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Geometric dilation of closed planar curves: New lower bounds ☆

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Abstract

Given two points on a closed planar curve, C, we can divide the length of a shortest connecting path in C by their Euclidean distance. The supremum of these ratios, taken over all pairs of points on the curve, is called the geometric dilation of C. We provide lower bounds for the dilation of closed curves in terms of their geometric properties, and prove that the circle is the only closed curve achieving a dilation of $\pi/2$, which is the smallest dilation possible. Our main tool is a new geometric transformation technique based on the perimeter halving pairs of C.

Keywords: Computational geometry; Convex geometry; Convex curves; Dilation; Detour; Lower bound; Halving pair; Halving pair transformation

1. Introduction

Let C be a simple closed curve in the Euclidean plane. For any two points, p and q, on C let $d_C(p,q)$ denote the minimum length of one of the two curve segments of C connecting p to q. Moreover, let $|pq|$ denote the Euclidean distance of the two points. Then the *geometric dilation*, $\delta(C)$, of C is defined as

$$\delta(C) := \sup_{p,q\in C,\, p\neq q} \frac{d_C(p,q)}{|pq|}. \tag{1}$$

One should note that the geometric dilation is a concept substantially different from the standard graph-theoretic dilation usually studied in the context of spanners [5,6,12], where only the point set of all vertices is considered.

With the geometric dilation defined above, two natural problems arise. From a structural point of view, we would like to have general upper or lower bounds for the dilation of curves, in terms of their elementary geometric properties. From an algorithmic point of view, we would like to efficiently compute the dilation of a concrete curve, given a finite description.

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So far, only the case of open curves and polygonal chains has received some attention. Icking et al. [16] and Aurenhammer et al. [3] established tight upper bounds to the dilation of open planar curves that cannot meander too wildly. The computation of the geometric dilation (then called detour) was first studied in the conference version of [10], where an $O(n \log n)$ approximation algorithm for n-link polygonal chains in the plane was presented. Exact algorithms with a running time in $O(n \text{ polylog } n)$ were independently given by Agarwal et al. [1] and by Langerman et al. [20]; a forthcoming joint paper [2] presents randomized algorithms with running time in $O(n \log n)$ for planar polygonal chains, $O(n \log^2 n)$ for planar trees and cycles, and $O(n^{16/9+\epsilon})$ for polygonal chains in 3D.

The interest in structural results on closed curves is rather recent. Ebbers et al. [9] studied the question of embedding a finite point set into a planar network of low geometric dilation. Among other things it was shown, using Cauchy's surface area formula, that each closed curve has a geometric dilation of at least $\pi/2$, the dilation of the circle.

The concept of geometric dilation was independently introduced by Gromov [14,15] under the notion of *distortion*. He also mentioned the above lower bound property in [14]. By Kusner and Sullivan [19] distortion found its application in knot theory. For more recent results see e.g. Denne et al. [7,8].

In this paper we provide structural results on the geometric dilation of simple planar closed curves. Our main tool is a new transformation technique for closed curves, based on their *halving pairs*.[2] Given a convex curve, C, and a direction, v, there is a unique v-oriented pair of points on C that cut the perimeter of C into two parts of the same length. Similar to the well-known central symmetrization used in convex geometry [11,17], we take the segment connecting this halving pair, and translate its midpoint to the origin. By applying this procedure for each possible direction v, a new curve, C^*, results. By construction, C^* is centrally symmetric. As we shall prove in Section 7, C^* is again convex, and of dilation at most $\delta(C)$.

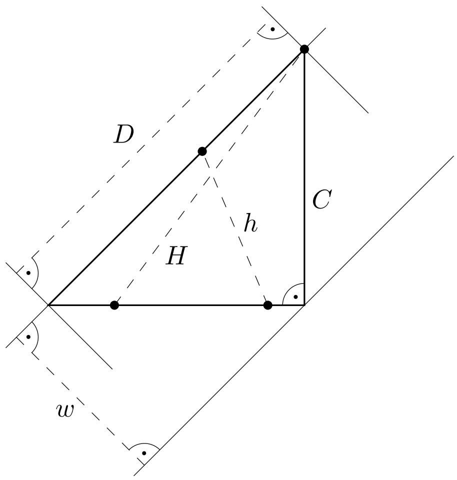

Fig. 1. Diameter D, width w and minimum and maximum halving distance h and H of an isosceles right-angled triangle.

In Section 8 we are using the new halving pair transformation and the central symmetrization method to derive the following new lower bounds on the dilation of a convex closed curve C:

$$\delta(C) \geqslant \arcsin\left(\frac{w}{D}\right) + \sqrt{\left(\frac{D}{w}\right)^2 - 1}, \qquad \delta(C) \geqslant \arcsin\left(\frac{h}{H}\right) + \sqrt{\left(\frac{H}{h}\right)^2 - 1}.$$

Here, w denotes the width of C, D denotes the diameter of C, and h (resp. H) is the minimum (resp. maximum) halving distance of C (see Fig. 1).

These lower bounds attain a minimum value of $\pi/2$ if and only if $w = D$ (resp. $h = H$) holds. Although the circle is not the only closed curve satisfying $w = D$ or $h = H$, we shall derive that only the circle attains dilation $\pi/2$.

[2] Halving pairs were introduced as "partition pairs" in [9].

