can assume that the δ-close to $i \circ f$ embedding $g : X \hookrightarrow Q$ is PL. Without loss of generality f is surjective, hence we can assume $\dim Y \leqslant n \leqslant m - 3$. Then by 3.1(a) and 3.5(a) there is a PL embedding $j : Y \hookrightarrow Q$ and a pseudo-isotopy H_t, taking j onto i. For any $\gamma > 0$ we can assume that H_t moves points less than γ and $j \circ f$ is γ-close to $i \circ f$.

The PL embedding g is $(\gamma + \delta)$-close to the PL map $j \circ f$, and one could attempt to apply Theorem 1.6(a) here. But this is impossible, for one cannot make γ as small as required keeping $j \circ f$ unchanged. The solution is to use uniform continuity (as in the proof of 3.6(a)).

Let $U \subset Q \times I$ be a closed neighborhood of $H_1 \circ j(Y) \times [0,1)$ in $Q \times I$ such that $U \cap Q \times 1 = H_1 \circ j(Y) \times 1$. Then $H|_U$ is injective, and since U is compact, the map $H^{-1}|_U : U \to H^{-1}(U)$ is uniformly continuous. Clearly, for each $t_0 < 1$ the number $\delta > 0$ can be chosen so that the image of the embedding $g \times t_0 : X \to Q \times t_0$ lies in U. Now $g \times t_0$ is $(1 - t_0 + \delta)$-close to $(i \circ f) \times 1$, therefore for each $\beta > 0$ the numbers t_0, δ can be chosen so that $g' = H_{t_0}^{-1} \circ g$ is β-close to $j \circ f = H_1^{-1} \circ (i \circ f)$. The map $G_t = H_{t_0}^{-1} \circ H_{t_0(1-t)}$, $t \in I$, yields a γ-ambient isotopy taking g onto g'.

By 1.6(a), for any $\alpha > 0$ the number β can be chosen so that g' is taken onto $j \circ f$ by an α-pseudo-isotopy $F_t : Q \to Q$. Then the 'diagonal' $(\alpha + \gamma)$-pseudo-isotopy $\Phi_t = H_t \circ F_t : Q \to Q$ takes g' onto $i \circ f$. Since g is γ-ambient isotopic to g', there is an ε-pseudo-isotopy taking g onto $i \circ f$, provided $2\gamma + \alpha < \varepsilon$. $\square$

5. Proof of Theorem 1.13

Definition. A subcomplex Y of a simplicial complex X is said to be *locally of codimension* $\geqslant k$ in X, if every n-simplex of Y faces some $(n+k)$-simplex of X [55]. We call

$$S(f) = \overline{\{x \in X \mid f^{-1}f(x) \neq x\}}$$

the *singular set* of a map $f : X \to Y$.

Lemma 5.1 [41,10]. *Let X^n be a compact polyhedron and Q^m a PL manifold, $m - n \geqslant 3$, and $p : X \times I \to X$, $P : Q \times I \to Q$ the projections. For any PL embedding $F : X \times I \hookrightarrow Q \times I$ and any $\varepsilon > 0$ there is a PL level-preserving ε-homeomorphism $H : Q \times I \to Q \times I$ such that $S(P \circ H \circ F)$ is locally of codimension $\geqslant 2$ in $X \times I$.*

Moreover, one can choose H so that $p|_{S(P\circ H\circ F)}$ is non-degenerate. Furthermore, the preimage of any point under $P\circ H\circ F$ contains at most $\varphi(n)=[(n+1)/3]+1$ points.

Definition. Let us think of the second factor of $Q\times I$ as of height (that is, a point (q_1,t_1) lies below a point (q_2,t_2) if $q_1=q_2$ and $t_1<t_2$). If $X\subset Q\times I$, let $\operatorname{sh}X$ denote a *shadow* of X, the set of points of $Q\times I$ lying below some point of X.

We say that a collapse $X\searrow Y$ in $Q\times I$ is a *simple sunny collapse*, if no point of $X\setminus Y$ lies in $\operatorname{sh}X$. A sequence of simple sunny collapses is called a *sunny collapse* [87]. Let us say that a sunny collapse is *m-complex*, if it consists of at most m simple sunny collapses. Repeating the same for $\overline{X\setminus Y}$ instead of $X\setminus Y$, we define a (simple/m-complex) *stable* sunny collapse [58].

Example 5.2 (*Compare to* [87, Remark on p. 510]). Let us illustrate the relation between sunny collapsing and unknotting. Evidently, I collapses onto 0. Let $F:I\to I^3$ be a PL embedding such that $F(i)\subset\operatorname{Int}(I^2\times i)$, $i=0,1$. It turns out that if a collapse $F(I)\searrow F(0)$ is sunny, F is unknotted. Indeed, define a PL isotopy $H_t:I\hookrightarrow I^3$ by $s\mapsto F(s)$ for $s\leqslant 1-t$ and by mapping $(1-t,1]$ linearly onto points lying above $F(1-t)$. Then $H_0=F$ and H_1 is linear. Clearly, H is locally flat, hence by [74] it extends to a PL ambient isotopy, which 'unknots' F.

Surprisingly, if $F:I\hookrightarrow I^3$ maps 0 into $\operatorname{Int}(I^2\times 1)$ and 1 into $\operatorname{Int}(I^2\times 0)$, then F can be knotted even if there is a sunny collapse $F(I)\searrow F(0)$. However, in all other cases of PL embeddings $F:(I,\partial I)\hookrightarrow(I^3,\partial I^3)$ existence of a sunny collapse $F(I)\searrow F(0)$ implies that F is unknotted. Indeed, in the case $F(1)\subset\overline{\partial I^3\setminus(I^2\times 0)}$ we use that $\operatorname{N}(0,I)$ is not overshadowed by I to shift $F(0)$ upwards into $(\partial I^2)\times 1$. Then we apply the above construction of H_t for $t\leqslant 1-\varepsilon$, where $\varepsilon>0$ is the minimal distance between vertices in $F(I)$. Now $H_{1-\varepsilon}$ consists of two linear pieces, hence is unknotted. To manage with the case $F(0)\subset\overline{\partial I^3\setminus(I^2\times 1)}$, notice that a collapse $F(I)\searrow F(0)$ is sunny, iff sunny is the analogous collapse $U\circ F\circ u(I)\searrow U\circ F\circ u(0)$, where $u:I\to I$ and $U:I^3\to I^3$ are defined by $t\mapsto 1-t$ and $(r,s,t)\mapsto(r,s,1-t)$, respectively.

Lemma 5.3 [41, Lemma 2]. *Let X be a simplicial complex, Q a combinatorial manifold, $p:X\times I\to X$ and $P:Q\times I\to Q$ simplicial projections. Let $G:X\times I\hookrightarrow Q\times I$ be a simplicial embedding satisfying the conclusion of* 5.1 *and such that $G(X\times 0)\subset Q\times 0$. Then there is a sunny collapse $G(X\times I)\searrow G(X\times 0)$ such that*

(i) $\operatorname{tr}Z\times I\subset\operatorname{N}(Z\times I,X\times I)$ *for any simplex Z of X.*

Speaking informally, the main idea of the proof of Lemma 5.3 was to use codimension 2 (that is, connectedness of $G(X\times I\setminus S)$) to have a simultaneous collapsing access to all the m-simplices of $G(S)$ successively for $m=n-1,\dots,0$, which enabled to collapse them in the order they overshadow each other. See [87, proof of Lemma 9] for a detailed proof of a similar statement.

Addendum to 5.3. *The sunny collapse $G(X\times I)\searrow G(X\times 0)$ can be chosen $\psi(n)=\frac{1}{6}(n+7)^2$-complex.*

Proof. Arrange the simplices of $K = G(S)$ in the following order. Assuming that the order is defined in the case $\dim K \leqslant m$, define it when $\dim K = m$, as follows. Let first go all the top-dimensional non-overshadowed simplices, then all the top-dimensional once overshadowed and so on, up to the top-dimensional simplices, overshadowed by $\varphi(m) - 1$ ones. After that put all at most $(m-1)$-dimensional simplices of K, arranged in the order given by the inductive assumption. The proof of Lemma 5.3 actually allows to collapse the simplices of K in any order of decreasing dimension, given in advance, particularly in the above. Clearly, the obtained collapse is sunny, and since $\psi(n) \geqslant \varphi(n-1) + \cdots + \varphi(0)$, it is $\psi(n)$-complex. $\square$

It turns out that any sunny collapse can be improved to a stable sunny one. We prove this by following the given collapse with a slight but precisely calculated lag.

Lemma 5.4. *Let Q be a combinatorial manifold, $P: Q \times I \to Q$ a simplicial projection and $K_0 \supset \cdots \supset K_N$ be a sequence of subcomplexes of Q such that* $\operatorname{sh} K_i \cap K_0 \subset K_{i+1}$ *for each $i \leqslant N$, where $K_{N+1} = \emptyset$, and suppose that $P|_{K_0}$ is non-degenerate. Then there is a sequence $K_0 = U_0 \supset \cdots \supset U_M = K_N$ of subpolyhedra of Q such that* $\operatorname{sh} U_j \cap U_0 \subset \operatorname{Int} U_{j+1}$ *for each $j \leqslant M$, where $U_{M+1} = \emptyset$ and*

(i) *$K_0 \searrow \cdots \searrow K_i \searrow \cdots \searrow K_N$ (simplicially) implies $U_0 \searrow \cdots \searrow U_j \searrow \cdots \searrow U_M$;*
(ii) *the trace of any simplex Z of K_0 under $U_0 \searrow U_M$ lies in that under $K_0 \searrow K_N$.*

Actually, in the application of Lemma 5.4 the hypothesis of (i) will be fulfilled; we allow it not to be fulfilled only to carry out induction in the proof of 5.4. The prototypes of Lemma 5.4 can be found in [40, proof of Proposition 5.1] and [58, Lemma 4.1]. To prove Lemma 5.4 we need a couple of preliminary observations.

Claim 5.5. *There is a second derived subdivision αK_0 of K_0 such that for any subcomplex Y of K_0, the inclusion* $\operatorname{sh} Y \cap K_0 \subset Y$ *implies*

$$\operatorname{sh} \mathrm{N}(Y, \alpha K_0) \cap K_0 \subset \operatorname{Int} \mathrm{N}(Y, \alpha K_0).$$

Proof. Let K_0' be the barycentrically derived subdivision of K_0 and construct a derived subdivision αK_0 of K_0' as follows. For each simplex A of K_0 define a map $f_A : A \to \mathbb{R}^1$ by $\partial A_j \mapsto -1$, $\widehat{A} \mapsto \varphi \circ \Pi(a)$ and extending linearly, where $\widehat{A}$ denotes the barycenter of A, $\Pi : Q \times I \to I$ denotes the projection, and φ maps $[0,1]$ linearly onto $[\frac{1}{100}, 1]$. Let $\mathcal{F} = (A_0 \supsetneq \cdots \supsetneq A_m)$ run over the flags of simplices in K_0 and let $B_i = B_i(\mathcal{F}) = \widehat{A}_i * \cdots * \widehat{A}_m$. Then B_0 runs over the simplices of K_0'. Define a derivation point of B_0 by $d(B_0) = \widehat{A}_0 * \widehat{B}_1 \cap f_{A_0}^{-1}(0)$, unless $m = 0$. The subdivision αK_0 is defined.

It is easy to see that $d(B_m) = B_m$ lies in a subcomplex Y of K_0 if and only if $d(B_0) * \cdots * d(B_m)$ (or, equivalently, $d(B_0)$) lies in $\mathrm{N}(Y, \alpha K_0)$. Notice that $\widehat{B}_0 \in \widehat{A}_0 * \widehat{B}_1$ and

$$f_{A_0}(\widehat{B}_0) = \frac{f_{A_0}(\widehat{A}_0) - m}{m+1} < 0$$

(unless $m = 0$), hence $\widehat{B}_0 \in \operatorname{Int}(d(B_0) * \widehat{B}_1)$. An induction on m implies that $\widehat{B}_0 \in \operatorname{Int}(d(B_0) * \cdots * d(B_m))$, and moreover, that if $x \in \operatorname{Int}(d(B_0) * \widehat{B}_1)$ then $x \in \operatorname{Int}(d(B_0) * \cdots * d(B_m))$.

Now suppose that $d = d(B_0)$ overshadows a point d^* of K_0. Then there are a flag $\mathcal{F}^* = (A_0^* \supsetneqq \cdots \supsetneqq A_m^*)$ and the simplices $B_i^* = B_i(\mathcal{F}^*)$, overshadowed respectively by a flag $\mathcal{F} = (A_0 \supsetneqq \cdots \supsetneqq A_m)$ and the simplices $B_i = B_i(\mathcal{F})$ (in the sense that each each point, say, of A_0^* is overshadowed by, or coincides with a point of A_0) and such that $d^* \in \widehat{A}_0^* * \widehat{B}_1^*$. If $d \in \mathrm{N}(Y, \alpha K_0)$ then $B_m \subset Y$, and since B_m overshadows, or equals, to B_m^*, we obtain $B_m^* \subset Y$. Consequently $d(B_0^*) * \cdots * d(B_0^*) \subset \mathrm{N}(Y, \alpha K_0)$. Finally, $\widehat{A}_0^* \neq \widehat{A}_0$, hence $f_{A_0^*}(\widehat{A}_0^*) < f_{A_0}(\widehat{A}_0)$, therefore $f_{A_0^*}(d^*) < f_{A_0}(d)$ and $f_{A_0^*}(d^*) < 0$. Thus by the above $d^* \in \operatorname{Int} \mathrm{N}(Y, \alpha K_0)$. □

Claim 5.6. *Let K be a simplicial complex and A its simplex.*

(a) *If $V \supset W$ in $\operatorname{lk}(A, K'')$ then $\mathrm{N}(\partial A, K'') \cup A * V \searrow \mathrm{N}(\partial A, K'') \cup A * W$. Moreover, $\operatorname{tr} Z \subset Z$ for any simplex Z of K.*

(b) *If $K \searrow L$ simplicially, then $K \searrow \mathrm{N}(L, K'')$. Moreover, the trace of any simplex Z of K under the second collapse lies in that under the first.*

Proof. (a) Suppose that a simplex A (strictly) faces a simplex B. Since a ball collapses onto its face, $\mathrm{N}(A, B'') \searrow \mathrm{N}(A, \partial B'') \cup \mathrm{N}(\partial A, B'')$. Applying this to B running over the simplices of K which are faced by A and meet $V \setminus W$, in order of decreasing dimension, we obtain the required collapse.

