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Multiple Solutions for $2m^{\text{th}}$-Order Sturm-Liouville Boundary Value Problems

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Abstract—Growth conditions are imposed on f such that the following boundary value problem: $(-1)^m y^{(2m)} = f(t,y)$, $\alpha_{i+1}y^{(2i)}(0) - \beta_{i+1}y^{(2i+1)}(0) = \gamma_{i+1}y^{(2i)}(1) + \delta_{i+1}y^{(2i+1)}(1) = 0$, $0 \le i \le m-1$, has an arbitrary number of positive solutions.

Keywords—Boundary value problem, Multiple solutions.

1. INTRODUCTION

Boundary value problems for even-order differential equations can arise, especially for fourth-order equations, in applications such as

(a) modeling a number of axially loaded beams fastened together with boundary conditions involving displacement (or deflection at ends), velocity (or vibration at ends), bending moments, and shear forces; see [1],

(b) modeling behavior of a compressed beam subjected to a load causing buckling with the stipulation that the ends are constrained to remain straight and there is zero end shear stress (such as deflection of girders in multilevel buildings as well as deflection of flat-bed trailers in tractor-trailer trucks); see [2], and

(c) modeling the effects of soil settlement on elastically bedded building girders loaded by concentrated forces; see [3].

Meirovitch [4] used higher even-order boundary value problems in studying the open-loop control of a distributed structure whose undamped behavior is governed by

$$Lw(x) + m(x)w(x) = f(x), \qquad 0 < x < L,$$

$w(x)$ is displacement at a point x in the structure, L is a homogeneous differential stiffness operator of order $2p$, $m(x)$ is the mass density, and $f(x)$ is a distributed control force. The

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solution $w(x)$ is subject to boundary conditions

$$B_i w(x) = 0, \qquad x = 0, L, \quad 1 \le i \le p,$$

where the B_i are differential operators of maximum order $2p-1$.

In this paper, with $m \ge 1$ fixed, we shall be concerned with the existence of multiple positive solutions of the $2m^{\text{th}}$-order ordinary differential equation

$$(-1)^m y^{(2m)} = f(t,y), \qquad 0 \le t \le 1, \tag{1.1}$$

satisfying the boundary conditions

$$\begin{aligned} \alpha_{i+1} y^{(2i)}(0) - \beta_{i+1} y^{(2i+1)}(0) = 0, \qquad 0 \le i \le m-1, \\ \gamma_{i+1} y^{(2i)}(1) + \delta_{i+1} y^{(2i+1)}(1) = 0, \qquad 0 \le j \le m-1, \end{aligned} \tag{1.2}$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $f(t,y) \not\equiv 0$ on any subinterval of $[0,1]$, and for all $0 < y < \infty$, and for each $1 \le i \le m$, $\alpha_i, \beta_i, \gamma_i, \delta_i \ge 0$ such that

$$\rho_i \equiv \gamma_i \beta_i + \alpha_i \gamma_i + \alpha_i \delta_i > 0. \tag{1.3}$$

When $m = 1$, the boundary value problem (1.1),(1.2) has received much attention in determining conditions on f for which there are either at least one, at least two, or at least three positive solutions. Some of those results, along with excellent lists of references are contained in [5–7]. In addition, when $m = 1$, Erbe and Tang [8] gave sufficient conditions on f for the existence of any number of positive solutions of (1.1),(1.2).

For $m > 1$, recent attention has been devoted to multiple solutions of (1.1) satisfying usually boundary conditions of the conjugate or right focal types; see, for example, [9–12], and the recent book by Agarwal, O'Regan and Wong [13] is an excellent treatise on positive solutions. The techniques for obtaining multiple solutions in most of the above cited papers have involved applications of fixed-point theorems due to Krasnosel'skii [14] and to Leggett and Williams [15].