In order to prove these results, some technical preparations are necessary. In Section 3 we show that the boundary of the convex hull of any simple closed curve C does never have a bigger dilation than C. Then, in Section 4, we make sure that the geometric dilation of a closed convex curve is, in fact, attained by one of its halving pairs. While these facts are not surprising, their exact proofs need some care. Section 5 contains some facts on the dilation of centrally symmetric curves. For the convenience of the reader, we review, in Section 6, the essential properties of the central symmetrization transformation.

2. Definitions and basic properties

Throughout this paper we consider finite, simple,[3] piecewise continuously differentiable curves in the Euclidean plane. A subset $C \subset \mathbb{R}^2$ is such a curve, if it is the image of an injective, continuous, piecewise continuously differentiable function $c : [0, 1] \to \mathbb{R}^2$ or $c : \mathbb{S}^1 \to \mathbb{R}^2$, where $\mathbb{S}^1 := \{p \in \mathbb{R}^2 \mid |p| = 1\}$ denotes the *unit circle*. In the first case it is an *open* curve, in the second case we call it *closed curve* or *cycle*. The *length* of C is given by $|C| := \int |\dot{c}(t)|\,\mathrm{d}t$. Let C° denote the open bounded region encircled by C, called its *interior domain*. The curve C is *convex* if its interior domain C° is convex.

Let C be a curve, and let p and q be two distinct points of C. By $d_C(p, q)$ we denote the C-*distance* of p and q, i.e. the length of the shortest path on C connecting p and q. The dilation $\delta_C(p, q)$ of p and q in C is the ratio of their C-distance and their Euclidean distance. As stated already in the introduction, the (*geometric*) *dilation* of C is the supremum of these dilation values.

Sometimes we will consider an *arc-length parametrization* of a curve C, a continuous, bijective, piecewise continuously differentiable mapping of an interval I onto C so that $|\dot{\bar{c}}(t)| = 1$ almost everywhere. Such a parametrization often helps to simplify the dilation analysis because for any open curve C and values $t_1, t_2 \in I$ we have $d_C(\bar{c}(t_1), \bar{c}(t_2)) = |t_2 - t_1|$. For closed curves the same equation holds if additionally $|t_2 - t_1| \leqslant |C|/2$ is granted.

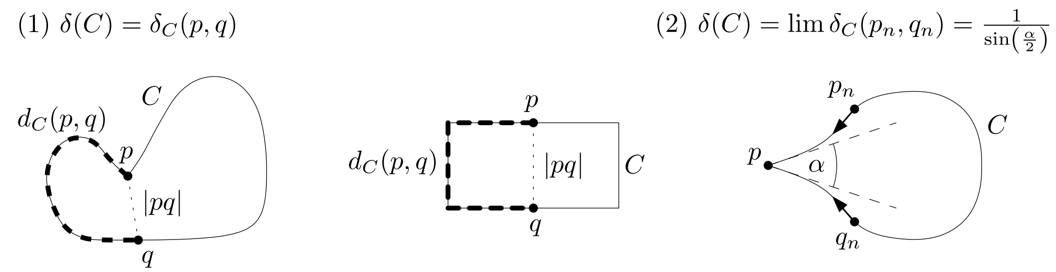

Fig. 2. The maximum dilation $\delta(C)$ is attained by a pair of points (p, q) or by the limit of pairs of points (p_n, q_n) approaching the same point p from opposite sides.

First, we observe that with the exception of one special case the geometric dilation of any cycle C is attained by a pair of points of C:

Lemma 1. *Let C be an arbitrary open or closed curve. Then, at least one of the following cases occurs*:

(1) *The maximum dilation $\delta(C)$ is attained by a pair of points $p, q \in C$, $p \neq q$, i.e. $\delta(C) = \delta_C(p, q)$.*
(2) *The maximum dilation $\delta(C)$ is the limit of dilation values of a sequence of pairs $(p_n, q_n)_{n\in\mathbb{N}} \subset C \times C$, so that $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are approaching the same point $p \in C$ from opposite sides. Let $\alpha \in [0, \pi]$ denote the angle enclosed by the tangents in p. Then,*

$$\delta(C) = \lim_{n\to\infty} \delta_C(p_n, q_n) = \begin{cases} \frac{1}{\sin(\alpha/2)} & \alpha \in]0, \pi], \\ \infty & \alpha = 0. \end{cases}$$

If C is convex, case (1) *must occur.*

[3] A curve is called *simple* if it has no self-intersections.

Proof. Let $(p_n, q_n)_{n\in\mathbb{N}}$ be a maximum sequence, i.e. $\delta_C(p_n, q_n) \nearrow \delta(C)$. Because C is compact in $\mathbb{R}^2$, so is $C \times C$ in $\mathbb{R}^4$. There is a sub-sequence we will from now on denote by $(p_n, q_n)_{n\in\mathbb{N}}$ converging to a pair $(p, q) \in C \times C$.

The continuity of $d_C(.,.)$ implies $d_C(p_n, q_n) \to d_C(p, q)$. And by continuity of $|.|$ we obtain $|p_n q_n| \to |pq|$. If $p \neq q$, this implies $\delta_C(p_n, q_n) \to \delta_C(p, q)$.