(b) For each elementary collapse $K_i \searrow K_{i+1}$ it suffices to prove that $N(K_i, K'') \searrow N(K_{i+1}, K'')$. Suppose that $K_i \searrow K_{i+1}$ goes from A_i along B_i. Apply the full collapse of (a) first to $A = A_i$ and then to $A = B_i$ to obtain a collapse $\mathrm{N}(K_i, K'') \searrow \mathrm{N}(K_{i+1}, K'') \cup K_i$. Finally, since ball collapses onto its face, $V \cup \mathrm{N}(W, K'') \searrow \mathrm{N}(W, K'')$. By the moreover part of (a) and since $\operatorname{tr} A_i \subset B_i$ under the last collapse, the trace of any simplex Z of K_0 under the obtained collapse $K \searrow \mathrm{N}(L, K'')$ lies in that under $K \searrow L$. □

Proof of 5.4. Assume that 5.4 is proved for $\dim K_0 < n$ and prove it for $\dim K_0 = n$. We will construct a descending sequence of subpolyhedra U_* (with several indices) in K_0, arranged lexicographically, so that the lexicographic unwrapping of indices yields the required sequence $U_0 \supset \cdots \supset U_M$.

Let αK_0 be the subdivision given by 5.5. Define $U_i = K_i \cup \mathrm{N}(K_{i+1}, \alpha K_0)$, $i \leqslant N$ and insert between them $U_{i,0} = \mathrm{N}(K_{i+1}, \alpha K_0)$, $i < N$. Then $U_0 = K_0$, $U_N = K_N$ and $U_i \setminus U_{i,0} \subset K_i \setminus K_{i+1}$. By 5.5 and since $\operatorname{sh} K_i \cap K_0 \subset K_{i+1} \subset \operatorname{Int} U_{i,0}$, we obtain $\operatorname{sh} U_i \cap U_0 \subset \operatorname{Int} U_{i,0}$. By 5.6(b), $K_i \searrow K_{i+1}$ implies $K_i \searrow \mathrm{N}(K_{i+1}, \alpha K_i)$, or, equivalently, $U_i \searrow U_{i,0}$, and the trace of any simplex Z of K under the last collapse lies in that under the first.

It remains to insert subpolyhedra in between $U_{i-1,0}$ and U_i. Let $A_1, \ldots, A_T$ be the simplices of $K_i \setminus K_{i+1}$, arranged in an order of decreasing dimension and put $B_j = \bigcup\{A_k \mid$

$k > j\}$. Define $U_{ij} = K_i \cup \mathrm{N}(K_{i+1} \cup B_j, \alpha K_0)$, $j \leqslant T$, then $U_{i,0}$ is same as above and $U_{i,T} = U_{i+1}$. By 5.5, $\operatorname{sh} U_{ij} \cap U_0 \subset \operatorname{Int} U_{ij}$ for each $j \leqslant T$. Unfortunately

$$\operatorname{sh} U_{ij} \cap U_0 \not\subset \operatorname{Int} U_{i,j+1}$$

in general, so we should insert yet more subpolyhedra. Put $L_l = \operatorname{lk}(A_j, K_l)$ for each $l \leqslant i$. Since $\operatorname{sh} K_l \cap K_0 \subset K_{l+1}$, we have that $\operatorname{sh} L_l \cap L_0 \subset L_{l+1}$. Now $\dim L_0 < n$ and we can apply the inductive hypothesis to obtain a sequence of subpolyhedra $L_0 = V_0 \supset \cdots \supset V_R = L_i$ such that $\operatorname{sh} V_k \cap V_0 \subset \operatorname{Int}_{V_0} V_{k+1}$, $k \leqslant R$, where $V_{R+1} = \emptyset$. Here 'Int_{V_0}' denotes topological interior in V_0. Since $P|_{K_0}$ is simplicial and non-degenerate,

$$\operatorname{sh}(A_j * V_k) \cap \operatorname{Int}(A_j * V_0) \subset \operatorname{Int}(A_j * V_{k+1}).$$

We put $W_k = A_j * V_k \cap \mathrm{N}(A_j, \alpha K_0)$ for $k \leqslant R$, then $W_0 \cup \mathrm{N}(\partial A_j, \alpha K_0) = \mathrm{N}(A_j, \alpha K_0)$ and $W_R \subset K_i$. Furthermore, $\operatorname{sh} W_k \cap W_0$ lies in

$$\begin{aligned}
&\operatorname{sh}(A_j * V_k) \cap \operatorname{sh}\mathrm{N}(A_j, \alpha K_0) \cap \mathrm{N}(A_j, \alpha K_0)\\
&\quad \subset \operatorname{sh}(A_j * V_k) \cap \operatorname{Int}\mathrm{N}(A_j, \alpha K_0)\\
&\quad \subset \operatorname{sh}(A_j * V_k) \cap \big(\operatorname{Int}(A_j * V_0) \cap \operatorname{Int}\mathrm{N}(A_j, \alpha K_0) \cup \operatorname{Int}\mathrm{N}(\partial A_j, \alpha K_0)\big)\\
&\quad \subset \operatorname{Int}(A_j * V_{k+1}) \cap \operatorname{Int}\mathrm{N}(A_j, \alpha K_0) \cup \operatorname{Int}\mathrm{N}(\partial A_j, \alpha K_0)\\
&\quad = \operatorname{Int}\big(W_{k+1} \cup \mathrm{N}(\partial A_j, \alpha K_0)\big).
\end{aligned}$$

Finally, define $U_{ijk} = U_{i,j+1} \cup W_k$, $k \leqslant R$. Then by the above $U_{ij,0} = U_{ij}$ and $U_{ij,R} = U_{i,j+1}$, while $\operatorname{sh} U_{ijk} \cap U_0 \subset \operatorname{Int} U_{ij,k+1}$. By 5.6(a) U_{ijk} collapses onto $U_{ij,k+1}$ for all $k < R$ and $\operatorname{tr} Z \subset Z$ under this collapse for any simplex Z of K_0. $\square$

Addendum to 5.4. *M can be chosen equal to $\xi(N, n) = N^{n+1} n!$.*

Proof. Prove this by induction on n. Clearly, we can choose $M = N$ if $n = 0$. Since M originally depends on an arbitrarily great number T, we should redefine the subpolyhedra U_* so that it does not. Notice that since $\operatorname{sh} A_j \cap K_0 \subset K_{i+1}$ for each simplex A_j of $K_i \setminus K_{i+1}$, by 5.5, $\operatorname{sh}\mathrm{N}(A_j, \alpha K_0) \cap K_0 \subset \operatorname{Int}\mathrm{N}(K_{i+1} \cup A_j, \alpha K_0)$. Hence

$$\operatorname{sh} W_k \cap K_0 \subset \operatorname{Int}\big(W_{k+1} \cup \mathrm{N}(K_{i+1} \cup \partial A_j, \alpha K_0)\big),$$

where $W_k = W_k(A_j)$ is defined for each simplex A_j of $K_i \setminus K_{i+1}$ as in the proof of 5.4, $k \leqslant R$ (by the inductive hypothesis we can choose $R = \xi(N, n-1)$ to be the same for all A_j). We redefine the subpolyhedra U_* by

$$\begin{aligned}
U_{ij} &= K_i \cup \mathrm{N}\big(K_{i+1} \cup (K_i \setminus K_{i+1})^{(n-j)}, \alpha K_0\big), && j \leqslant n,\\
U_{ijk} &= U_{i,j+1} \cup \big\{W_k(A_l) \mid \dim A_l = n - j\big\}, && k \leqslant R.
\end{aligned}$$

Then by the above $\operatorname{sh} U_{ijk} \cap U_0 \subset U_{ij,k+1}$ and the statement is fulfilled for the new sequence of subpolyhedra U_*, while there are only $M = NnR$ of them. Hence we can choose $\xi(N, n) = Nn\xi(N, n-1) = N^{n+1} n!$. $\square$

Proof of 1.13(a). Let $F : X \times I \to Q \times I$ be the given concordance between $F_0 = f$ and $F_1 = g$. Write $G = H \circ F$ and $S = S(P \circ G)$. Let $\zeta(n) = \xi(\psi(n), n)$ and $\delta =$

$\varepsilon/(5\zeta(n)+1)$. Subdivide $Q \times I$ and $X \times I$ so that G, p and P are simplicial and $\gamma = \operatorname{mesh}(Q \times I)$ is less than $\min(\delta, \frac{1}{3}\delta_{3.3(a)})$, where $\delta_{3.3(a)}$ is obtained from Theorem 3.3(a) for $f_{3.3(a)} = f$ and $\varepsilon_{3.3(a)} = \delta$.

Apply Lemma 5.4 to the sunny collapse of Lemma 5.3 to obtain a stable sunny collapse $G(X \times I) = U_0 \searrow \cdots \searrow U_M = G(X \times 0)$. By Addenda, it consists of at most $\zeta(n)$ simple stable sunny collapses. Also, $\operatorname{tr} G(Z \times 1) \subset N_\gamma(G(Z \times I))$ for any simplex Z of X. Since F is a δ-concordance and H is arbitrarily, say, δ-close to the identity, G is a 2δ-concordance. Hence $\operatorname{tr} G(Z \times 1) \subset \mathrm{N}_{2\delta+\gamma}((G_0 \times \mathrm{id}_I)(Z \times I))$.

By the simple stable condition, the projection $P : Q \times I \to Q$, restricted to $\overline{U_i \setminus U_{i+1}} \cup \operatorname{im}_i G(X \times 1)$, where im_i denotes the image under the first i collapses, is a homeomorphism for each $i < M$. Notice that

$$\overline{U_i \setminus U_{i+1}} \cup \operatorname{im}_i G(X \times 1) \subset U_i \setminus (\operatorname{Int} U_{i+1} \setminus X \times 1)$$

(the inclusion follows by an induction on i). Since $\overline{U_i \setminus U_{i+1}} \cup \operatorname{im}_i G(X \times 1)$ collapses onto $\operatorname{im}_{i+1} G(X \times 1)$, there is a sequence of at most $\zeta(n)$ collapses

$$P\big(\overline{U_i \setminus U_{i+1}} \cup \operatorname{im}_i G(X \times 1)\big) \searrow P\big(\operatorname{im}_{i+1} G(X \times 1)\big), \tag{$*$}$$

each of diameter at most $2\delta + \gamma < 3\delta$. Subdivide Q so that these collapses are simplicial. Our next goal is to obtain a sequence of isotopies, using the following

Lemma 5.7 [62, Proposition 1], [64, proof of Theorem 11]. *Let Q be a combinatorial manifold, V and W be its subcomplexes such that V collapses simplicially to W. Then there is a PL ambient isotopy H_t of Q, $H_0 = \mathrm{id}_Q$ such that for any subcomplex Z of V and arbitrary $t \in I$*

(i) $H_1\mathrm{N}(V, Q'') = \mathrm{N}(W, Q'')$;
(ii) *H_t is the identity outside* $\mathrm{N}(\mathrm{N}(\text{vertices in } (V \setminus W)', Q''), Q''')$;
(iii) $H_1\mathrm{N}(Z, Q'') \subset \mathrm{N}(\operatorname{im} Z, Q'')$;
(iv) $H_t\mathrm{N}(\mathrm{N}(Z, Q''), Q''') \subset \mathrm{N}(\mathrm{N}(\operatorname{tr} Z, Q''), Q''')$.

Addendum to 5.7 [62, Corollary 2]. *If the diameter of the collapse is less than α and* $\operatorname{mesh} Q < \gamma$*, then H moves points less than $\alpha + 2\gamma$.*

Proof of 1.13(a) (*continued*). By 5.3, 5.4 and 5.7(iii), there is a sequence of ambient isotopies h_t^i of Q such that for any simplex Z of X

$$h_1^i\mathrm{N}\big(P(\operatorname{im}_i G(Z \times 1)), Q''\big) \subset \mathrm{N}\big(P(\operatorname{im}_{i+1} G(Z \times 1)), Q''\big).$$

Let h_t be the stacked composition of $h_t^0, \dots, h_t^M$. Then by the addendum to 5.7 h_t is a composition of at most $\zeta(n)$ of 5δ-ambient isotopies. An induction on i implies

$$h_1\mathrm{N}\big(P \circ G(Z \times 1), Q''\big) \subset \mathrm{N}\big(P(\operatorname{im}_M G(Z \times 1)), Q''\big).$$

By 5.1 the homeomorphism H can be chosen arbitrarily, say, $\frac{1}{4}\gamma$-close to the identity. By 5.3(i) and 5.4(ii), $\operatorname{im}_M G(Z \times 1) \subset \mathrm{N}(G(Z \times 0), G(X \times 0))$. Thus,

$$\begin{aligned} h_1 \circ P \circ F(Z \times 1) &\subset h_1 \mathrm{N}\bigl(P \circ G(Z \times 1), Q''\bigr) \\ &\subset \mathrm{N}\bigl(G\bigl(\mathrm{N}(Z \times 0, X \times 0)\bigr), Q''\bigr) \\ &\subset \mathrm{N}_{2\gamma}\bigl(F(Z \times 0)\bigr), \end{aligned}$$

that is, $h_1 \circ P \circ F(a \times 1) \subset \mathrm{N}_{3\gamma}(F(a \times 0))$ for any $a \in X$. Recalling that $f = F_0$ and $g = P \circ F_1$ are the given embeddings, we obtain that $h_1 \circ g$ is 3γ-close to f.

Finally, we use 3.3(a) to obtain a δ-ambient isotopy φ_t, taking $h_1 \circ g$ onto f. Since $(5\zeta(n) + 1)\delta = \varepsilon$, the stacked composition of h_t and φ_t is an ε-ambient isotopy, taking g onto f. □

In the proof of 1.13(b) we will need the following observation.

Lemma 5.8. *For each positive integer n there is a number $\rho(n)$ such that the following holds. Let X^n be a compact polyhedron, Q^m a PL manifold, $m - n \geqslant 3$, and $f, g : X \hookrightarrow Q$ two PL δ-concordant embeddings.*

Given $\beta > 0$, there are PL ambient isotopies $H_t^1, \dots, H_t^{\rho(n)}$ such that for each $i = 1, \dots, \rho(n)$, $H_0^i = \mathrm{id}_Q$ and the isotopy H_t^i has support in the disjoint union of sets of diameter $< 7\delta$, and the composition $H_1^{\rho(n)} \circ \dots \circ H_1^1 \circ g$ is β-close to f.

Proof. This is clear from the proof of 1.13(a), provided the following modification is made. (We use the notation from the proof of 1.13(a).)