The techniques in this work will follow along the lines of those introduced by Erbe and Tang [8] when $m = 1$, and then were extended in [16] for the case when $m > 1$ and $\alpha_i = \alpha_{i+1}$, $\beta_i = \beta_{i+1}$, $\gamma_i = \gamma_{i+1}$, $\delta_i = \delta_{i+1}$, to yield symmetric solutions. For this work, their techniques will be used in conjunction with certain properties of the Green's function, $G_i(t,s)$, $1 \le i \le m$, for each of the boundary value problems

$$-u'' = 0, \tag{1.4}$$

$$\begin{aligned} \alpha_i u(0) - \beta_i u'(0) = 0, \\ \gamma_i u(1) + \delta_i u'(1) = 0. \end{aligned} \tag{1.5}$$

Growth conditions will be imposed on $h(t,y) = f(t,y)/y$, $0 \le t \le 1$, $0 < y < \infty$, which yield the existence of any number of positive solutions of (1.1),(1.2) that lie in nested annular-like regions. In that direction, we define the extended real-valued functions $h_0(t)$ and $h_\infty(t)$ by

$$\begin{aligned} h_0(t) &= \lim_{y \to 0^+} h(t,y), \\ h_\infty(t) &= \lim_{y \to \infty} h(t,y). \end{aligned}$$

The respective cases, when f is superlinear (at both $y = 0$ and $y = \infty$) or f is sublinear (at both $y = 0$ and $y = \infty$), have been studied in [6,8,17], for $m = 1$. Recently (yet for $m = 1$), Lian, Wong and Yeh [18] assumed particular smallness or largeness conditions, not as restrictive as superlinearity nor sublinearity, on $h(t,y)$ at $y = 0$ and $y = \infty$ to obtain multiple positive solutions.

2. EXISTENCE OF A POSITIVE SOLUTION

In this section, we apply a Krasnosel'skii [14] fixed-point theorem for operators which are of an expansion/compression type with respect to an annular region in a cone.

THEOREM 2.1. *Let $\mathcal{E}$ be a Banach space, and let $\mathcal{K} \subset \mathcal{E}$ be a cone in $\mathcal{E}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{E}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_1$, *and* $\|Au\| \geq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_2$, *or*
(ii) $\|Au\| \geq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_1$, *and* $\|Au\| \leq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_2$.

Then A has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We will apply Theorem 2.1 to a completely continuous operator whose kernel is the Green's function for

$$(-1)^m y^{(2m)} = 0, \tag{2.1}$$

satisfying boundary conditions (1.2). For $1 \leq i \leq m$, let $G_i(t,s)$ be the Green's function for (1.4),(1.5). Then, for $1 \leq i \leq m$,

$$G_i(t,s) = \frac{1}{\rho_i} \begin{cases} (\gamma_i + \delta_i - \gamma_i t)(\beta_i + \alpha_i s), & 0 \leq s \leq t \leq 1, \\ (\beta_i + \alpha_i t)(\gamma_i + \delta_i - \gamma_i s), & 0 \leq t \leq s \leq 1, \end{cases}$$

where ρ_i is defined by (1.3). Next, we set

$$H_1(t,s) = G_1(t,s),$$

and for $2 \leq j \leq m$, we recursively define

$$H_j(t,s) = \int_0^1 H_{j-1}(t,r))G_j(r,s)\,dr, \qquad 0 \leq s,t \leq 1. \tag{2.2}$$

Then $H_m(t,s)$ is the Green's function for (2.1),(1.2). It is rather straightforward that, for $1 \leq i \leq m$,

$$0 \leq G_i(t,s) \leq G_i(s,s), \qquad 0 \leq t,s \leq 1, \tag{2.3}$$

and

$$0 < \sigma_i G_i(s,s) \leq G_i(t,s), \qquad \frac{1}{4} \leq t \leq \frac{3}{4}, \quad 0 < s < 1, \tag{2.4}$$

where

$$\sigma_i = \min\left\{ \frac{\gamma_i + 4\delta_i}{4(\gamma_i + \delta_i)}, \frac{\alpha_i + 4\beta_i}{4(\alpha_i + \beta_i)} \right\} < 1.$$

If we define

$$L_i = \int_0^1 G_i(r,r)\,dr, \qquad 1 \leq i \leq m$$

and

$$K_i = \int_{1/4}^{3/4} G_i(r,r)\,dr, \qquad 1 \leq i \leq m,$$

it follows that

$$0 \leq H_m(t,s) \leq \prod_{j=1}^{m-1} L_j G_m(s,s), \qquad 0 \leq s,t \leq 1 \tag{2.5}$$

and

$$H_m(t,s) \geq \sigma_m \prod_{j=1}^{m-1} \sigma_j K_j G_m(s,s), \qquad \frac{1}{4} \leq t \leq \frac{3}{4}, \quad 0 \leq s \leq 1. \tag{2.6}$$

Inequalities (2.5) and (2.6) will play a fundamental role in the growth constraints we impose on f which yield positive and multiple solutions of (1.1),(1.2).