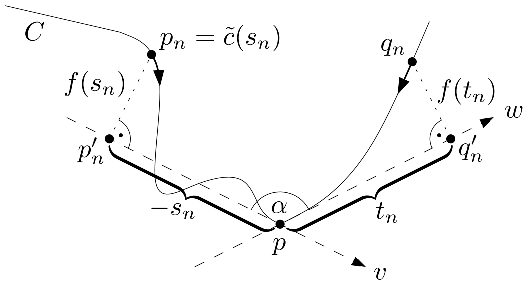

Fig. 3. A dilation maximum sequence $(p_n, q_n)_{n\in\mathbb{N}}$ converging to a single point.

Now we consider the case $p = q$. See Fig. 3. We choose an arbitrary orientation of C. Let $v, w \in \mathbb{S}^1$ be the left- and right-sided normalized derivative vectors of C in p. We use the following parametrization $\tilde{c}:]-\delta, \delta[\to C$ of C in a neighborhood of p. Let $q \in C$ be an arbitrary point in this neighborhood. If q lies to the left of p, then we consider the orthogonal projection q' of q onto the v-tangent through p. Then, $\tilde{c}^{-1}(q) := -|pq'|$. If q lies to the right of p, we analogously denote by q' the orthogonal projection of q onto the w-tangent through p and define $\tilde{c}^{-1}(q) := |pq'|$. Because of the one-sided differentiability of C we can choose $\delta > 0$ small enough so that $\tilde{c}$ is well-defined and continuous at 0.

Obviously, the parameter t is the component of $\tilde{c}(t) - p$ parallel to v (w resp.). Let $f(t)$ denote the orthogonal component oriented in such a way that turning v (w resp.) by 90° anti-clockwise results in a positive $f(t)$. We define $s_n := \tilde{c}^{-1}(p_n)$ and $t_n := \tilde{c}^{-1}(q_n)$. By continuity and injectivity of $\tilde{c}$ we get $s_n \to 0$ and $t_n \to 0$.

By definition, f is continuously differentiable in a neighborhood of p, and we have $f'(t) \to 0$ for $t \to 0$. For any given $\varepsilon > 0$ we can choose a small $\delta > 0$ so that $|f'(t)| < \varepsilon$ for every $t \in]-\delta, \delta[$. And we can choose N big enough such that every s_n and t_n, $n \geqslant N$, are contained in $]-\delta, \delta[$.

If $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ approach p from the same side, these n-values satisfy

$$\delta_C(p_n, q_n) = \frac{|\int_{s_n}^{t_n} \sqrt{1 + (f'(t))^2}\,\mathrm{d}t|}{\sqrt{|t_n - s_n|^2 + |f(t_n) - f(s_n)|^2}} \leqslant \frac{\sqrt{1+\varepsilon^2}|t_n - s_n|}{|t_n - s_n|} \to 1.$$

But if $\delta(C) = \lim \delta_C(p_n, q_n) = 1$, we have $\delta_C(p, q) = 1$ for every pair of points. Hence, case (1) applies, too.

If there is a sub-sequence $(p_n, q_n)_{n\in\mathbb{N}}$ such that p_n and q_n are always on opposite sides of p, by renaming we can build a sequence of the same dilation limit where p_n is always left of p ($s_n < 0$) and q_n is right of p ($t_n > 0$). Again, for a given $\varepsilon > 0$ we can choose a big $N \in \mathbb{N}$ so that for every $n \geqslant N$, $\tilde{c}(t)$ and $f(t)$ are well-defined and continuously differentiable on $[s_n, t_n]$ and $|f'(t)| < \varepsilon$. Clearly, this implies $|f(t)| \leqslant \int_0^t |f'(t)| \leqslant \varepsilon|t|$ on the same interval. The shortest-path distance $d_C(p_n, q_n)$ is bounded by

$$(t_n - s_n) \leqslant d_C(p_n, q_n) = \int_{s_n}^{t_n} \sqrt{1 + \left(f'(t)\right)^2}\,\mathrm{d}t \leqslant \sqrt{1+\varepsilon^2}(t_n - s_n).$$

We apply the triangle inequality to get upper and lower bounds for $|p_n q_n|$ and obtain

$$|p_n q_n| \leqslant |p'_n q'_n| + \left|f(s_n)\right| + \left|f(t_n)\right| \leqslant |p'_n q'_n| + \varepsilon(t_n - s_n) \quad \text{and}$$
$$|p_n q_n| \geqslant |p'_n q'_n| - \left|f(s_n)\right| - \left|f(t_n)\right| \geqslant |p'_n q'_n| - \varepsilon(t_n - s_n),$$

which implies

$$\frac{t_n - s_n}{|p'_n q'_n| + \varepsilon(t_n - s_n)} \leqslant \delta_C(p_n, q_n) \leqslant \frac{\sqrt{1+\varepsilon^2}(t_n - s_n)}{|p'_n q'_n| - \varepsilon(t_n - s_n)}.$$

Hence, $\lim \delta_C(p_n, q_n) = \lim(t_n - s_n)/|p'_n q'_n|$ which is the limit of the dilation values of (p'_n, q'_n) on the polygonal path built by the v- and the w-tangent through p. It is easy to show that the fraction $(t_n - s_n)/|p'_n q'_n|$ cannot be bigger than $1/\sin(\alpha/2)$ where $\alpha \in]0, \pi[$ is the angle enclosed by the v- and the w-tangent. And this maximum is attained by values $s_n = -t_n$. Case $\alpha = 0$ yields $\lim \delta_C(p_n, q_n) = \infty$.