We can divide each ith collapse $(*)$, $i = 1, \dots, \zeta(n)$, which we denote for simplicity by $K_i \searrow L_i$, into $n + 1$ collapses $K_i \searrow K_{1,i} \cup L_i \searrow \dots \searrow K_{n,i} \cup L_i \searrow L_i$, where $K_{ij} = \mathrm{tr}_{K_i \searrow L_i} P(\mathrm{im}_i\, G(X \times 1)^{(n-j)})$. For each $i = 1, \dots, \zeta(n)$, $j = 0, \dots, n$, the set $K_{ij} \setminus K_{i,j+1}$ is the disjoint union of the sets $T_{i,Z} \setminus K_{i,j+1}$, where $T_{i,Z}$ denotes $\mathrm{tr}_{K_i \searrow L_i} P(\mathrm{im}_i\, G(Z \times 1))$ and Z runs over the $(n - j)$-simplices of X.

By 5.3(i) and 5.4(ii), each $T_{i,Z}$ is of diameter at most $\delta + 4\gamma$. Now the sets $\mathrm{N}(\mathrm{N}(\text{vertices in } (T_{i,Z} \setminus K_{i,j+1})', Q''), Q''')$ are each of diameter at most $\delta + 6\gamma < 7\delta$, and are disjoint for distinct $(n - j)$-simplices Z of X. Hence applying Lemma 5.7 for each $i = 1, \dots, \zeta(n)$, $j = 0, \dots, n$ to the collapse $K_{ij} \cup L_i \searrow K_{i,j+1} \cup L_i$, we obtain a PL ambient isotopy $H_t^{(n+1)(i-1)+j+1}$ with support in the disjoint union of the sets each of diameter at most 7δ. As in the proof of 1.13(a) it follows that $H_1^{(n+1)\zeta(n)} \circ \dots \circ H_1^1 \circ g$ is 3γ-close to f. Finally, the statement follows if we put $\rho(n) = (n + 1)\zeta(n)$ and $\gamma = \operatorname{mesh} Q < \frac{\beta}{3}$. □

Proof of 1.13(b). Put $\delta = \varepsilon/(29\rho(n) + 2)$. Let $\beta = \min(\delta, \frac{1}{3}\delta_{3.3(\mathrm{b})})$, where $\delta_{3.3(\mathrm{b})}$ is obtained from 3.3(b) for $f_{3.3(\mathrm{b})} = f$ and $\varepsilon_{3.3(\mathrm{b})} = \delta$. By 3.1(a) and 3.5(a), f is topologically β-isotopic to a PL embedding f_{PL}. Let $\alpha = \min(\delta, \frac{1}{2}\delta_{3.3(\mathrm{b})})$, where $\delta_{3.3(\mathrm{b})}$ is obtained from 3.3(b) for $f_{3.3(\mathrm{b})} = g$ and $\varepsilon_{3.3(\mathrm{b})} = \delta$. By 3.1(a) and 3.5(a), g is topologically α-isotopic to a PL embedding g_{PL}. Now g_{PL} is TOP $(\alpha + \beta + \delta)$-concordant to f_{PL}. Hence by (the relative case of) 3.1(a) the embeddings f_{PL} and g_{PL} are PL 4δ-concordant.

Apply Lemma 5.8 to obtain a sequence of PL ambient isotopies $H_t^1, \dots, H_t^{\rho(n)}$ such that for each $i = 1, \dots, \rho(n)$ the isotopy H_t^i has support in the disjoint union of sets of diameter $< 28\delta$, and the composition $H_1^{\rho(n)} \circ \cdots \circ H_1^1 \circ f_{\mathrm{PL}}$ is β-close to g_{PL}. Each H_t^i has a compact support, hence is uniformly continuous. Let $\gamma_{\rho(n)} = \beta$, and assuming that γ_i is defined, define γ_{i-1} to be a number such that under H_1^i any γ_{i-1}-close points are thrown into $\frac{1}{2}\gamma_i$-close points.

Put $\gamma = \min(\gamma_0, \alpha)$ and γ-approximate g_{PL} by a smooth embedding $g_0 : X \hookrightarrow Q$. By 3.3(b), g_0 is smoothly δ-ambient isotopic to g. Now g_0 is topologically isotopic to the embedding $H_1^1 \circ g_0$ by means of the isotopy $H_t^1 \circ g_0$, supported by the disjoint union U of sets of diameter $< 28\delta$. The embedding $H_1^1 \circ g_0$ is smooth outside U. Hence by (the relative case of) 3.1(b) and by (the relative case of) 3.5(b) this embedding is TOP $\frac{1}{2}\gamma_1$-isotopic, fixing the exterior of an arbitrarily small neighborhood U' of U, to a smooth embedding $g_1 : X \hookrightarrow Q$. The embedding g_1 is γ_1-close to $H_1^1 \circ g_{\mathrm{PL}}$, therefore by (the relative case of) 3.4(b) g_1 is smoothly isotopic to g_0 by an isotopy g_t, $t \in I$, fixing the exterior of an arbitrarily small neighborhood U'' of U'. Hence g_t extends to a smooth ambient isotopy $G_t : Q \to Q$ with support in arbitrarily small neighborhood U''' of U'', such that $G_0 = \mathrm{id}_Q$ and $G_1 \circ g_0 = g_1$. We can assume that U''' is the disjoint union of sets of diameter $< 29\delta$. Consequently g_0 is smoothly 29δ-ambient isotopic to g_1.

Repeating the same construction for $i = 2, 3, \ldots, \rho(n)$, we obtain a sequence of smooth embeddings $g_2, \ldots, g_{\rho(n)} : X \hookrightarrow Q$ such that g_i and g_{i+1} are smoothly 29δ-ambient isotopic for each $i = 0, \ldots, \rho(n) - 1$ and such that $g_{\rho(n)+1}$ is β-close to $H_1^{\rho(n)} \circ \cdots \circ H_1^1 \circ g_{\mathrm{PL}}$. Therefore $g_{\rho(n)+1}$ is 3β-close to f. Hence by 3.3(b) $g_{\rho(n)+1}$ and f are smoothly δ-ambient isotopic. Thus g is smoothly δ-ambient isotopic to g_0, which, in turn, is smoothly $29\delta\rho(n)$-ambient isotopic to $g_{\rho(n)}$, which is smoothly δ-ambient isotopic to f. $\square$

6. Proofs of Theorems 1.12 and 1.16

Proof of 1.12. We proceed with the first and the 'moreover' parts simultaneously (in the first part, let $\varepsilon > 0$ be any number). Let $F : X \times I \to Q \times I$ be the given (PL) pseudo-concordance. The proof splits into two cases.

PL case. (Compare to [74, proof of Lemma 4.23 on level-preserving collars].) Fix some triangulations of $X \times I$, $Q \times I$ such that mesh $Q < \varepsilon/2$ and F and the projections $p : X \times I \to X$, $P : Q \times I \to Q$ are simplicial. Let $(X \times I)'$, $(Q \times I)'$ denote derived subdivisions of $X \times I$, $Q \times I$ which project simplicially onto the barycentrically derived subdivisions X', Q' of X, Q. For each simplex A of $X \times I$ (respectively $Q \times I$), we denote by d_A its derivation point in $(X \times I)'$ (respectively in $(Q \times I)'$). Let $\gamma > 0$ be so small that no vertex of $(X \times I)'$ lies in $X \times (1-\gamma, 1)$ and no vertex of $(Q \times I)'$ lies in $Q \times (1-\gamma, 1)$. Then for each simplex A of $X \times I$ (respectively $Q \times I$) meeting $X \times 1$ (respectively $Q \times 1$) in a simplex B, the join $d_A * d_B$ meets $X \times \{1-\gamma\}$ (respectively $Q \times \{1-\gamma\}$) precisely in one point, which we denote by d_A^+.

We define a new PL pseudo-concordance $F_+ : X \times I \hookrightarrow Q \times I$ as follows. Put $F_+|_A = F|_A$ for any simplex A not meeting $X \times \{1-\gamma\}$. Let $A_1, \ldots, A_M$ be the simplices of $X \times I$ meeting $X \times \{1-\gamma\}$, arranged in some order of increasing dimension. Assuming that $F_+|_{\partial A_i}$ is defined, define $F_+|_{A_i}$ by $d^+_{A_i} \mapsto d^+_{F(A_i)}$ and extending linearly. Then $F_+^{-1}(Q \times \{1-\gamma\}) = X \times \{1-\gamma\}$ (moreover, $F_+|_{X\times[1-\gamma,1]}$ is level-preserving, but we do not use this fact). Also, $p \circ F_+^{-1}(B \times \{1-\gamma\}) = p \circ F_+^{-1}(B \times 1) = f^{-1}(B)$ for each dual cell B of Q. Let h denote the unique embedding $X \hookrightarrow Q$ such that $h \times \{1-\gamma\} = F_+|_{X\times\{1-\gamma\}}$. By Lemma 4.2, h is PL $\frac{1}{2}\varepsilon$-pseudo-isotopic to f, and this completes the proof of the first part.

To prove the 'moreover' part, notice that $F_+|_{X\times[0,1-\gamma]}$ yields a PL δ-concordance between g, h. By 1.13(a) δ can be chosen so that g is PL $\frac{1}{2}\varepsilon$-ambient isotopic to h.

TOP case. In the 'moreover' part we can assume (by 3.6(b), 3.5(a) and 3.1(a)) that the embedding $g : X \hookrightarrow Q$ is PL. By the non-compact relative case of Theorem 3.1(a) the embedding $F|_{X\times[0,1)}$ can be assumed PL.

Put $t_0 = t_1 = 0$. Assuming t_i, $i > 1$, to be already defined, put

$$t_{i+1} = \begin{cases} \sup \Pi \circ F(X \times [0, \frac{1}{2}(1-t_i)]) & \text{if } i \text{ is even,} \\ \sup \pi \circ F^{-1}(Q \times [0, \frac{1}{2}(1-t_i)]) & \text{if } i \text{ is odd,} \end{cases}$$

where $\pi : X \times I \to I$, $\Pi : Q \times I \to I$ denote the projections. Clearly, $t_i < t_{i+1} < 1$ for all $i = 0, 1, \ldots$. The main property of t_i's is: for even i

$$F\big(X \times [t_{i-1}, t_{i+1}]\big) \subset Q \times [t_{i-2}, t_{i+2}].$$

Let $4\gamma = \delta_{3.7}$ be given by Theorem 3.7 for $\varepsilon_{3.7} = \frac{1}{2}$. Define a PL homeomorphism $\lambda : [0,1) \to [0,+\infty)$ by mapping $t_i \mapsto \gamma(i-1)$ for all odd i, an extending linearly. Analogously, define a PL homeomorphism $\mu : [0,1) \to [0,+\infty)$ by mapping $t_i \mapsto \gamma i$ for all even i, an extending linearly. Let

$$G = \big(\lambda^{-1} \times \mathrm{id}_X\big) \circ F \circ (\mu \times \mathrm{id}_Q) : X \times [0,+\infty) \to Q \times [0,+\infty).$$

We obtain that $G(X \times [\gamma(i-2), \gamma i]) \subset Q \times [\gamma(i-2), \gamma(i+2)]$ for each even $i > 0$. Then by Theorem 3.7 there exists an ambient isotopy taking G onto an embedding $G_+ : X \times [0,+\infty) \to Q \times [0,+\infty)$ such that $G_+^{-1}(Q \times i) = X \times i$ for all $i = 1, 2, \ldots$ (in the 'moreover' part for $i = 0$ in addition).

Let us assume i to run over the positive integers in the proof of the first part, and over the nonnegative integers in the 'moreover' part. Let $\alpha_i = \delta_{1.13(\mathrm{a})}$ be given by Theorem 1.13(a) for $\varepsilon_{1.13(\mathrm{a})} = \varepsilon/2^{i+1}$. (In the 'moreover' part we put $\delta = \alpha_0/2$ in addition.) Since F is continuous, we can choose a sequence of integers n_i such that $P \circ G|_{X\times[n_i,+\infty)}$ is $\frac{1}{2}\alpha_i$-close to $f \times \mathrm{id}_{[n_i,+\infty)}$, where $P : Q \times \mathbb{R}^1 \to Q$ denotes the projection. (In the 'moreover' part the hypothesis allows to take $n_0 = 0$.) By the furthermore part of 3.7 we can assume without loss of generality that $P \circ G_+|_{X\times[n_i,+\infty)}$ is $\frac{1}{2}\alpha_i$-close to $P \circ G|_{X\times[n_i,+\infty)}$ for each i. Therefore $G_+|_{X\times[n_i,n_{i+1}]}$ is an α_i-concordance. Hence by 1.13(a) for each i the embeddings $f_i = P \circ G_+|_{X\times n_i}$ and f_{i+1} are PL $\frac{\varepsilon}{2^{i+1}}$-ambient isotopic. It follows (cf. [46, Lemma 1]) that f_1 (f_0 in the 'moreover' part) can be taken onto f by an ε-pseudo-isotopy. $\square$

Proof of 1.16. (a) We prove PL and DIFF cases simultaneously. Suppose that an embedding $g' : X \hookrightarrow Q$ is taken onto f by a pseudo-isotopy $H'_t : Q \to Q$. By 3.1 and 3.5(a) there is a PL (smooth) embedding $g : X \hookrightarrow Q$, which is taken onto g' by a pseudo-isotopy $G_t : Q \to Q$. Then the 'diagonal' pseudo-isotopy $H_t = H'_t \circ G_t$ takes g onto f.

(b) The PL case was actually proved in the above proof of 1.12, TOP case. Or it can be proved analogously to the below proof of the DIFF case:

DIFF case. Let $H'_t : Q \to Q$ be the given pseudo-isotopy and put $h_t = H'_t \circ g$, where $g : X \hookrightarrow Q$ is the given smooth embedding. Let α_i, $i = 1, 2, \ldots,$ be a monotonely decreasing sequence of reals (defined below) and let t_i, $i = 1, 2, \ldots,$ be such that h_{t_i} is α_i-close to f. By 3.1(b) and 3.5(b), h_{t_i} is TOP α_i-isotopic to a smooth embedding g_i, $i = 1, 2, \ldots$. Therefore g_i and g_{i+1} are TOP $3\alpha_i$-isotopic for each $i = 1, 2, \ldots,$ and g is TOP isotopic to g_1. By the relative case of 3.1(b) and by 1.13(b), α_i can be chosen so that g_i and g_{i+1} are smoothly 2^{-i-1}-ambient isotopic, while g and g_1 are smoothly ambient isotopic. It follows that g can be taken onto f by a pseudo-isotopy H_t which is smooth whenever $t \in [0, 1)$. □

Remark 6.1. We point out one useful observation following from Theorem 1.16 and the relative version of Theorem 1.12 (which is proved analogously to 1.12, using the relative version of 1.13(a)). Suppose that $m - n \geqslant 3$, X^n is a compact polyhedron, Q^m a PL manifold, and $f : X \to Q$ a map which PL embeds a subpolyhedron Z of X. It turns out that if f is isotopically realizable, then there is an embedding $g : X \hookrightarrow Q$, agreeing with f on Z, and a pseudo-isotopy $H_t : Q \to Q$, taking g onto f and keeping $g(Z)$ fixed.