Now, let the Banach space $\mathcal{E} = C[0,1]$ equipped with norm $\|y\| = \max_{0 \leq t \leq 1} |y(t)|$, $y \in C[0,1]$. Let

$$M = \sigma_m \prod_{j=1}^{m-1} \frac{\sigma_j K_j}{L_j} < 1,$$

and then define a cone $\mathcal{K} \subset \mathcal{E}$ by

$$\mathcal{K} = \left\{ v \in \mathcal{E} \mid v(t) \geq 0 \text{ on } [0,1] \text{ and } \min_{1/4 \leq t \leq 3/4} v(t) \geq M\|v\| \right\}.$$

In obtaining solutions of (1.1),(1.2) which are positive with respect to the cone $\mathcal{K}$, we seek a fixed point of the completely continuous integral operator $A : \mathcal{E} \to \mathcal{E}$ defined by

$$(Ay)(t) = \int_0^1 H_m(t,s) f(s, y(s))\, ds.$$

From inequalities (2.5) and (2.6), it is immediate that $A : \mathcal{K} \to \mathcal{K}$.

For notational convenience, define the constants

$$\eta = \left(\prod_{j=1}^{m} L_j \right)^{-1},$$

$$\mu = \left(\prod_{j=1}^{m} \sigma_j K_j \right)^{-1}.$$

The growth restrictions on f which will yield the existence of positive and multiple solutions are as follows.

(C1) There exists a $p > 0$ such that $f(t,y) \leq \eta p$ for $0 \leq t \leq 1$ and $0 \leq y \leq p$.

(C2) There exists a $q > 0$ such that $f(t,y) \geq \mu q$ for $1/4 \leq t \leq 3/4$ and $Mq \leq y \leq q$.

THEOREM 2.2. *Suppose there exist positive numbers $p \neq q$ such that Condition (C1) is satisfied with respect to p and Condition (C2) is satisfied with respect to q. Then (1.1),(1.2) has a positive solution y such that $\|y\|$ lies between p and q.*

PROOF. Without loss of generality, we may assume $0 < p < q$. Define open sets

$$\Omega_p = \{ y \in C[0,1] \mid \|y\| < p \}$$

and

$$\Omega_q = \{ y \in C[0,1] \mid \|y\| < q \}.$$

Then $0 \in \Omega_p \subset \Omega_q$. Now, for $y \in \mathcal{K} \cap \partial\Omega_p$, so that $\|y\| = p$, we have from (2.5) that, for $0 \leq t \leq 1$,

$$\begin{aligned} Ay(t) &= \int_0^1 H_m(t,s) f(s,y(s))\, ds \\ &\leq \int_0^1 \prod_{j=1}^{m-1} L_j G_m(s,s) f(s,y(s))\, ds \\ &\leq \prod_{j=1}^{m-1} L_j \eta p \int_0^1 G_m(s,s)\, ds \\ &= p \\ &= \|y\|. \end{aligned}$$

Thus, $\|Ay\| \leq \|y\|$, for $y \in \mathcal{K} \cap \partial\Omega_p$.