If C is convex, and $(p_n, q_n) \in C \times C$ is a dilation maximum sequence converging to a single point p, we can choose $\tilde{p}$ and $\tilde{q}$ close to p satisfying $-s := |\tilde{p}' p| = |\tilde{q}' p| =: t$ and $|\tilde{p}\tilde{q}| \leqslant |\tilde{p}'\tilde{q}'|$. Hence,

$$\delta_C(\tilde{p}, \tilde{q}) \geqslant \frac{t-s}{|\tilde{p}'\tilde{q}'|} \overset{\text{see above}}{=} \frac{1}{\sin(\alpha/2)} \overset{\text{see above}}{\geqslant} \lim_{n\to\infty} \delta_C(p_n, q_n) = \delta(C).$$

Therefore, for convex cycles there is always a pair of distinct points of C attaining maximum dilation. □

We mentioned before that the halving pairs play a crucial role within the dilation analysis of closed curves. Therefore, we want to define them more formally.

Definition 2 *(Halving Pair).* Let $p \in C$ be a point on a closed curve C. Then the unique *halving partner* $\hat{p} \in C$ of p is characterized by $d_C(p, \hat{p}) = |C|/2$. We say that $(p, \hat{p})$ is a *halving pair* of C, see Fig. 4.

In order to define the halving distance, the following observation is important.

Lemma 3. *For every direction $v \in \mathbb{S}^1 = \{p \in \mathbb{R}^2 \mid |p| = 1\}$ there exists a halving pair $(p, \hat{p})$, i.e. $p - \hat{p} = |p - \hat{p}|v$. For convex cycles, this halving pair is unique.*

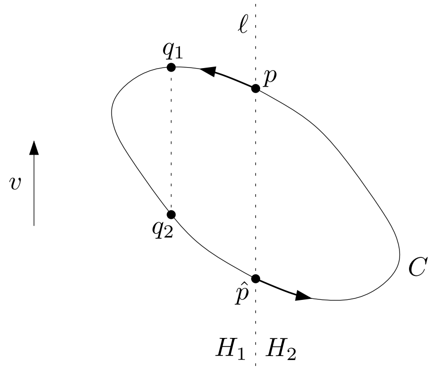

Fig. 4. By moving p and $\hat{p}$ we get a halving pair for every direction. These halving pairs are unique for convex cycles.

Proof. Consider Fig. 4. Clearly, we can find one halving pair $(p, \hat{p})$ of C. Next, we move p and $\hat{p}$ continuously on C so that they keep their C-distance $d_C(p, \hat{p}) = |C|/2$. Eventually, p reaches the former position of $\hat{p}$ and, at the same time, $\hat{p}$ reaches the former position of p. If we continue, finally, both points return to their starting position. Because C is simple, in between, the pair must have attained all possible directions.

Now, let C be convex and let $v \in \mathbb{S}^1$ be an arbitrary direction. Let $(p, \hat{p})$ be a halving pair of direction v. By convexity, the line ℓ through this pair can only intersect with C twice, namely in p and $\hat{p}$. It divides the plane $\mathbb{R}^2$ into two half-planes H_1 and H_2.

Let $(q_1, q_2) \in C^2$ be a pair of points lying in the same half-plane, say H_1. Then, it cannot be a halving pair, since half of the length of C is contained in H_2, and there is some additional length needed to reach q_1 and q_2. Thus, every halving pair $(q, \hat{q})$ except $(p, \hat{p})$ contains one point of H_1 and one point of H_2. It cannot have direction v. □

We keep this in mind while defining the halving distance and comparing it to well-known breadth measures.

Definition 4 *(Breadth Measures).* (See Fig. 5(a)) Let C be a closed curve, and let $v \in \mathbb{S}^1$ be an arbitrary direction.

(1) The *v-length* of C is the maximum distance of a pair of points with direction v, i.e. $l_C(v) := \max\{|pq| \mid p, q \in C, q - p = |q - p|v\}$.

(2) The v-*breadth* (v-*width*) of C is the distance of the two supporting lines of C perpendicular to v, i.e. $b_C(v) := \max_{p\in C} p\cdot v - \min_{p\in C} p\cdot v$ where $p\cdot v$ denotes the scalar product.
(3) The v-*halving distance*, $h_C(v)$, of C is the distance of the halving pair with direction v. Fig. 5(b) and Lemma 3 show that this is defined properly only for convex curves.
(4) The *diameter* $D(C) := \max_{v\in\mathbb{S}^1} l_C(v)$ of C is the maximum v-length. The *width* $w(C) := \min_{v\in\mathbb{S}^1} l_C(v)$ of C is the minimum v-length.
(5) The *maximal halving distance* is denoted by $H(C) := \max_{v\in\mathbb{S}^1} h_C(v)$. For non-convex curves we can use an arc-length parametrization $\bar{c}$ to get a proper definition: $H(C) := \max_{t\in[0,|C|[} |\bar{c}(t) - \bar{c}(t+|C|/2)|$. Analogously, the *minimal halving distance* is $h(C) := \min_{t\in[0,|C|[} |\bar{c}(t) - \bar{c}(t+|C|/2)|$.