Indeed, by 1.16 there is a PL embedding g' and a pseudo-isotopy H'_t, PL whenever $t \in [0, 1)$ and taking g' onto f.

From the non-compact case of Theorem 3.2(a) it follows that $H'_t \circ g'|_Z$, regarded as a level-preserving embedding $Z \times I \to Q \times I$, can be topologically isotoped (not level-preserving in general) onto the embedding $f|_Z \times \mathrm{id}_I$. Thus we obtain a pseudo-concordance from an embedding to f, keeping Z fixed, and it remains to apply the relative version of 1.12, TOP case.

Notice that if Z is a manifold or $m - n \geqslant 4$, instead of Theorem 3.2(a) one could apply its parametric version [64,53,86], thus making the application of the relative version of 1.12 no longer necessary.

7. Proof of Criterion 1.7

The following is the controlled version of [37, Theorem 1(R), p. 21].

Theorem 7.1. *For each $\varepsilon > 0$ and a positive integer n there exists $\delta > 0$ such that the following holds. Let X^n be a compact polyhedron, $(Q^m, \partial Q)$ a PL manifold, $m \geqslant 3(n + 1)/2$, and $f : X \to Q$ a PL map which embeds a subpolyhedron Z of X.*

If f^2 is equivariantly δ-homotopic to an isovariant map H_1 by a homotopy $H : X^2 \times I \to Q^2$ which is isovariant on $Z^*_f \times I$, then f is PL ε-homotopic keeping Z fixed to an embedding $g : X \hookrightarrow Q$ such that g^2 is isovariantly ε-homotopic to H_1.

If $f : X \to Q$ is a map into a manifold and $Z \subset X$, we denote by Z^*_f the set $Z^2 \cup (Z \cap f^{-1}\partial Q) \times X \cup X \times (Z \cap f^{-1}\partial Q)$.

Compare 7.1 to [70, pre-limit version of 1.2].

Theorem 7.1 follows from 7.2 and 7.4 below, where $U_{7.4} = X^2$ and $U_{7.2} = V_{7.4}$.

Theorem 7.2. *For each $\varepsilon > 0$ and a positive integer n there exists $\delta > 0$ such that the following holds. Let X^n be a compact polyhedron, $(Q^m, \partial Q)$ a PL manifold, $m \geqslant 3(n+1)/2$ and $f : X \looparrowright Q$ a PL immersion which embeds a subpolyhedron Z.*

Suppose that f^2 is equivariantly δ-homotopic to an isovariant map H_1 by a homotopy $H : X^2 \times I \to Q^2$ which is isovariant on $(U \cup Z^*_f) \times I$, where U is some neighborhood of Δ_X in X^2.

Then f is PL ε-regularly homotopic, keeping Z fixed, to an embedding $g : X \hookrightarrow Q$ such that g^2 is isovariantly ε-homotopic to H_1.

Proof. This is a modification of [37, proof of Theorem 3] in the spirit of [70]. Familiarity with [37, proof of Proposition 9] is assumed.

Fix a triangulation of X such that f is simplicial, Z is a subcomplex, and the diameter of each simplex is less than δ. Arrange the simplices of X so that first go that in $Z \cap f^{-1}\partial Q$ in some order of increasing dimension, then the rest in Z in like order, and then the rest in X in like order. Equip X^2 with a cell structure induced by the triangulation of X.

Now 7.2 follows from the case $(p, q) = (n, n)$ of the below statement.

Claim 7.3. *Let p, q be some integers, $-1 \leqslant p \leqslant q \leqslant n$. There is a positive integer c_{pq} such that the following holds.*

The immersion f is PL $c_{pq}\delta$-regularly homotopic, keeping L_p fixed, to an immersion $f_{pq} : X \looparrowright Q$ such that $f_{pq}(A) \cap f_{pq}(B) = \emptyset$ for any cell $A \times B$ of L_{pq}, and such that f^2_{pq} is equivariantly $c_{pq}\delta$-homotopic to H_1 by a homotopy $H_{pq} : X^2 \times I \to Q^2$ which is isovariant on $(V \cup Z^*_f \cup L_{pq}) \times I$.

Here $L_p = X^{(p)} \cup Z$ and $L_{pq} = L_p \times L_q \cup L_q \times L_p \cup L_{p-1} \times X \cup X \times L_{p-1}$.

Proof. The case $(p, q) = (-1, -1)$ follows from the hypothesis of 7.2, assuming $c_{-1,-1} = 1$. The transition from (p, n) to $(p + 1, p + 1)$ can be regarded as that from $(p + 1, p)$ to $(p + 1, p + 1)$. We thus assume 7.3(p, q) and prove 7.3$(p, q + 1)$.

If two simplices A, B, where $A \times B$ is a cell of $L_{p,q+1}$, are mapped by f_{pq} sufficiently far from each other (namely, so that the minimal distance between points in $f_{pq}(A)$, $f_{pq}(B)$ is greater than $2c_{pq}\delta$), then f_{pq} embeds $A \cup B$ and H_{pq} is isovariant on $(A \times B \cup B \times A) \times I$.

On the other hand, let J_{pq} denote the set of pairs (A_i, A_j) of simplices of X such that $\dim A_i = p$, $\dim A_j = q + 1$, A_j is not a simplex of Z, if $p = q$ then A_i precedes A_j in the ordering, and such that $\operatorname{diam} f_{pq}(A_i \cup A_j) < 4c_{pq}\delta$.

By [70, 3.2] (which obviously generalizes for embedding into a PL manifold) we can assume that there is a collection of PL balls $B_{ij} \subset Q$, where (A_i, A_j) run over J_{pq}, such that $\operatorname{diam} B_{ij} < 16c_{pq}\delta$, $B_{ij} \cap B_{kl} = \emptyset$ whenever $(i,j) \neq (k,l)$, $f_{pq}(A_i) \cap f_{pq}(A_j)$, if nonempty, lies in $\operatorname{Int} B_{ij}$, and B_{ij} meets A_i, A_j in PL balls $B_i^p \subset \operatorname{Int} A_i$, $B_j^{q+1} \subset \operatorname{Int} A_j$.

Now the argument of [37, proof of Proposition 9] can be applied to each pair $(A_i, A_j) = (\sigma^p, \sigma^{q+1})$ independently, so that there is a sequence of ambient isotopies H_{ij}, H'_{ij} of Q, supported by arbitrarily small neighborhoods N_{ij} of balls B_{ij} (which can be chosen still disjoint from each other and of diameter $< 17c_{pq}\delta$).

The immersion $f_{p,q+1}$ is defined by $f_{p,q+1} = H'_{ij} \circ H_{ij} \circ f_{pq}$ on $N_{ij} \cap \operatorname{st}(A_j, X')$ and by $f_{p,q+1} = f_{p,q}$ elsewhere, in particular, on $N_{ij} \cap \operatorname{st}(A_i, X')$. It follows from the construction of H, H' in [37] that $f_{p,q+1}(A) \cap f_{p,q+1}(B)$ is empty for any simplices A of $L_p = X^{(p)} \cup Z$ and B of L_{q+1} and for any simplices A of L_{p-1} and B of X. Also $f_{p,q+1}$ is $17c_{pq}\delta$-regularly homotopic to f_{pq}, keeping Z fixed.

The equivariant homotopy $H_{p,q+1}$ is defined analogously to as F' in [37, proof of Proposition 9], and it follows from the construction of F' in [37] that $H_{p,q+1}$ is isovariant on $(V \cup Z^*_f \cup L_{pq}) \times I$.

From the obtained control of $f_{p,q+1}$ and the construction of F' in [37] it follows that $H_{p,q+1}$ moves points less than $17\sqrt{2}c_{pq}\delta$.

Hence if we take $c_{p,q+1} > (17\sqrt{2}+1)c_{pq}$, all conditions are satisfied. □

Addendum to 7.2. *Let $H' : X^2 \times I \to Q^2$ denote the isovariant homotopy between g^2 and H_1, let $r_t : X \to Q$ denote the regular homotopy between f and g, and let $G : X^2 \times I \to Q^2$ be defined by $G_t = (r_{1-2t})^2$ for $t \in [0, \frac{1}{2}]$ and $G_t = H_{2t-1}$ for $t \in [\frac{1}{2}, 1]$.*

Then G and H' are equivariantly ε-homotopic by a homotopy isovariant on $(X^2 \times \partial I \cup (V \cup Z^*_f) \times I) \times I$, where V is some smaller neighborhood of Δ_X in X^2.

This addendum is needed to carry out induction in the proof of 7.4 below. It follows from the proofs of 7.2 and [37, Proposition 9]. (The similar addenda to 7.1 and 7.4 can be also shown to hold, but are not required in this paper.)

Theorem 7.4. *Let X^n be a compact polyhedron, $(Q^m, \partial Q)$ be a PL manifold, $m \geqslant 3(n+1)/2$, and $f : X \to Q$ be a PL map immersing a subpolyhedron Z of X.*

Suppose that f^2 is equivariantly homotopic to map H_1 which is isovariant on a neighborhood U of Δ_X by a homotopy $H : X^2 \times I \to Q^2$ which is isovariant on $(Z^*_f \cap U) \times I$.

Then for any $\varepsilon > 0$, f is PL ε-homotopic, keeping Z fixed, to an immersion $g : X \looparrowright Q$ such that g^2 is equivariantly homotopic to H_1 by a homotopy which is ε-close to H and isovariant on $(V \cup (Z^*_f \cap U)) \times I$ for some smaller neighborhood V of Δ_X.

We minimize the program in [37] by using a different method. The idea is to construct a PL immersion by inductive gathering of certain PL regular homotopies from immersions to embeddings. (This idea traces back to discussions of Skopenkov and the author, and was first explicitly realized by Skopenkov; see [83].) More precisely, the PL immersion will be constructed via an inductive application of (Theorem 7.2 + Addendum).

Actually we do not use the control in this application, that is, we could apply simply ([37, Theorem 3] + non-controlled Addendum). Hence the proof below of 7.4, together with Harris' original proof of [37, Theorem 3], yields a new short proof of [37, Theorems 1, 2]. It seems that an application of the control in (Theorem 7.2 + Addendum) would be necessary in a proof of a version of 7.4 with C^1-control. Perhaps 7.4 can be also proved by a modification of [37, proof of Proposition 11] in the spirit of [70].

We prove explicitly only the case $Z = \emptyset$. Lack of Z_f^* allows to use the following convention: we call a map $f : X^2 \to Q^2$ (a homotopy $h : X^2 \times I \to Q^2$) *locally isovariant* if there is a neighborhood W of Δ_X in X^2 such that $f|_W$ (respectively $h|_{W\times I}$) is isovariant.

Proof. Without loss of generality we can assume that f is non-degenerate. Triangulate X and Q so that f is simplicial and diameter of each dual cell C of Q is less than $\varepsilon/\sqrt{2}$.

Let $C_1, \dots, C_M$ be the dual cells of Q arranged in an order of increasing dimension, and let $C_{i1}, \dots, C_{iM_i}$ be the dual cones of X, whose disjoint union is $f^{-1}(C_i)$. Each cell C_i is dual to a simplex of Q which we denote by A_i, whose barycenter we denote by b_i, and whose link in Q' we denote by B_i, so that $b_i * B_i = C_i$. We also write $D_i = b_i^2 * B_i^2 \subset C_i^2$. We define A_{ij}, b_{ij}, B_{ij} and D_{ij} analogously.

We write $\mathcal{S}$ for $\{C_i \times C_j, b_i^2, D_i \mid i, j = 1, \dots, M\}$. We arrange C_{ij} lexicographically, and denote $L_{pq} = C_{11} \cup \dots \cup C_{pq}$.

We use the following notation. By $\varepsilon(b * B)$ we denote, for any $\varepsilon \in (0, 1]$ and any cone $b * B = B \times [0,1]/_{B\times 1}$, its subpolyhedron $B \times [1-\varepsilon, 1]/_{B\times 1}$. By $H_{\varepsilon(b*B)}$ we denote the natural homeomorphism $(b * B \setminus \varepsilon(b * B), B) \to (B \times I, B \times 0)$.

Theorem 7.4 follows from 7.5 and the case $p = M$, $q = M_M$ of 7.8.

Claim 7.5. *For each $\delta > 0$ there is a locally isovariant homotopy $G : X^2 \times I \to Q^2$, δ-close to H and such that $G_1 = H_1$.*

Proof. Clearly, $H|_{\Delta_X \times I}$ can be extended (analogously to the Borsuk Lemma) to some locally isovariant homotopy $H' : X^2 \times I \to Q^2$ with $H'_1 = H_1$. There is a neighborhood W of Δ_X in X^2 such that $H'|_{W\times I}$ is δ-close to $H|_{W\times I}$.

Moreover, if δ is sufficiently small, $H'|_{(\operatorname{Fr} W)\times I}$ is equivariantly homotopic, by the 'linear' homotopy, to $H|_{(\operatorname{Fr} W)\times I}$. Hence by the Borsuk Lemma we can assume without loss of generality (but with possible decrease of the neighborhood W' of Δ_X in X^2 such that $H'|_{W'\times I}$ is isovariant) that $H'|_{(\operatorname{Fr} W)\times I} = H|_{(\operatorname{Fr} W)\times I}$.

Define G by $G = H'$ on $W \times I$ and by $G = H$ elsewhere, then G is locally isovariant and δ-close to H. $\square$

Lemma 7.6. *Let X^n be a finite simplicial complex, $(Q^m, \partial Q)$ a combinatorial manifold, $m - n \geqslant 2$, $f : X \to Q$ a simplicial map and C a union of some top-dimensional dual cells of Q. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that the following holds.*

Suppose that $G : X^2 \to Q^2$ is a locally isovariant map, δ-close to f^2 and such that $G^{-1}(C_1 \times C_2) = (f^2)^{-1}(C_1 \times C_2)$, where C_i run over (dual cells of C) $\cup \{Q\}$. Then G is locally isovariantly ε-homotopic, keeping C^2 fixed and preserving $C \times Q \cup Q \times C$, to *a map $F : X^2 \to Q^2$ such that $F^{-1}(C_1 \times C_2) = (f^2)^{-1}(C_1 \times C_2)$ for each dual cells C_1, C_2 of Q.*

The proof of Lemma 7.6 modulo Lemma 7.7(a) is analogous to the proof of Lemma 4.1 modulo Theorem 3.7 and we leave it for the reader.