Similarly, if $y \in \mathcal{K} \cap \partial\Omega_q$, so that $\|y\| = q$ and $Mq \le y(s) \le q$, for $1/4 \le s \le 3/4$, we have from (2.6) that, for $1/4 \le t \le 3/4$,

$$\begin{aligned} Ay(t) &= \int_0^1 H_m(t,s) f(s, y(s))\, ds \\ &\ge \int_{1/4}^{3/4} H_m(t,s) f(s, y(s))\, ds \\ &\ge \int_{1/4}^{3/4} H_m(t,s) \mu q\, ds \\ &\ge \mu q \sigma_m \prod_{j=1}^{m-1} \sigma_j K_j \int_{1/4}^{3/4} G_m(s,s)\, ds \\ &= q \\ &= \|y\|, \end{aligned}$$

and so $\|Ay\| \ge \|y\|$, for $y \in \mathcal{K} \cap \partial\Omega_q$. By Theorem 2.1, A has a fixed point $y \in \mathcal{K} \cap (\overline{\Omega}_q \setminus \Omega_p)$, which is a positive solution of (1.1),(1.2) such that $p \le \|y\| \le q$. ∎

COROLLARY 2.1. *The boundary value problem (1.1),(1.2) has a positive solution provided, either*

(C3) $h_0(t) < \eta$, *for* $0 \le t \le 1$, *and* $h_\infty(t) > \mu/M$, *for* $1/4 \le t \le 3/4$, *or*
(C4) $h_0(t) > \mu/M$, *for* $1/4 \le t \le 3/4$, *and* $h_\infty(t) < \eta$, *for* $0 \le t \le 1$.

PROOF. Suppose first that (C3) holds. Then, there exist sufficiently small $p > 0$ and sufficiently large $q > 0$ such that

$$\frac{f(t,y)}{y} \le \eta, \qquad 0 \le t \le 1, \quad 0 < y \le p$$

and

$$\frac{f(t,y)}{y} \ge \frac{\mu}{M}, \qquad \frac{1}{4} \le t \le \frac{3}{4}, \quad y \ge Mq.$$

Thus,

$$f(t,y) \le \eta y \le \eta p, \qquad 0 \le t \le 1, \quad 0 \le y \le p$$

and

$$f(t,y) \ge \frac{\mu}{M} y \ge \mu q, \qquad \frac{1}{4} \le t \le \frac{3}{4}, \quad Mq \le y \le q.$$

In particular, both (C1) and (C2) hold, and (1.1),(1.2) has positive solutions by Theorem 2.2.

For the remainder of the proof, assume that (C4) holds. Then, there are $0 < p < q$ such that

$$\frac{f(t,y)}{y} \ge \frac{\mu}{M}, \qquad \frac{1}{4} \le t \le \frac{3}{4}, \quad 0 < y \le p, \tag{2.7}$$

$$\frac{f(t,y)}{y} \le \eta, \qquad 0 \le t \le 1, \quad y \ge q. \tag{2.8}$$

From (2.7) it follows that

$$f(t,y) \ge \frac{\mu}{M} y \ge \mu p, \qquad \frac{1}{4} \le t \le \frac{3}{4}, \quad Mp \le y \le p,$$

so that (C2) is satisfied with respect to p.

In dealing with inequality (2.8), we wish to show that (C1) is satisfied. To that end, there are two cases to consider:

(a) $f(t,y)$ is bounded, or
(b) $f(t,y)$ is unbounded.

CASE (a). Suppose there exists $N > 0$ such that $f(t, y) \leq N$, for $0 \leq t \leq 1$ and $0 \leq y < \infty$. By (2.8), there is an $r \geq \max\{q, N/\eta\}$ such that $f(t,y) \leq N \leq \eta r$, for $0 \leq t \leq 1$ and $0 \leq y \leq r$, and thus, (C1) is satisfied with respect to r.

CASE (b). For the setting that f is unbounded, there exist $t_0 \in [0,1]$ and $\bar{r} \geq q$ such that $f(t,y) \leq f(t_0, \bar{r})$, for $0 \leq t \leq 1$ and $0 \leq y \leq \bar{r}$. Then, $f(t,y) \leq f(t_0,\bar{r}) \leq \eta\bar{r}$, for $0 \leq t \leq 1$ and $0 \leq y \leq \bar{r}$, and (C1) is satisfied with respect to $\bar{r}$.

Thus in both Cases (a) and (b), Condition (C1) is satisfied, and Theorem 2.2 yields the conclusion. ∎

3. ANY NUMBER OF POSITIVE SOLUTIONS

In this section, we show that any number of positive solutions of (1.1),(1.2) can be obtained when appropriate combinations of assumptions like (C1)–(C4) are imposed on f. We begin the pattern by establishing the existence of at least two positive solutions.