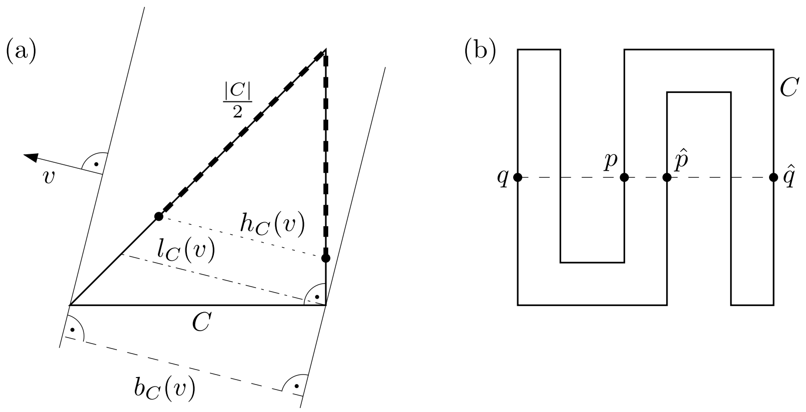

Fig. 5. (a) The v-breadth $b_C(v)$, v-length $l_C(v)$ and v-halving distance $h_C(v)$. (b) There can be more than one halving pair with direction v for non-convex cycles, e.g. $(p,\hat{p})$ and $(q,\hat{q})$.

Sometimes it is more convenient to consider the angle $\alpha \in [0, 2\pi)$ instead of the corresponding direction $(\cos\alpha, \sin\alpha) \in \mathbb{S}^1$. We will use both expressions synonymously. E.g. $h_C(\alpha)$ denotes $h_C((\cos\alpha, \sin\alpha))$.

Width and diameter can also be defined using the v-breadth values which is proved e.g. in [21, p. 76], [13, (1.5)], respectively:

Lemma 5. *Let C be a simple closed curve, then $D(C) = \max_{v\in\mathbb{S}^1} b_C(v)$. If C is convex, then $w(C) = \min_{v\in\mathbb{S}^1} b_C(v)$.*

Lemma 5 might raise the question whether one of the equations $H(C) = D(C)$ or $h(C) = w(C)$ holds at least for convex closed curves. However, already the isosceles right-angled triangle of Fig. 1 is a counter-example.

The next lemma gives another straight forward relation between the three breadth measures which follows immediately from the definitions (compare to Fig. 5(a)).

Lemma 6. *Let C be a convex closed curve, and let $v \in \mathbb{S}^1$ be an arbitrary direction. Then the following inequalities hold*: $h_C(v) \leqslant l_C(v) \leqslant b_C(v)$.

As proved in [9], the two-dimensional version of Cauchy's surface area formula cited below, leads to the general lower dilation bound of $\pi/2$.

Lemma 7 *(Cauchy). Let C be a closed convex curve. Then, its length is given by*

$$|C| = \int_0^\pi b_C(\alpha)\,\mathrm{d}\alpha.$$

A proof of the n-dimensional formula can be found e.g. in [11]. It uses the notion of mixed volumes. An easier approach dealing only with the two-dimensional case is described on the first pages of [22]. Lemma 7 immediately implies a first lower bound to the dilation of arbitrary closed curves, see [9].

Corollary 8. [9] *The dilation of any closed curve C is bounded by $\delta(C) \geqslant \pi/2$.*

Proof. Because the length of the boundary of the convex hull cannot be bigger than the original length and because of $h(C) \leqslant h_C(v) \leqslant b_C(v) = b_{\partial\mathrm{ch}(C)}(v)$ for every direction $v \in \mathbb{S}^1$, we get

$$|C| \geqslant \left|\partial\mathrm{ch}(C)\right| \overset{\text{Lem. 7}}{=} \int_0^\pi b_{\partial\mathrm{ch}(C)}(\alpha)\,\mathrm{d}\alpha \geqslant \pi h(C). \tag{2}$$

The definitions of dilation and halving pairs imply $\delta(C) \geqslant |C|/2h(C) \overset{(2)}{\geqslant} \pi/2$. □

Note that this property was independently proved by Gromov in [14], and dealt with in the context of knot theory by Kusner and Sullivan in [19].

3. Non-convex closed curves

Within this section we want to compare the dilation $\delta(C)$ of any closed curve C to the dilation $\delta(\partial\mathrm{ch}(C))$ of (the boundary of) its convex hull. We will prove that the inequality $\delta(\partial\mathrm{ch}(C)) \leqslant \delta(C)$ holds. On the other hand, clearly, no such inequality can hold for the other direction (see Fig. 6).

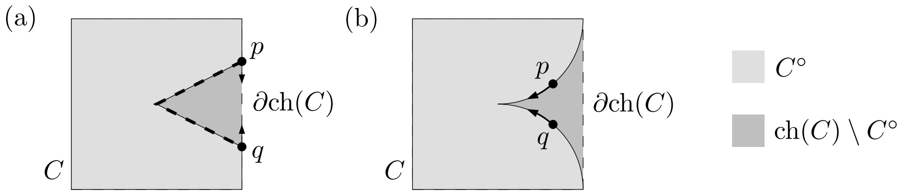

Fig. 6. The dilation $\delta(C)$ of a non-convex closed curve C can get arbitrarily big, even infinite, while $\delta(\partial\,\mathrm{ch}(C))$ stays bounded.