Lemma 7.7. *Suppose that $m - n \geqslant 1$.*

(a) *Let X^n be a compact polyhedron, Y its collared subpolyhedron, $(Q^m, \partial Q)$ a PL manifold and $J_0 = \{(-\infty, 0], 0, [0, +\infty)\}^2$. Any locally isovariant map*

$$f : (X^2, X \times Y \cup Y \times X) \times [-1, 1]^2 \to \big(Q^2, \partial(Q^2)\big) \times \mathbb{R}^2$$

such that $f^{-1}((\partial Q)^2 \times J) = Y^2 \times (J \cap [-1, 1]^2)$ for each $J \in J_0$ is locally isovariantly homotopic, keeping $(\partial Q)^2 \times \mathbb{R}^2$ fixed and preserving $\partial(Q^2) \times \mathbb{R}^2$, to a map g such that $g^{-1}(Q^2 \times J) = X^2 \times (J \cap [-1, 1]^2)$ for each $J \in J_0$.

(b) *Let $c * X^n$ be cone over a compact polyhedron, $c * Q^m$ cone over a PL sphere or a PL ball, and*

$$f : \big((c * X)^2, X \times (c * X) \cup (c * X) \times X\big) \to \big((c * Q)^2, Q \times (c * Q) \cup (c * Q) \times Q\big)$$

a locally isovariant map such that $f^{-1}(Q^2) = X^2$. Then f is locally isovariantly homotopic, keeping $(Q \times (c * Q) \cup (c * Q) \times Q)$ fixed, to a mapping g such that $g^{-1}(c^2 * Q^2, c^2) = (c^2 * X^2, c^2)$.

(c) *Let X^n be a compact polyhedron, Y its subpolyhedron, $(Q^m, \partial Q)$ a PL manifold,*

$$J_d = \big\{\{(x, y) \in \mathbb{R}^2 \mid x \geqslant y\}, \Delta_{\mathbb{R}}, \{(x, y) \in \mathbb{R}^2 \mid x \leqslant y\}\big\},$$

K an equivariant subpolyhedron of $\mathbb{R}^2$. Any locally isovariant map $f : X^2 \times [-1, 1]^2 \to Q^2 \times \mathbb{R}^2$ such that

$$f^{-1}\big(\partial(Q^2) \times J_\Delta\big) = (X \times Y \cup Y \times X) \times \big(J_\Delta \cap [-1, 1]^2\big)$$

and

$$f^{-1}\big(Q^2 \times (K \cap J_\Delta)\big) = X^2 \times \big(K \cap J_\Delta \cap [-1, 1]^2\big)$$

for each $J_\Delta \in J_d$, is locally isovariantly homotopic, keeping $\partial(Q^2) \times \mathbb{R}^2 \cup Q^2 \times K$ fixed, to a map g such that $g^{-1}(Q^2 \times J_\Delta) = X^2 \times (J_\Delta \cap [-1, 1]^2)$ for each $J_\Delta \in J_d$.

Lemma 7.7 can be regarded as a homotopy-theoretic version of Theorem 3.7. We reduce (a) and (b) to (c), which is proved analogously to the geometric proof of the Freudental Suspension Theorem.

Proof of 7.7. (a) Let us write $R = \partial(Q^2)$ and $Z = Y \times X \cup X \times Y$. Without loss of generality $f^{-1}(R \times J) = Z \times (J \cap [-1, 1]^2)$ for each $J \in J_0$.

By 7.7(c) with $K = (-\infty, 0] \times [0, +\infty) \cup [0, +\infty) \times (-\infty, 0]$ there is a locally isovariant homotopy $f_t : Z \times [-1, 1]^2 \to R \times \mathbb{R}^2$, preserving $R \times J$ for each $J \in J_0$, from $f|_{Z \times [-1,1]^2}$ to a map f_1 such that $f_1^{-1}(R \times J_\Delta) = Z \times (J_\Delta \cap [-1, 1])$ for each $J_\Delta \in J_d$.

Hence (using collars on Z and R) without loss of generality we can assume that $f^{-1}(R \times J_\Delta) = Z \times (J_\Delta \cap [-1,1])$ for each $J_\Delta \in J_d$.

Then we can apply 7.7(c) ($K = \emptyset$) to obtain a locally isovariant homotopy, keeping $R \times \mathbb{R}^2$ fixed, from f to a map h such that $h^{-1}(Q^2 \times J_\Delta) = X^2 \times (J_\Delta \cap [-1,1])$ for each $J_\Delta \in J_d$.

There is no obstruction in homotoping h onto required g keeping $R \times \mathbb{R}^2$ fixed.

(b) We consider the case $Q = S^m$, since the case $Q = B^m$ is its corollary. By general position we can assume that $f^{-1}(c^2) = c^2$, by [37, Appendix] we can assume that f is PL, and using pseudo-radial projection [74] we can assume that $f^{-1}(\varepsilon(c * S^m)^2) = \varepsilon(c * X)^2$ for sufficiently small $\varepsilon > 0$. Let

$$h : \partial\varepsilon(c * X)^2 \to X \times (c * X) \cup (c * X) \times X,$$
$$H : \partial\varepsilon\big(c * S^m\big)^2 \to S^m \times \big(c * S^m\big) \cup \big(c * S^m\big) \times S^m$$

be the natural homeomorphisms. The map

$$F = H \circ f|_{\partial\varepsilon(c*X)^2} \circ h^{-1} : X \times (c * X) \cup (c * X) \times X \to S^m \times \big(c * S^m\big) \cup \big(c * S^m\big) \times S^m$$

is isovariant. Notice that $X \times (c * X) \cup (c * X) \times X$ is homeomorphic to $X * X$, i.e., to $X \times X \times I/(X \times 0 \cup X \times 1)$, the product $X \times X$ being thrown onto the middle section $X \times X \times \frac{1}{2}$ of the join. Now the pair

$$\big(S^m * S^m \setminus \big(\Delta_{S^m} \times \tfrac{1}{2}\big), \big(S^m \times S^m \setminus \Delta_{S^m}\big) \times \tfrac{1}{2}\big)$$

with induced involution $(x, y, t) \leftrightarrow (y, x, 1 - t)$ equivariantly deformation retracts onto $\nabla_{S^m} \times (I, \frac{1}{2})$ (where ∇_{S^m} is the *antidiagonal* $\{(x, -x) \in S^m \times S^m\} \cong S^m$) with involution $(z, t) \leftrightarrow (-z, 1-t)$. If r denotes the retraction, $r \circ F|_{X*X \setminus \Delta_X \times \frac{1}{2}}$ is equivariantly homotopic to a map R such that $R^{-1}(\nabla_{S^m} \times \frac{1}{2}) = X \times X \times \frac{1}{2}$. Since r is homotopy invertible, it follows that there is an isovariant homotopy F_t from F to a map F_+ such that $F_+^{-1}((S^m)^2) = X^2$.

Let us consider the map

$$f_+ = H^{-1} \circ F_+ \circ h : \partial\frac{\varepsilon}{2}(c * X)^2 \to \partial\frac{\varepsilon}{2}\big(c * S^m\big)^2.$$

Let us extend f_+ linearly over $\frac{\varepsilon}{2}(c * X)^2$, by $f_+ = f$ on $(c * X)^2 \setminus \varepsilon(c * X)^2$ and using F_t on $\varepsilon(c * X)^2 \setminus \frac{1}{2}\varepsilon(c * X)^2$ (more precisely, by $f_+ = H^{-1}_{\varepsilon(c*S^m)^2} \circ F_{2t} \circ H_{\varepsilon(c*X)^2}$). Then f_+ is locally isovariant,

$$f_+^{-1}\left(\frac{\varepsilon}{2}\big(c * S^m\big)^2, \frac{\varepsilon}{2}\big(c^2 * \big(S^m\big)^2\big)\right) = \left(\frac{\varepsilon}{2}(c * X)^2, \frac{\varepsilon}{2}\big(c^2 * X^2\big)\right),$$

and F_t yields a locally isovariant homotopy from f to f_+. To obtain the required map g, it remains to apply 7.7(c) with $K = 1 \times [-1,1] \cup [-1,1] \times 1 \cup [-1, -1 + \varepsilon]^2$ to $f_+|_{(c*X)^2 \setminus \frac{\varepsilon}{2}(c*X)^2}$.

(c) Let us write R for $\partial(Q^2) \times \mathbb{R}^2 \cup Q^2 \times K$ and Z for

$$(X \times Y \cup Y \times X) \times \mathbb{R}^2 \cup X^2 \times K.$$

By [37, Appendix] we can assume that f is PL, meanwhile by [52, Theorem 1] there exists a PL tangent bundle to Q, i.e., a collection of PL open ball pairs $(B_i^{2m}, D_i^m) \subset (Q^2, \Delta_Q)$ such that for each i there exists a PL homeomorphism $H_i : (B_i, D_i) \to (\mathbb{R}^{2m}, \mathbb{R}^m)$, making the following diagram commutative:

$$\begin{tikzcd}
\mathbb{R}^{2m} \arrow[r, "H_i"] \arrow[d, "\text{projection}"'] & B_i \arrow[r, "\text{inclusion}"] & Q^2 \arrow[d, "(q,r)\mapsto(q,q)"] \\
\mathbb{R}^m \arrow[r, "H_i"] & D_i \arrow[r, "\text{inclusion}"] & \Delta_Q
\end{tikzcd}$$

Fix equivariant triangulations on $X^2 \times [-1,1]^2$ and $Q^2 \times \mathbb{R}^2$ in which f and the projection $\delta : Q^2 \times \mathbb{R}^2 \to Q^2 \times \Delta_{\mathbb{R}}$ are simplicial. We define

$$\widehat{Q} = \mathrm{N}\big(\Delta_{Q\times\mathbb{R}}, Q^2 \times \mathbb{R}^2\big), \qquad \widehat{X} = \mathrm{N}\big(\Delta_{X\times[-1,1]}, X^2 \times [-1,1]^2\big),$$
$$\widehat{Q}_0 = \widehat{Q} \cap Q^2 \times \Delta_{\mathbb{R}}, \qquad \widehat{X}_0 = \widehat{X} \cap X^2 \times \Delta_{[-1,1]}.$$

Notice that $\widehat{X}$ is a connected component of $f^{-1}(\widehat{Q})$. We are to modify f so that $\widehat{X}_0$ is a connected component of $f^{-1}(\widehat{Q}_0)$.

We have that $f^{-1}(\mathrm{Fr}\,\widehat{Q} \cap R) = \mathrm{Fr}\,\widehat{X} \cap Z$. It suffices to homotop $f|_{\mathrm{Fr}\,\widehat{X}}$ onto a map

$$F_+ : \big(\mathrm{Fr}\,\widehat{X}, \mathrm{Fr}\,\widehat{X}_0\big) \to \big(\mathrm{Fr}\,\widehat{Q}, \mathrm{Fr}\,\widehat{Q}_0\big)$$

by a sufficiently small homotopy F_t keeping $\widehat{Q} \cap R$ fixed. For if F_t is so small that each simplex is left by it in the same $B_i \times \mathbb{R}^2$, then it can be extended 'linearly' (by an induction on joins $Y * Z$, where Y and Z run, in orders of increasing dimension, over simplices of Δ_Q and $\mathrm{lk}(Y, Q) \cap \mathrm{Fr}\,\widehat{X}$, respectively) to an isovariant homotopy f_t taking $f|_{\widehat{X}}$ onto a map

$$f_+ : \big(\widehat{X}, \widehat{X}_0\big) \to \big(\widehat{Q}, \widehat{Q}_0\big).$$

To arrange such a homotopy F_t, let us consider the analogues of the north and the south poles:

$$\widehat{Q}^n = \big(\mathrm{Fr}\,\widehat{Q}\big) \cap \Delta_Q \times \big((-\infty) \times (+\infty), 0^2\big],$$
$$\widehat{Q}^s = \big(\mathrm{Fr}\,\widehat{Q}\big) \cap \Delta_Q \times \big[0^2, (+\infty) \times (-\infty)\big),$$
$$\widehat{X}^n = \big(\mathrm{Fr}\,\widehat{X}\big) \cap \Delta_X \times \big[(-1) \times 1, 0^2\big],$$
$$\widehat{X}^s = \big(\mathrm{Fr}\,\widehat{X}\big) \cap \Delta_X \times \big[0^2, 1 \times (-1)\big].$$

Since δ is simplicial, these are subcomplexes of $\mathrm{Fr}\,\widehat{Q}, \mathrm{Fr}\,\widehat{X}$, while $\mathrm{Fr}\,\widehat{X} \setminus (\widehat{X}^n \cup \widehat{X}^s)$ is equivariantly homeomorphic to the cylinder $C = \mathrm{Fr}\,\widehat{X}_0 \times (0,1)$ equipped with the involution $((x,y,s),t) \leftrightarrow ((y,x,s),-t)$. We can assume (adjusting K, if necessary, without loss of generality) that the homeomorphism takes $Z \cap (\mathrm{Fr}\,\widehat{X} \setminus (\widehat{X}^n \cup \widehat{X}^s))$ onto $Z_0 \times (0,1)$ for some subpolyhedron Z_0 of $\mathrm{Fr}\,\widehat{X}_0$.

By general position [10] the inequality $(2n-m+1)+(n+1) < 2n+2$ implies that $f^{-1}(\widehat{Q}^n)$ does not meet $\widehat{X}^n \cup \widehat{X}^s$, while $(2n-m+1)+(2n-m+1) < 2n+2$ implies that $f^{-1}(\widehat{Q}^n \cup \widehat{Q}^s)$, regarded as a subset of $\mathrm{Fr}\,\widehat{X}_0 \times (0,1)$, is not self-overshadowing. Therefore there is an equivariant isotopy $h_t : \mathrm{Fr}\,\widehat{X} \to \mathrm{Fr}\,\widehat{X}$, keeping $\widehat{X}^n \cup \widehat{X}^s \cup Z_0 \times (0,1)$ fixed and preserving the generators of the cylinder C, and taking $f^{-1}(\widehat{Q}^n)$ (respectively $f^{-1}(\widehat{Q}^s)$)

into a small neighborhood of $\widehat{X}^n$ (respectively $\widehat{X}^s$). The homotopy $H_t = f \circ h_t^{-1}$ carries $f|_{\mathrm{Fr}\,\widehat{X}}$ (equivariantly and keeping $R \cap \mathrm{Fr}\,\widehat{Q}$ fixed) onto a map $f_1 : \mathrm{Fr}\,\widehat{X} \to \mathrm{Fr}\,\widehat{Q}$, such that $f_1^{-1}(\widehat{Q}^n)$ (respectively $f_1^{-1}(\widehat{Q}^s)$) is close to $\widehat{X}^n$ (respectively to $\widehat{X}^s$).