THEOREM 3.1. *The boundary value problem (1.1),(1.2) has at least two positive solutions, y_1 and y_2, if (C1) is satisfied for some $p > 0$, and in addition, both*

$$h_0(t) > \frac{\mu}{M}, \quad \frac{1}{4} \leq t \leq \frac{3}{4} \qquad \text{and} \qquad h_\infty(t) > \frac{\mu}{M}, \quad \frac{1}{4} \leq t \leq \frac{3}{4}. \tag{3.1}$$

Moreover, $0 < \|y_1\| < p < \|y_2\|$.

PROOF. Somewhat along the lines of the proof of Corollary 2.1, there exist $0 < p_1 < p < p_2$ for which

$$f(t,y) \geq \mu p_1, \qquad \frac{1}{4} \leq t \leq \frac{3}{4}, \quad Mp_1 \leq y \leq p_1$$

and

$$f(t,y) \geq \mu p_2, \qquad \frac{1}{4} \leq t \leq \frac{3}{4}, \quad Mp_2 \leq y \leq p_2.$$

By Theorem 2.2, there exist solutions, y_1 and y_2, of (1.1) and (1.2) satisfying $0 < p_1 < \|y_1\| < p < \|y_2\| < p_2$. ∎

In a completely analogous manner, the next result is also obtained.

THEOREM 3.2. *The boundary value problem (1.1),(1.2) has at least two positive solutions, y_1 and y_2, if (C2) is satisfied for some $q > 0$, and in addition, both*

$$h_0(t) < \eta, \quad 0 \leq t \leq 1 \qquad \text{and} \qquad h_\infty(t) < \eta, \quad 0 \leq t \leq 1. \tag{3.2}$$

Moreover, $0 < \|y_1\| < q < \|y_2\|$.

To set the pattern for the manner in which an arbitrary number of positive solutions are obtained, we state an existence result for at least three positive solutions.

THEOREM 3.3. *Suppose Condition (C3) (or respectively, Condition (C4)), is satisfied, and suppose there exist $0 < p_1 < p_2$ such that (C1) holds with respect to $p = p_2$ (respectively, with respect to $p = p_1$), and (C2) holds with respect to $q = p_1$ (respectively, with respect to $q = p_2$). Then, the boundary value problem (1.1),(1.2) has at least three positive solutions, y_1, y_2, and y_3, satisfying $0 < \|y_1\| < p_1 < \|y_2\| < p_2 < \|y_3\|$.*

We now state sufficient conditions under which there are n positive solutions of (1.1),(1.2), for any $n \in \mathbb{N}$. We state these results in terms of whether n is odd or even.

THEOREM 3.4. *Let $n = 2k+1$, where $k \in \mathbb{N}$, be given. Suppose Condition (C3) (or respectively, Condition (C4)), is satisfied, and suppose there exist $0 < p_1 < \cdots < p_{n-1}$ such that (C2) holds (respectively, (C1) holds), with respect to p_{2i-1}, $1 \leq i \leq k$, and (C1) holds (respectively, (C2) holds), with respect to p_{2i}, $1 \leq i \leq k$. Then, the boundary value problem (1.1),(1.2) has at* *least n positive solutions, $y_1, y_2, \ldots, y_n$, satisfying $0 < \|y_1\| < p_1 < \|y_2\| < p_2 < \cdots < \|y_{n-1}\| < p_{n-1} < \|y_n\|$.*

THEOREM 3.5. *Let $n = 2k$, where $k \in \mathbb{N}$, be given. Suppose (3.1) (or respectively, (3.2)), is satisfied, and suppose there exist $0 < p_1 < \cdots < p_{n-1}$ such that (C1) holds (respectively, (C2) holds), with respect to p_{2i-1}, $1 \le i \le k$, and (C2) holds (respectively, (C1) holds), with respect to p_{2i}, $1 \le i \le k-1$. Then, the boundary value problem (1.1),(1.2) has at least n positive solutions, $y_1, y_2, \ldots, y_n$, satisfying $0 < \|y_1\| < p_1 < \|y_2\| < p_2 < \cdots < \|y_{n-1}\| < p_{n-1} < \|y_n\|$.*

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