Lemma 9. *Let $C \subset \mathbb{R}^2$ be a closed curve. Then the dilation of the boundary of its convex hull is not bigger than the original dilation, $\delta(\partial\mathrm{ch}(C)) \leqslant \delta(C)$.*

Proof. We will prove that for any pair of points $(p,q) \in \partial\mathrm{ch}(C)^2$ on the boundary of the convex hull we can find a corresponding pair of points $(\tilde{p},\tilde{q}) \in C \times C$ on the original cycle not having smaller dilation, i.e. $\delta_C(\tilde{p},\tilde{q}) \geqslant \delta_{\partial\mathrm{ch}(C)}(p,q)$. We distinguish 3 cases:

Case 1: $p,q \in \partial\mathrm{ch}(C) \cap C$

In this case we pick $\tilde{p} := p$ and $\tilde{q} := q$. Obviously, $d_{\partial\mathrm{ch}(C)}(p,q) \leqslant d_C(p,q)$ holds and this implies $\delta_{\partial\mathrm{ch}(C)}(p,q) \leqslant \delta_C(p,q) = \delta_C(\tilde{p},\tilde{q})$.

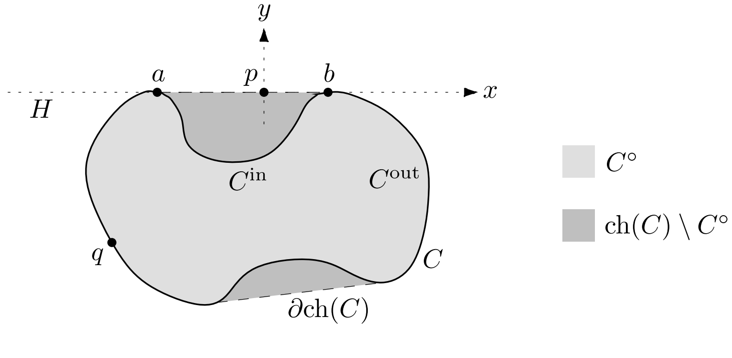

Fig. 7. Case 2: $p \in \partial\mathrm{ch}(C) \setminus C$ and $q \in \partial\mathrm{ch}(C) \cap C$.

Case 2: $p \in \partial\text{ch}(C) \setminus C$ and $q \in \partial\text{ch}(C) \cap C$

Let $\overline{ab}$ be the line segment of $\partial\text{ch}(C)$ so that $p \in \overline{ab}$ and $\overline{ab} \cap C = \{a, b\}$ (see Fig. 7). Let C^{in} denote the path on C connecting a and b that is contained in the interior of the convex hull, and let $C^{\text{out}} := C \setminus C^{\text{in}}$ be the other path on C connecting a and b. We can assume that a and b are located on the x-axis, p is the origin, the x-coordinates are ordered by $a_x < p_x = 0 < b_x$, and that C is contained in the lower half-plane $H := \{(x, y) \in \mathbb{R}^2 \mid y \leqslant 0\}$.

Then, q is contained in H. By the conditions of Case 2 the point q cannot be part of $C^{\text{in}} \subset C \setminus \partial\text{ch}(C)$. Hence, $q \in C^{\text{out}}$. Because C is simple, C^{out} cannot intersect with C^{in} and by definition of $\overline{ab}$ it cannot intersect with $\overline{ab}$. Thus C^{out} cannot enter the region bounded by $\overline{ab} \oplus C^{\text{in}}$. The remaining area where q could be located, can be divided into three regions R_1, R_2 and R_3 (see Fig. 8).

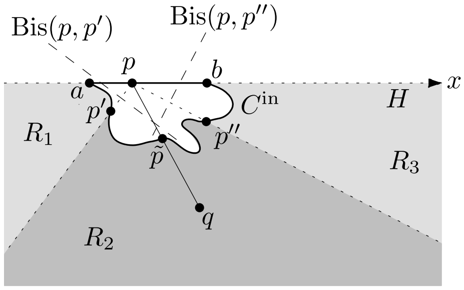

Fig. 8. The regions R_1, R_2 and R_3 containing q.

Let p' (p'' resp.) be the point on C^{in} satisfying $d_C(a, p') = |ap|$ or $d_C(b, p'') = |pb|$, respectively. Then, the x-coordinates satisfy $p'_x < 0 = p_x < p''_x$. Hence, the two rays emanating from p through p' and p'' resp. divide H into three parts. If we remove the closed region bounded by $C_b^a \oplus \overline{ab}$, we get three regions (from left to right) R_1, R_2 and R_3 whose union contains q.

If $q \in R_2$, then $\overline{pq}$ intersects with C^{in} at a point $\tilde{p}$ between p' and p''. It follows that $|\tilde{p}q| \leqslant |pq|$ and

$$\begin{aligned}
d_C(\tilde{p}, q) &= \min\bigl(d_C(\tilde{p}, a) + d_C(a, q), d_C(\tilde{p}, b) + d_C(b, q)\bigr)\\
&\geqslant \min\bigl(d_C(p', a) + d_C(a, q), d_C(p'', b) + d_C(b, q)\bigr)\\
&\geqslant \min\bigl(d_{\partial\text{ch}(C)}(p, a) + d_{\partial\text{ch}(C)}(a, q), d_{\partial\text{ch}(C)}(p, b) + d_{\partial\text{ch}(C)}(b, q)\bigr)\\
&= d_{\partial\text{ch}(C)}(p, q). \qquad (3)
\end{aligned}$$

And we conclude that $\delta_{\partial\text{ch}(C)}(p, q) \leqslant \delta_C(\tilde{p}, q)$. Choosing $\tilde{q} := q$ completes the proof in this sub-case.