By stretching a neighborhood of $\widehat{Q}^n$ over the 'hemisphere' $(\mathrm{Fr}\,\widehat{Q}) \cap Q^2 \times \{(x,y) \mid x \leqslant y\}$ (and similarly for the symmetric neighborhood of $\widehat{Q}^s$) we obtain a homotopy (equivariant, keeping $R \cap \mathrm{Fr}\,\widehat{Q}$ fixed) onto a map f_2 such that $f_2(\mathrm{Fr}\,\widehat{X}_0) \subset \mathrm{Fr}\,\widehat{Q}_0$. Final straightening in the spirit of the Alexander trick yields an equivariant homotopy, keeping $R \cap \mathrm{Fr}\,\widehat{Q}$ fixed, onto the required map $f_+ : \mathrm{Fr}\,\widehat{X} \to \mathrm{Fr}\,\widehat{Q}$ such that $f_+^{-1}(\mathrm{Fr}\,\widehat{Q}_0) = \mathrm{Fr}\,\widehat{X}_0$. □

Claim 7.8. *Let F denote the map, obtained from Lemma* 7.6 *for $G_{7.6} = G_0$ and $C_{7.6} = \emptyset$, and refined by application of Lemma* 7.7(b) *to cones $c * X_{7.6}$ running over all dual cones C_{ij}. Then for each $p = 0, \dots, M$, $q = 0, \dots, M_p$ the following holds.*

The map f is PL homotopic to a map $f_{pq} : X \to Q$, which immerses L_{pq}, by a homotopy $h_{pq} : X \times I \to Q$ such that $h_{pq}^{-1}(C_i) = f^{-1}(C_i) \times I$ for each $i \leqslant M$.

Moreover, f_{pq}^2 is equivariantly homotopic to F by a homotopy $F_{pq} : X^2 \times I \to Q^2$ such that $F_{pq}^{-1}(S) = (f^2)^{-1}(S) \times I$ for each $S \in \mathcal{S}$, and which is locally isovariant on $L_{pq}^2 \times I$.

Proof. By lexicographic induction on p, q. The base $p, q = 0$ follows by taking $f_{00} = f$ and constructing F_{00} inductively by the Alexander trick. (Given polyhedra $Y \supset Y_1$, $Z \supset Z_1$ and maps $\Psi_0, \Psi_1 : c * Y \to c * Z$ such that $\Psi_i^{-1}(c) = c$ and $\Psi_i^{-1}(c * Z_1) = c * Y_1$, then any homotopy $\psi_t : Y \to Z$ between $\Psi_0|_Y$, $\Psi_1|_Y$ such that $\psi^{-1}(Z_1) = Y_1 \times I$ can be extended to a homotopy $\Psi_t : c * Y \to c * Z$ between Ψ_0, Ψ_1 such that $\Psi^{-1}(Z) = Y \times I$, $\Psi^{-1}(c * Z_1) = (c * Y_1) \times I$, and $\Psi_t^{-1}(c) = c \times I$.)

To prove the inductive step, notice that $f_{p,q-1}|_{B_{pq}} : B_{pq} \to B_p$ is an immersion, and that the homotopy $F_{p,q-1}|_{B_{pq}^2 \times I} : B_{pq}^2 \times I \to B_p^2$ is locally isovariant.

Also we have that $F_{p,q-1}|_{B_{pq}^2 \times 1} = F|_{B_{pq}^2}$ is locally isovariantly homotopic to an isovariant map. Indeed, define a homotopy $\Psi : B_{pq}^2 \times I \to B_p^2$ by

$$\Psi = \pi_{B_p^2} \circ H_{\varepsilon D_p} \circ F \circ H_{\varepsilon D_{pq}}^{-1},$$

where ε is so small that εD_{pq} lies in a neighborhood W of Δ_X in $X \times X$ such that $F|_W$ is isovariant. Then by 7.2 there is a PL regular homotopy $r : B_{pq} \times I \to B_p \times I$ from $f_{p,q-1}|_{B_{pq}}$ to an embedding r_1. We define f_{pq} by $H_{\varepsilon C_p}^{-1} \circ r \circ H_{\varepsilon C_{pq}}$ on $C_{pq} \setminus \varepsilon C_{pq}$, conewise on εC_{pq}, and by $f_{p,q-1}$ outside C_{pq}. Since r is a PL regular homotopy and r_1 is a PL embedding, $f_{pq}|_{C_{pq}}$ is a PL immersion.

Also f_{pq} agrees with $f_{p,q-1}$ on B_{pq}, hence $f_{pq}|_{C_{pq}}$ is linearly homotopic, keeping B_{pq} fixed, to $f_{p,q-1}|_{C_{pq}}$. We denote this homotopy by $l : C_{pq} \times I \to C_p$, and we denote its extension over X by identity by $L : X \times I \to Q$, so that $L_0 = f_{p,q-1}$ and $L_1 = f_{pq}$. Then L together with $h_{p,q-1}$ yield the required PL homotopy $h_{pq} : X \times I \to Q$ between f and f_{pq}.

Since r is level-preserving, $l^{-1}(tC_p) = tC_{pq} \times I$ for each $t \in I$. It follows that $(l^2)^{-1}(D_p) = (D_{pq})$ and hence $(L^2)^{-1}(S) = (f^2)^{-1}(S) \times I$ for each $S \in \mathcal{S}$. Now L^2 and $F_{p,q-1}$ together give an equivariant homotopy $F_{p,q-1}^+ : X^2 \times I \to Q^2$ between $(f_{pq})^2$

and F such that $(F^+_{p,q-1})^{-1}(S) = (f^2)^{-1}(S) \times I$ for each $S \in \mathcal{S}$, and which is locally isovariant on $L^2_{p,q-1} \times I$. We want to make it locally isovariant on $L^2_{pq} \times I$.

We define an equivariant homotopy $\Phi : B^2_{pq} \times [-1, 2] \to B^2_p$ by

$$\Phi_t = \begin{cases} r_{-t}, & t \in [-1, 0]; \\ (F_{p,q-1}|_{B^2_{pq} \times I})_t, & t \in [0, 1]; \\ \Psi_{t-1}, & t \in [1, 2]. \end{cases}$$

By 7.2 and Addendum, Φ is locally isovariantly homotopic to an isovariant map Ξ_1 by a homotopy $\Xi : B^2_{pq} \times [-1, 2] \times I \to B^2_p$, which is in addition isovariant on $B^2_{pq} \times \{-1, 2\} \times I$. Denote $B^2_p \times I \cup D_p \times \partial I$ by E_p. Define an embedding $\varphi_p : B^2_p \times [-1, 2] \hookrightarrow E_p$ by the identity on $B^2_p \times [0, 1]$, by $H^{-1}_{\varepsilon D_p \times 0} \circ (t \mapsto -t)$ on $B^2_p \times [-1, 0]$, and by $H^{-1}_{\varepsilon D_p \times 1} \circ (t \mapsto t-1)$ on $B^2_p \times [1, 2]$. Define analogously E_{pq} and φ_{pq}. Then twice extending the homotopy $\varphi_p \circ \Xi_t \circ \varphi^{-1}_{pq}$ conewise, we obtain a locally isovariant homotopy $\Xi^+ : E_{pq} \times I \to E_p \times I$ from $F^+_{p,q-1}|_{E_{pq}} : E_{pq} \to E_p$ to an isovariant map Ξ^+_1.

We define the required F_{pq} by $F^+_{p,q-1}$ outside C^2_{pq}, by $H^{-1}_{\varepsilon(D_p \times I)} \circ \Xi^+ \circ H_{\varepsilon(D_{pq} \times I)}$ on $(D_{pq} \times I) \setminus \varepsilon(D_{pq} \times I)$, conewise on $\varepsilon(D_{pq} \times I)$, and by the relative Alexander trick on the rest of $C^2_{pq} \times I$. Then F_{pq} is locally isovariant on $L^2_{p,q-1} \times I$ and on $C^2_{pq} \times I$. Also $F^{-1}_{pq}(S) = (f^2)^{-1}(S) \times I$ for all $S \in \mathcal{S}$, and $\Delta_{L_{pq}} \subset L^2_{p,q-1} \cup C^2_{pq}$, therefore F_{pq} is locally isovariant on $L^2_{pq} \times I$. □

Proof of 1.7(a+). The PL case follows immediately from 7.1, because any equivariant map, δ-close to a given equivariant map f^2, is equivariantly δ-homotopic to f^2, provided $\delta > 0$ is sufficiently small.

The TOP case follows from the PL case and simplicial approximation. □

It is convenient to call a homotopy $h_t : X^2 \to Q^2$ *pseudo-isovariant* if h_t is isovariant for $t < 1$. If $f_0, f_1 : X \to Q$ are maps, let us say that an equivariant (δ-)homotopy $\varphi_t : X^2 \to Q^2$ between f^2_0 and f^2_1 (δ-)*holonomic* if the homotopy φ_t is homotopic, in the class of equivariant (δ-)homotopies between f^2_0 and f^2_1, to a homotopy $(f_t)^2$, where $f_t : X \to Q$ is some homotopy between f_0 and f_1.

Corollary 7.9. *For each $\varepsilon > 0$ and a positive integer n there exists $\delta > 0$ such that the following holds. Let X^n be a compact polyhedron, Q^m a PL manifold, and $m > 3(n+1)/2$.*

(a) *If $f, g : X \hookrightarrow Q$ are PL embeddings such that f^2 and g^2 are isovariantly δ-holonomically homotopic, then f and g are PL ε-ambient isotopic.*
(b) *If $f : X \hookrightarrow Q$ is a PL map and $g : X \to Q$ a PL embedding such that f^2 and g^2 are pseudo-isovariantly δ-holonomically homotopic, then f and g are PL ε-pseudo-isotopic.*

Compare Corollary 7.9(a) to [70, Conjecture 1.9b]. Notice that the statement of 7.9(a) is concerned with embeddings only, although its proof and applications contain the idea of map realization. Actually 7.1 and 7.9(a) are not only controlled versions of Harris' results, but also generalize them (by considering an appropriate metric on Q). Notice that from 7.1 and 7.9(a) at once follow 3.1(a) and 3.2(a) restricted to the metastable range, respectively (compare to [37, Corollary 4]).

Observe that if $Q=\mathbb{R}^m$ or if δ is allowed to depend on the metric on Q, the holonomy condition is fulfilled automatically and can be dropped from the hypotheses of 7.9.

Proof. (a) This is analogous to [37, proof of Corollary 1] (but not to [82, proof of Theorems 1.0e and 1.1c]).

Let $\varphi_t : X^2 \to Q^2$ be the given isovariant δ-homotopy between f^2 and g^2. Let $\Omega_{s,t} : X^2 \to Q^2$ be the given homotopy, in the class of equivariant δ-homotopies between f^2 and g^2, from φ_t to $(f_t)^2$, where $f_t : X \to Q$ is a homotopy between f and g. Let $F : X \times I \to Q \times I$ be the PL map defined by $F(x,t) = (f_t(x), t)$. Define an equivariant homotopy $\Psi_t : (X \times I)^2 \to (Q \times I)^2$ from $F^2 = \Psi_0$ to a map Ψ_1 by

$$\Psi_t(x,u,y,v) = \left(F\left(x, u\left(1-\frac{t}{2}\right)+v\frac{t}{2}\right), F\left(y, v\left(1-\frac{t}{2}\right)+u\frac{t}{2}\right)\right).$$

Then

$$\Psi_1(x,u,y,v) = \left(F\left(x, \frac{u+v}{2}\right), F\left(y, \frac{u+v}{2}\right)\right),$$

which equals $((f_{(u+v)/2})^2(x,y), u, v)$. Define an equivariant homotopy $\Phi_t : (X \times I)^2 \to (Q \times I)^2$ from $\Psi_1 = \Phi_0$ to an isovariant map Φ_1 by

$$\Phi_t\big((x,u,t,v)\big) = \big(\Omega_{t,(u+v)/2}(x,y), u, v\big).$$

Together Ψ_t and Φ_t yield an equivariant 3δ-homotopy H_t, which is isovariant on $(X \times \partial I) \times (X \times I) \cup (X \times I) \times (X \times \partial I)$, from F^2 to an isovariant map $H_1 = \Phi_1$. Then by 7.1 for any $\gamma > 0$, δ can be chosen so that F is γ-homotopic, fixing $X \times \partial I$, to a PL embedding, i.e., to a PL $(\gamma+\delta)$-concordance between f and g. By 1.13(a), $\gamma+\delta$ can be chosen so that f and g are PL ε-ambient isotopic.

(b) Let F be as above, and use the argument above to obtain an equivariant homotopy $H_t : (X \times I)^2 \to (Q \times I)^2$, isovariant on

$$(X \times 0) \times (X \times I) \cup (X \times I) \times (X \times 0),$$

between $F^2 = H_0$ and a map H_1, isovariant on

$$\big(X \times [0,1)\big) \times (X \times I) \cup (X \times I) \times \big(X \times [0,1)\big).$$

Introduce an equivalence relation σ on $X \times I$ by

$$(x,t) \sim (y,s) : t = s = 1 \quad \text{and} \quad f(x) = f(y).$$

Then the quotient space $M = X \times I/_{\sigma}$ is a compact polyhedron (the mapping cylinder of f) and the identifying map $\Sigma : X \times I \to M$ is PL. By the construction of H_t, if $(x,u) \sim (x',u')$ and $(y,v) \sim (y',v')$, then H_t maps (x,u,y,v) and (x',u',y',v') to the same point of Q^2. Hence there is the (unique) homotopy $H_t/_{\sigma} : M^2 \to (Q \times I)^2$ such that

$(H_t/_\sigma)\circ\Sigma^2 = H_t$. Furthermore, H_t maps (x,u,y,v) to Δ_Q, where either of u, v equals to 1, only if $(x,u)\sim(y,v)$. Therefore $H_t/_\sigma$ is isovariant on

$$(X\times\partial I/_\sigma)\times(X\times I/_\sigma)\cup(X\times I/_\sigma)\times(X\times\partial I/_\sigma)$$

and $H_1/_\sigma$ is isovariant everywhere. Hence by 7.1 for any $\gamma>0$, δ can be chosen so that $F/_\sigma = H_0/_\sigma : M\to Q\times I$ is γ-homotopic, fixing $X\times\partial I/_\sigma$, to a PL embedding, i.e., to a PL $(\gamma+\delta)$-pseudo-concordance between f and g. By 1.12 (the PL case), $\gamma+\delta$ can be chosen so that f is PL ε-pseudo-isotopic to g. □

Proof of 1.7(b+). Let $\chi_t : X^2\to Q^2$ denote the equivariant δ-homotopy such that $\chi_1 = f^2$, $\chi_0 = g^2$ and χ_t is isovariant for $t<1$. The proof splits into two cases.

PL case. Since δ is allowed to depend on the metric on Q, the PL case follows at once from 7.9(b). □

Remark 7.10. Notice that to prove only the PL case of 1.7(b), rather than that of 1.7(b+), one can simplify the argument above by using the non-controlled version of 7.9(b), in whose proof one can use the first part of 1.12 instead of its 'moreover' part, and the Harris' Theorem itself instead of its controlled version 7.1. This idea provides a somewhat simpler proof (not using the control in Harris' Theorem) of the PL case of 1.7(a) for $m>3(n+1)/2$ (instead of $m\geqslant 3(n+1)/2$).

TOP case. By 3.6(b), 3.5(a) and 3.1(a) we can assume that the embedding g is PL.

For each positive integer k let $\alpha_k>0$ be some number (defined below). Choose $t_k\in(0,1)$ so that χ_t moves points less than $\frac12\alpha_k$ for $t\in[t_k,1]$. Then χ_t is equivariantly α_k-homotopic to f_k^2 for any PL map f_k, which is $\frac12\alpha_k$-homotopic to f, provided α_k is sufficiently small.

Then by 7.1, for any $\beta_k>0$, α_k can be chosen so that there is a PL embedding $\varphi_k : X\hookrightarrow Q$, $\frac13\beta_k$-close to f and such that φ_k^2 is isovariantly $\frac13\beta_k$-homotopic to χ_{t_k}. Also, χ_{t_k} is isovariantly $\frac13\beta_k$-homotopic to $\chi_{t_{k+1}}$ (provided $\frac13\beta_k\geqslant\frac12\alpha_k$), which, in turn, is isovariantly $\frac13\beta_k$-homotopic to φ_{k+1}^2 (provided $\beta_{k+1}<\beta_k$). Thus, φ_k^2 and φ_{k+1}^2 are isovariantly β_k-homotopic.

Hence by 7.9(a) for any $\gamma_k>0$, the number $\beta_k=\beta_k(\gamma_k,n,\text{metric on }Q)$ can be chosen so that φ_k and φ_{k+1} are PL γ_k-ambient isotopic. Similarly, for any $\gamma_0>0$, the number $\delta+\frac13\beta_1$ can be chosen so that φ_1 and g are PL γ_0-isotopic. Thus if we take $\gamma_k=\varepsilon/2^{k+1}$, where $k=0,1,2,\ldots$, then composing these isotopies for all k, we obtain an ε-pseudo-isotopy taking f onto g.

8. Proof of Corollary 1.8

In view of Criterion 1.7, it suffices to prove the following:

Lemma 8.1. *Let X^n be a compact polyhedron, Q^m a PL manifold, $\varphi : X^2\to Q^2$ an equivariant map. If either*

(a) *$S=\varphi^{-1}(\Delta_Q)$ has an equivariant mapping cylinder neighborhood N_1 in X^2, or*

(b) $m = 2n + 1$, *then for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any isovariant map* ψ, δ*-close to* φ, *is* ε*-homotopic to* φ *by a homotopy, isovariant for* $t < 1$.

Recall that for some neighborhood U of Δ_Q in $Q \times Q$ the projection τ onto the first factor is a PL (in general not vector) bundle over Q, called tangent bundle in the PL category [52]. For each $t \in [0, 1]$ let us denote by U_t the total space of some subbundle of this bundle with each fiber U_{pt} of diameter $< t$, cf. [52].

Proof of 8.1(a). By the definition, N_1 is equivariantly homeomorphic, by a homeomorphism G, to the mapping cylinder $N \times I/_\sim$ of an equivariant map $g : N \to S$, where $(n_1, t) \sim (n_2, s)$ denotes '$g(n_1) = g(n_2)$ and $s = t = 0$'. Let N_t be the G-preimage of $N \times [0, t]/_\sim$ for any $t \in (0, 1]$ and N_{pt} be the G-preimage of $p \times [0, t]/_\sim$ for any $p \in N$, $t \in (0, 1]$. Let us write simply (p, t) for $G^{-1}((p, t)/_\sim) \in N_1$, if $t \in (0, 1]$. If $\delta > 0$ is sufficiently small, there exists a 'linear' equivariant δ-homotopy h_t between φ and ψ. Furthermore, by taking δ small enough we can achieve that $h_t^{-1}(\Delta_Q) \subset \operatorname{Int} N_1$ for each $t \in I$.

Let us construct an isovariant 2δ-homotopy ψ_t between $\psi_0 = \psi$ and ψ_1 such that ψ_1 is 2δ-close to φ and equals to φ outside N_1. Define two functions on triangles: for each $t \in [0, 1]$, let α_t map $[\frac{1}{3}, \frac{2}{3}t + 1(1-t)]$ linearly onto $[\frac{1}{3}, 1]$ and β_t map $[\frac{2}{3}t + 1(1-t), 1]$ linearly onto $[0, t]$. Put $\psi_t = \psi_0$ on $N_{1/3}$, $\psi_t(p, s) = \psi_0(p, \alpha_t(s))$ for $s \in [\frac{1}{3}, \frac{2}{3}t + 1(1-t)]$, $\psi_t(p, s) = h_{\beta_t(s)}(p, 1)$ for $s \in [\frac{2}{3}t + 1(1-t), 1]$, and $\psi_t = h_t$ outside N_1. Then ψ_t is clearly isovariant. Moreover, h_t is 2δ-close to φ, provided N_1 is so small that $\varphi(p, s)$ and $\varphi(p, t)$ are δ-close for any $p \in N$, $s, t \in (0, 1]$.

Choose N_1 so small that $\varphi(N_1)$ and $\psi_1(N_1)$ lie in U_1. Let $\alpha, \beta : [1, +\infty) \to (0, 1]$, be such homeomorphisms that $\varphi(N_{\alpha(t)}) \subset U_{\beta(t)}$ and $\operatorname{diam} \varphi(N_{p\alpha(t)})$, $\operatorname{diam} U_{p\beta(t)}$ are less than $r_t = \varepsilon/(11 * 2^{t+1})$ for each t. Let us write $(p, t), N_t, N_{pt}, U_t, U_{pt}$ instead of $(p, \alpha(t)), N_{\alpha(t)}, N_{p\alpha(t)}, U_{\beta(t)}, U_{p\beta(t)}$. If $\delta < \varepsilon/44$, the statement follows from

Claim 8.2. *Let* k *be a positive integer and* $\psi_k : X^2 \to Q^2$ *an isovariant map,* r_k*-close to* φ *and coinciding with* φ *outside* N_k. *Then* ψ_k *is isovariantly* $(11r_k)$*-homotopic to a map* ψ_{k+1}, r_{k+1}*-close to* φ *and coinciding with* φ *outside* N_{k+1}.

Proof. Define a homotopy $\Psi_t : X^2 \to Q^2$, $t \in I$, by $\Psi_0 = \psi_k$, the identity outside N_k, the identity on N_{k+2} and by

$$\Psi_t(p, s) = \begin{cases} \psi_k(p, s - (s-k) * 2t), & t \leqslant \frac{1}{2},\ k \leqslant s \leqslant k + \frac{3}{2}, \\ \psi_k(p, s - (k+2-s) * 3 * 2t), & t \leqslant \frac{1}{2},\ k + \frac{3}{2} \leqslant s \leqslant k+2, \\ \varphi(p, k + (s-k) * (2t-1)), & t \geqslant \frac{1}{2},\ k \leqslant s \leqslant k+1, \\ \varphi(p, k + (k + \frac{3}{2} - s) * 2 * (2t-1)), & t \geqslant \frac{1}{2},\ k+1 \leqslant s \leqslant k + \frac{3}{2}, \\ \psi_k(p, s - (k+2-s) * 3 * 1), & t \geqslant \frac{1}{2},\ k + \frac{3}{2} \leqslant s \leqslant k+2. \end{cases}$$

In other words, Ψ_t stretches $\psi_k(N_{k+3/2})$ over $\psi_k(N_k)$ and takes $\psi_k(\overline{N_k \setminus N_{k+3/2}})$ onto $\varphi(\overline{N_k \setminus N_{k+1}})$. The result is that Ψ_1 coincides with φ outside N_{k+1}. Clearly, Ψ_t is isovariant and moves points less than $3r_k$, meanwhile Ψ_1 is $3r_k$-close to φ.

We have that $\varphi(N_{k+1}) \subset U_{k+1}$. Homotop Ψ_1 isovariantly, fixing the exterior of N_{k+1}, 'linearly' towards φ onto a map Ψ_2 such that $\Psi_2(N_{k+1}) \subset U_{k+1}$. This homotopy moves points less than $3r_k$, and Ψ_2 is still $3r_k$-close to φ.

Now both $\varphi(N_{k+1})$ and $\Psi_2(N_{k+1})$ lie in U_{k+1}. Since tangent bundle is locally trivial, we can homotop Ψ_2 isovariantly, fixing the exterior of N_{k+1}, leaving the image of N_{k+1} inside U_{k+1}, and moving points less than $5r_k$, onto a map ψ_{k+1} such that $\tau \circ \psi_{k+1}|_{N_{k+1}} = \tau \circ \varphi|_{N_{k+1}}$. Then the images of any point under φ and ψ_{k+1} lie in the same $U_{p,k+1}$, which implies that ψ_{k+1} is r_{k+1}-close to φ. $\square$

Proof of 8.1(b). The following argument was partially inspired by [1, proof of Proposition 3.5]. Given a compactum C in the body of a simplicial complex K, we denote by $\mathrm{N}(C, K)$ the union of all simplices of K, meeting C. Let K_1 be a triangulation of X^2, and for each positive integer k let K_{k+1} be a subdivision of K_k such that for any simplex A of K_k $\operatorname{diam}\varphi(A) < n_k = \varepsilon/(14 * 2^{k+1})$ and $\varphi(A) \subset U_{n_k}$.

Let $S = \varphi^{-1}(\Delta_Q)$ and $N_k = \mathrm{N}(S, K_k)$, then $\varphi(N_k) \subset U_{n_k}$. If $\delta > 0$ is sufficiently small, there exists a 'linear' equivariant δ-homotopy h_t between φ and ψ such that $h_t^{-1}(\Delta_Q) \subset \operatorname{Int} N_1$ for each $t \in I$. By the equivariant Borsuk Lemma, ψ is isovariantly 2δ-homotopic to a map $\psi_1 : X^2 \to Q^2$ which coincides with φ outside N_1 and is 2δ-close to φ, provided $\operatorname{mesh} K_1 < \delta$. If $\delta < \varepsilon/56$, the statement follows from

Claim 8.3. *Let k be a positive integer and $\psi_k : X^2 \to Q^2$ an isovariant map, n_k-close to φ and coinciding with φ outside N_k. Then ψ_k is isovariantly $(14n_k)$-homotopic to a map ψ_{k+1}, n_{k+1}-close to φ and coinciding with φ outside N_{k+1}.*

Proof. Observe that in each simplex of N_k there is at least one simplex of N_{k+1}. Therefore $P_{k+1} = \overline{X^2 \setminus N_{k+1}}$ equivariantly strong deformation retracts onto $P_k \cup Y$, where Y^{2n-1} is an equivariant subpolyhedron of N_k of dimension $2n-1$. Let $r_t : P_{k+1} \to P_{k+1}$ denote the deformation retraction, so that $r_1(P_{k+1}) = P_k \cup Y$; we can assume that r_t moves points $< n_k$.

Now ψ_k coincides with φ on P_k and is n_k-close to φ elsewhere. Hence there is a 'linear' equivariant n_k-homotopy $g_t : X^2 \to Q^2$ from ψ_k to φ, keeping P_k fixed. By equivariant general position we can assume that $g_t|_{Y^{2n-1}}$ does not meet Δ_Q.

Define an equivariant $(3n_k)$-homotopy $\Psi_t : P_{k+1} \to Q^2 \setminus \Delta_Q$ between $\psi_k|_{P_{k+1}}$ and $\varphi|_{P_{k+1}}$ as follows. Put $\Psi_t = g_0 \circ r_{3t}$ for $t \in [0, \frac{1}{3}]$, $\Psi_t = g_{3t-1} \circ r_1$ for $t \in [\frac{1}{3}, \frac{2}{3}]$, and $\Psi_t = g_1 \circ r_{3-3t}$ for $t \in [\frac{2}{3}, 1]$. By the equivariant Borsuk Lemma Ψ_t extends to an isovariant $(4n_k)$-homotopy $\Psi_t : X^2 \to Q^2$ between ψ_k and a map Ψ_1 such that $\Psi_1 = \varphi$ on P_{k+1} and Ψ_1 is $4n_k$-close to φ. The rest of the proof goes as in 8.2. $\square$

Remark 8.4. One can give an alternative proof of 8.3 using equivariant obstruction theory (see [22]) as follows. Since the pair (P_{k+1}, P_k) is equivariantly homotopy equivalent to $(P_k \cup Y^{2n-1}, P_k)$, the equivariant cohomology group $H^{2n}_{\mathrm{eq}}(P_{k+1}, P_k, \pi)$ is zero for any group π. Since ψ_k and φ are close, and $\pi_i(\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \setminus \Delta_{\mathbb{R}^{2n+1}}) = 0$ for $i < 2n$, the difference cocycle $d(\psi_k, \varphi) \in H^i_{\mathrm{eq}}(P_{k+1}, P_k, \pi_i(Q^2 \setminus \Delta_Q))$ can be shown to vanish for $i < 2n$. Hence ψ_k can be isovariantly homotoped, keeping P_k fixed, so as to agree with φ on P_{k+1}.

Acknowledgements

I wish to thank P.M. Akhmet'ev, E.V. Ščepin and A.B. Skopenkov for suggesting the problem, attention, and help, which made the present work possible. I am especially indebted to Petya Akhmet'ev for numerous discussions which eventually led to the present understanding of the phenomena under study, and to N.B. Brodskij, A.N. Dranishnikov and R.V. Mikhailov for pointing out some gaps in my arguments. I would like to acknowledge D.A. Botin, A.V. Černavskij, I.A. Chubarov, U. Koschorke, W. Lewis, D.V. Millionščikov, V.M. Nezhinskij, D. Repovš, R.R. Sadykov, K.R. Salikhov, M.A. Štan'ko, E.G. Skl'arenko, Yu.A. Turygin and my scientific advisor Yu.P. Solovjov for useful discussions and important remarks.