If $q \in R_1$, we use $d_C(p', q) \geqslant d_{\partial\text{ch}(C)}(p, q)$ which follows analogously to (3). We will show that every point included in R_1 is not closer to p than to p'. This implies $|p'q| \leqslant |pq|$ and, finally, $\delta_{\partial\text{ch}(C)}(p, q) \leqslant \delta_C(p', q)$. Hence, we can choose $\tilde{p} := p'$ and $\tilde{q} := q$ to complete the proof.

We still have to prove that the whole region R_1 lies on the side of the bisector $\text{Bis}(p, p')$ belonging to p'. Note, that by construction of p' the point a lies on that side.

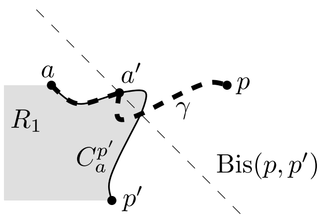

Fig. 9. R_1 must totally lie on the p'-side of $\text{Bis}(p, p')$.

We use a proof by contradiction. Assume that there is a point $d \in R_1$ which is closer to p than to p'. Then, the part of C^{in} connecting a with p', denoted by $C_a^{p'}$, has to intersect with the bisector $\text{Bis}(p, p')$ at least twice. Let a' be the first intersection point starting from a (see Fig. 9). We now mirror $C_{a'}^{p}$, the part of $C_a^{p'}$ linking a' with p', at $\text{Bis}(p, p')$ and denote the resulting path from a' to p by $C_{a'}^{p}$. Then, by concatenation we can define a path $\gamma := C_a^{a'} \oplus C_{a'}^{p}$

connecting a and p. By construction, it has the same length as the path $C_a^{p'}$, which by construction of p' has the length $|ap|$. Thus, $\gamma = \overline{ap}$, and this line segment intersects with the line $\mathrm{Bis}(p, p')$ at most in a single point. Hence, also $C_a^{p'}$ intersects with $\mathrm{Bis}(p, p')$ at most in a single point, contradicting our deduction that there must be at least two intersection points.

In the last sub-case of Case 2 the point q is contained in R_3. But then, we can argue analogously to the case $q \in R_1$.

Case 3: $p, q \in \partial\mathrm{ch}(C) \setminus C$

If p and q are located on the same line segment of $\partial\mathrm{ch}(C)$, we have $\delta_{\partial\mathrm{ch}(C)}(p, q) = 1 \leqslant \delta_C(p', q')$ for any $(p', q') \in \partial\mathrm{ch}(C)^2$.

In the remaining case, we can apply the step of Case 2 twice. First, consider the cycle $C' := \partial\mathrm{ch}(C) \setminus \overline{ab} \cup C^{\mathrm{in}}$ where $\overline{ab}$ is replaced by C^{in} in $\partial\mathrm{ch}(C)$ and everything is defined as in Case 2. Again, we can find a point $\tilde{p} \in C^{\mathrm{in}} \subset C'$ so that $\delta_{C'}(\tilde{p}, q) \geqslant \delta_{\partial\mathrm{ch}(C)}(p, q)$. Next, we can apply the arguments of Case 2 to the pair $(q, \tilde{p})$ instead of (p, q) and C' instead of $\mathrm{ch}(C)$. We get a point $\tilde{q} \in C$ so that $\delta_C(\tilde{p}, \tilde{q}) \geqslant \delta_{C'}(\tilde{p}, q) \geqslant \delta_{\partial\mathrm{ch}(C)}(p, q) = \delta(\partial\mathrm{ch}(C))$. □

4. Dilation maximum attained by halving pair

Agarwal et al. [1, Lemma 3.1] proved that the maximum dilation of planar closed polygonal curves is attained by a halving pair or by a pair of points consisting of at least one vertex. In this section we will show that an arbitrary convex, but not necessarily polygonal, closed curve contains a halving pair attaining maximum dilation.

To this end, we first generalize Lemma 1 of [10] which analyzes the angles at a dilation maximum (p, q) as shown in Fig. 10. To define the angles precisely, imagine a robot moving from q towards p on C. Let v_1 be the direction it is heading to when it arrives at p. And let v_2 be the direction it is heading to when he leaves p to continue its journey away from q. Then $\alpha_1 := \angle(v_1, \overline{pq})$ and $\alpha_2 := \angle(v_2, \overline{pq})$. And β_1 and β_2 are defined analogously for a robot moving away from p and taking a rest in q.

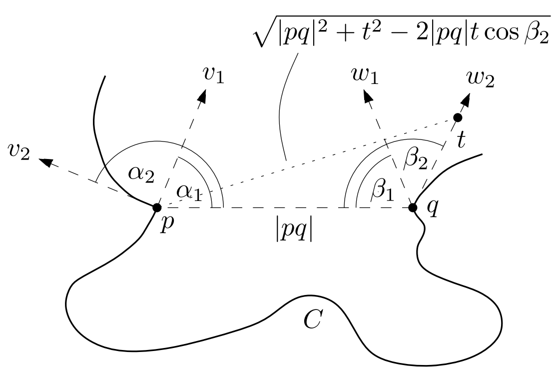

Fig. 10. Angles between the one-sided derivative vectors of a dilation maximum (p, q) and their connecting line segment $\overline{pq}$.