References

[1] S.M. Ageev, D. Repovš, E.V. Ščepin, On softness of Dranishnikov's map, Trudy Mat. Inst. Steklova 212 (1996) 7–32; English translation: Proc. Steklov Inst. Math. 212 (1996) 3–27.
[2] P.M. Akhmet'ev, On isotopic and discrete realizations of mappings of n-sphere to Euclidean space, Mat. Sb. 187 (7) (1996) 3–34; English translation: Math. USSR-Sb. 187 (1996) 951–980.
[3] P.M. Akhmet'ev, The generalized Daverman problem, in: Trudy Mezhd. Mat. Konf. k 90-letiju L.S. Pontryagina, VINITI, Moscow, 2000, pp. 1–11 (in Russian).
[4] P.M. Akhmet'ev, Embeddings of compacta, stable homotopy groups of spheres and singularity theory, Uspekhi Mat. Nauk 55 (3) (2000) 3–62; English translation: Russ. Math. Surv. 55 (2000) 405–462.
[5] P.M. Akhmet'ev, S.A. Melikhov, On isotopic realizability of continuous mappings, in: O.Ya. Viro, N.Yu. Netsvetaev (Eds.), Geometry and Topology, Part 5: Proc. Rokhlin Memorial, Zap. Nauch. Sem. POMI, Vol. 267, 2000, pp. 53–87 (in Russian); Erratum, Zap. Nauch. Sem. POMI, to appear.
[6] P.M. Akhmet'ev, D. Repovš, A.B. Skopenkov, Obstructions to approximating maps of surfaces into $\mathbb{R}^4$ by embeddings, Topology Appl. 123 (2002), to appear.
[7] H.W. Berkowitz, J. Dancis, PL approximations of embeddings and isotopies of polyhedra in the metastable range, in: J.C. Cantrell, C.H. Edwards Jr. (Eds.), Topology of Manifolds, Markham, Chicago, IL, 1970, pp. 341–352.
[8] R.H. Bing, A homogeneous indecomposable plane continuum, Duke Math. J. 15 (1948) 729–742.
[9] R.H. Bing, A simple closed curve that pierces no disk, J. Math. Pure Appl. 35 (1956) 337–343.
[10] R.H. Bing, Vertical general position, in: Geometric Topology, Lecture Notes in Math., Vol. 438, Springer, Berlin, 1975, pp. 16–42.
[11] R.H. Bing, J.M. Kister, Taming complexes in hyperplanes, Duke Math. J. 31 (1964) 491–512.
[12] J.L. Bryant, Approximating embeddings of polyhedra in codimension three, Trans. Amer. Math. Soc. 170 (1972) 85–95.
[13] J.L. Bryant, Triangulation and general position of PL diagrams, Topology Appl. 34 (1990) 211–233.

[14] J.L. Bryant, C.L. Seebeck III, Locally nice embeddings in codimension 3, Bull. Amer. Math. Soc. 74 (1968) 378–380.
[15] J.L. Bryant, C.L. Seebeck III, Locally nice embeddings in codimension 3, Quart. J. Math. 21 (1970) 265–272.
[16] S.E. Capell, J.L. Shaneson, Close codimension two embeddings of even-dimensional manifolds, Amer. J. Math. 97 (1975) 733–740.
[17] A.V. Černavskij, Topological embeddings of polyhedra in euclidean spaces, Dokl. Akad. Nauk 165 (1965) 1257–1260; English translation: Soviet Math. Dokl. 6 (1965) 1606–1610.
[18] A.V. Černavskij, Topological embeddings of manifolds, Dokl. Akad. Nauk 187 (6) (1969) 1247–1249; English translation: Soviet Math. Dokl. 10 (4) (1969) 1247–1249.
[19] A.V. Černavskij, Local contractibility of the homeomorphism group of a manifold, Mat. Sb. 79 (3) (1969) 307–356; English translation: Math. USSR-Sb. 8 (1969) 287–333.
[20] A.V. Černavskij, Piece-wise linear approximation of embeddings of cells and spheres in codimensions greater than two; Supplement: Piece-wise linear approximation of embeddings of manifolds in codimensions greater than two, Mat. Sb. 80 (1969) 339–364; Mat. Sb. 82 (1970) 499–500; English translation: Math. USSR-Sb. 9 (1969) 321–343; Math. USSR-Sb. 11 (1970) 465–466.
[21] R. Conelly, Unknotting close polyhedra in codimension 3, in: J.C. Cantrell, C.H. Edwards Jr. (Eds.), Topology of Manifolds, Markham, Chicago, IL, 1970, pp. 384–388.
[22] P.E. Conner, E.E. Floyd, Fixed point free involutions and equivariant maps, Bull. Amer. Math. Soc. 66 (1960) 416–441.
[23] R.J. Daverman, Decompositions of Manifolds, Academic Press, Orlando, FL, 1986.
[24] A.N. Dranishnikov, D. Repovš, E.V. Ščepin, On intersections of compacta of complementary dimension in Euclidean space, Topology Appl. 38 (1991) 237–253.
[25] R.D. Edwards, The equivalence of close piecewise-linear embeddings, Gen. Topology Appl. 5 (1975) 147–180.
[26] R.D. Edwards, Approximating codimension $\geqslant 3$ σ-compacta with locally homotopically unknotted embeddings, Topology Appl. 24 (1986) 95–122.
[27] R.D. Edwards, R.C. Kirby, Deformations of spaces of embeddings, Ann. of Math. 93 (1971) 63–88.
[28] S. Eilenberg, N.E. Steenrod, Foundations of Algebraic Topology, Princeton Univ. Press, Princeton, NJ, 1952.
[29] S. Ferry, Remarks on Steenrod homology, in: Novikov Conjecture, Index Theorems and Rigidity (Vol. 2), London Math. Soc. Lecture Note Series, Vol. 227, Cambridge Univ. Press, Cambridge, 1995, pp. 149–166.
[30] R.H. Fox, A quick trip through knot theory, in: M.K. Fort Jr. (Ed.), Topology of 3-Manifolds and Related Topics, Prentice-Hall, Englewood Cliffs, NJ, 1962, pp. 120–167.
[31] R.H. Fox, O.G. Harrold, The Wilder arcs, in: M.K. Fort Jr. (Ed.), Topology of 3-Manifolds and Related Topics, Prentice-Hall, Englewood Cliffs, NJ, 1962, pp. 184–186.
[32] B.I. Gray, Spaces of the same n-type for all n, Topology 5 (1966) 241–243.
[33] A. Haefliger, Plongements différentiables de variétés dans variétés, Comment. Math. Helv. 36 (1961) 47–82.
[34] A. Haefliger, Knotted $(4k-1)$-spheres in $6k$-space, Ann. of Math. 75 (1962) 452–466.
[35] A. Haefliger, Plongements différentiables dans le domain stable, Comment. Math. Helv. 37 (1962–63) 155–176.
[36] A.E. Harlap, Local homology and cohomology, homological dimension and generalized manifolds, Mat. Sb. 96 (3) (1975) 347–373; English translation: Math. USSR-Sb. 25 (3) (1975) 323–349.
[37] L.S. Harris, Intersections and embeddings of polyhedra, Topology 8 (1969) 1–26.
[38] M.W. Hirsch, Differential Topology, Springer, New York, 1976.

[39] W. Holsztyński, Approximation by homeomorphisms and solution of P. Blass problem on pseudo-isotopy, Proc. Amer. Math. Soc. 27 (3) (1971) 598–602.
[40] J.F.P. Hudson, Concordance, isotopy and diffeotopy, Ann. of Math. 91 (1970) 425–448.
[41] J.F.P. Hudson, W.B.R. Lickorish, Extending piecewise-linear concordances, Quart. J. Math. 22 (1971) 1–12.
[42] C. Kearton, W.B.R. Lickorish, Piecewise-linear critical levels and collapsing, Trans. Amer. Math. Soc. 170 (1972) 415–423.
[43] J.E. Keesling, D.C. Wilson, Embedding T^n-like continua in Euclidean space, Topology Appl. 21 (1985) 241–249.
[44] L.V. Keldyš, Topological embeddings and pseudoisotopy, Dokl. Akad. Nauk 169 (6) (1966) 1262–1265; English translation: Soviet Math. Dokl. 7 (1966) 1078–1082.
[45] L.V. Keldyš, Topological embeddings into a manifold and pseudoisotopy, Mat. Sb. 71 (4) (1966) 433–453; English translation: Amer. Math. Soc. Transl. 84 (1969) 253–275.
[46] L.V. Keldyš, Topological Embeddings in Euclidean Space, Trudy Mat. Inst. Steklova, Vol. 81, 1966; English translation: American Mathematical Society, Providence, RI, 1968.
[47] L.V. Keldyš, Pseudoisotopy for certain locally knotted simple arcs in E^3, Dokl. Akad. Nauk 200 (1) (1971) 21–23; English translation: Soviet Math. Dokl. 12 (1971) 1247–1249.
[48] R.C. Kirby, Smoothing locally flat imbeddings of differentiable manifolds, Topology 6 (1967) 207–220.
[49] R.C. Kirby, Lectures on Triangulations of Manifolds, UCLA Press, Los Angeles, CA, 1969.
[50] R.C. Kirby, L.C. Siebenmann, Foundational Essays on Topological Manifolds, Smoothing and Triangulations, Ann. Math. Studies, Vol. 88, Princeton Univ. Press, Princeton, NJ, 1977.
[51] S. Kojima, M. Yamasaki, Some new invariants of links, Invent. Math. 54 (3) (1979) 213–228.
[52] N.H. Kuiper, R.K. Lashof, Microbundles and bundles, Invent. Math. 1 (1966) 1–17.
[53] R. Lashof, Embedding spaces, Illinois J. Math. 20 (1976) 144–154.
[54] W. Lewis, The pseudo-arc, Bol. Soc. Mat. Mexicana 5 (1999) 25–77.
[55] W.B.R. Lickorish, The piecewise-linear unknotting of cones, Topology 4 (1965) 67–91.
[56] E. Lusk, Level-preserving approximations and isotopies and homotopy groups of spaces of embeddings, Illinois J. Math. 18 (1974) 147–159.
[57] W.S. Massey, Homology and Cohomology Theory, Marcel Dekker, New York, 1978.
[58] S.A. Melikhov, (Singular) link concordance implies link homotopy in codimension $\geqslant 3$, Preprint, 1998; announce, Pseudo-homotopy implies homotopy for singular links of codimension $\geqslant 3$, Uspekhi Mat. Nauk 55 (3) (2000) 183–184; English translation: Russ. Math. Surveys 55 (2000) 589–590.
[59] S.A. Melikhov, On isotopic realizability of maps factored through a hyperplane, Preprint.
[60] S.A. Melikhov, R.V. Mikhailov, Links modulo knots and the Isotopic Realization Problem Uspekhi Mat. Nauk 56 (2) (2001) 219–220; English translation, Russ. Math. Surveys, to appear.
[61] S.A. Melikhov, D. Repovš, A geometric filtration of links modulo knots, I, II, Preprint.
[62] R.T. Miller, Close isotopies on piecewise-linear manifolds, Trans. Amer. Math. Soc. 151 (1970) 597–628.
[63] R.T. Miller, Approximating codimension three embeddings, Ann. of Math. 95 (1972) 406–416.
[64] R.T. Miller, Fiber preserving equivalence, Trans. Amer. Math. Soc. 207 (1975) 241–268.
[65] R.T. Miller, Mapping cylinder neighborhoods of some ANR's, Ann. of Math. 103 (1976) 417–427.
[66] J. Milnor, Isotopy of links, in: S. Lefschetz, R.H. Fox, D. Spencer, J.W. Tucker (Eds.), Algebraic Geometry Topology: A Symposium in Honor of S. Lefschetz, Princeton Univ. Press, Princeton, NJ, 1957, pp. 208–306.
[67] P. Minc, Embedding of simplicial arcs into the plane, Topology Proc. 22 (1997) 305–340.
[68] F. Quinn, Ends of maps, III: dimensions 4 and 5 (includes the annulus conjecture), J. Differential Geom. 17 (1982) 503–521.

[69] D. Repovš, A.B. Skopenkov, Embeddability and isotopy of polyhedra in Euclidean spaces, Trudy Mat. Inst. Steklova 212 (1996) 163–178.
[70] D. Repovš, A.B. Skopenkov, A deleted product criterion for approximability of maps by embeddings, Topology Appl. 87 (1998) 1–19.
[71] D. Repovš, A.B. Skopenkov, E.V. Ščepin, On uncountable collections of continua and their span, Colloq. Math. 69 (1995) 289–296.
[72] D. Repovš, A.B. Skopenkov, E.V. Ščepin, On embeddability of $X \times I$ into Euclidean space, Houston J. Math. 21 (1) (1995) 199–204.
[73] C.P. Rourke, Embedded handle theory, concordance and isotopy, in: J.C. Cantrell, C.H. Edwards Jr. (Eds.), Topology of Manifolds, Markham, Chicago, IL, 1970, pp. 431–438.
[74] C.P. Rourke, B.J. Sanderson, Introduction to Piecewise-Linear Topology, Ergebn. der Math., Vol. 69, Springer, New York, 1972.
[75] E.V. Ščepin, M.A. Štan'ko, A spectral criterion for embeddability of compacta in Euclidean space, in: Proc. Leningrad Internat. Topology Conf., Nauka, Leningrad, 1983, pp. 135–142 (in Russian).
[76] J. Segal, A.B. Skopenkov, S. Spież, Embeddings of polyhedra in $\mathbb{R}^m$ and the deleted product obstruction, Topology Appl. 85 (1997) 1–10.
[77] L.C. Siebenmann, Topological manifolds, in: Proc. ICM Nice 1970, Vol. 2, Gauthier-Villars, Paris, 1971, pp. 133–163.
[78] L.C. Siebenmann, Approximating cellular maps by homeomorphisms, Topology 11 (1972) 271–294.
[79] K. Siekłucki, Imbedding of mappings, Bull. Acad. Polon. Sci. (Math. Ser.) 16 (2) (1968) 87–93.
[80] K. Siekłucki, Realization of mappings, Fund. Math. 65 (1969) 325–343.
[81] C.D. Sikkema, Pseudo-isotopies of arcs and knots, Proc. Amer. Math. Soc. 31 (1972) 615–616.
[82] A.B. Skopenkov, On the deleted product criterion for embeddability of manifolds in $\mathbb{R}^m$, Comment. Math. Helv. 72 (1997) 543–555.
[83] A.B. Skopenkov, On the Haefliger–Hirsch–Wu invariants for embeddings and immersions, Preprint.
[84] S. Spież, On pairs of compacta with $\dim(X \times Y) < \dim X + \dim Y$, Fund. Math. 135 (1990) 127–145.
[85] S. Spież, H. Toruńczyk, Moving compacta in $\mathbb{R}^m$ apart, Topology Appl. 41 (1991) 193–204.
[86] R.J. Stern, Classification theorems for parametrized families of smooth or piecewise linear manifold structures representing a submanifold, Illinois J. Math. 21 (1) (1977) 190–219.
[87] E.C. Zeeman, Unknotting combinatorial balls, Ann. of Math. 78 (1963) 501–526.