Lemma 10. *Let $C \subset \mathbb{R}^2$ be an open curve, and let $(p, q) \in C \times C$ be a local dilation maximum of C. Let the angles α_1, α_2, β_1, β_2 be defined as indicated in Fig.* 10.

Then, $\alpha_1 \leqslant \arccos(-1/\delta_C(p, q)) \leqslant \alpha_2$ and $\beta_1 \leqslant \arccos(-1/\delta_C(p, q)) \leqslant \beta_2$.

Proof. We consider an arc-length parametrization $\bar{c}(.)$ of C where $\bar{c}(0) = p$ and $\bar{c}(d_C(p, q)) = q$. For simplicity we define $T := d_C(p, q)$. Then, by the law of cosine for small $t > 0$ we have

$$\delta_C\big(p, \bar{c}(T + t)\big) = \frac{d_C(p, q) + t}{\sqrt{|pq|^2 + t^2 - 2t|pq|\cos\beta_2 \pm \mathrm{O}(t^2)}}.$$

Hence, the derivative at $t = 0$ equals

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta_C\big(p, \bar{c}(T + t)\big)\bigg|_{t=0} = \frac{|pq| + d_C(p, q)\cos\beta_2}{|pq|^2}.$$

This derivative cannot be strictly positive because in that case we could increase the dilation value by moving q slightly away from p. It follows

$$\beta_2 \geqslant \arccos\left(-\frac{|pq|}{\mathrm{d}_C(p,q)}\right) = \arccos\left(-\frac{1}{\delta_C(p,q)}\right).$$

By using analogous arguments, we get the inequalities bounding α_1, α_2 and β_1. □

Now we are ready to prove the result for closed convex curves:

Lemma 11. *Let C be a convex closed curve. Then, its maximum dilation is attained by a halving pair. This implies $\delta(C) = |C|/2h$.*

Proof. By Lemma 1 we know that the maximum dilation is attained by a pair of distinct points $(p,q) \in C \times C$. We will use a proof by contradiction and assume that there is no halving pair attaining maximum dilation. Then, let (p,q) be a pair of points attaining maximum dilation having biggest C-distance $d_C(p,q) < |C|/2$ among all such dilation maxima.

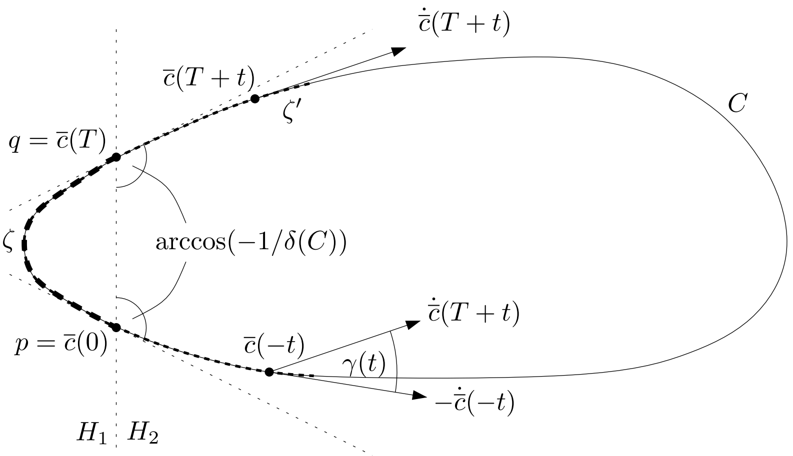

Fig. 11. The pair (p,q) can be moved towards a halving pair without reducing its dilation.

Consider Fig. 11. There exists a unique shortest path ζ on C connecting p and q. And this path can be extended a little to a path ζ', i.e. $\zeta \subset \zeta' \subset C$, so that for a small $\varepsilon > 0$ the neighborhoods $B_\varepsilon(p) \cap C$ and $B_\varepsilon(q) \cap C$ are contained in ζ', and the length $|\zeta'|$ of the extended path is still strictly smaller than $|C|/2$. The last property implies that for every pair of points $(p',q') \in \zeta' \times \zeta'$ their unique shortest connecting path on C is contained in ζ' which gives $\delta_C(p',q') = \delta_{\zeta'}(p',q')$. Hence, (p,q) is also a dilation maximum of ζ'.

By convexity of C, clearly, $\zeta \setminus \{p,q\}$ is fully contained in one of the open half-planes, say H_1, separated by the line through p and q, while the other path connecting p and q on C is contained in the other open half-plane H_2.

Let α_1, α_2, β_1 and β_2 be defined as in Lemma 10. The convexity condition implies that all the derivative vectors v_1, v_2, w_1 and w_2 are pointing into the open half-plane H_2, $\alpha_2 \leqslant \alpha_1$ and $\beta_2 \leqslant \beta_1$. Combining these inequalities with the result of Lemma 10 yields $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \arccos(-1/\delta(C))$.

Next, we will show a generalization of Lemma 2 in [10], namely, that in our situation the dilation value does not decrease if we move p and q simultaneously away from each other on ζ' with equal speed. This contradicts the fact that (p,q) is a dilation maximum having maximum C-distance among all dilation maxima, and the proof is completed.

For simplicity we again define $T := d_C(p,q)$. Let $\bar{c}$ denote an arc-length parametrization so that $\bar{c}(0) = p$ and $\bar{c}(T) = q$. If we move p and q simultaneously away from each other on ζ' both by a distance t, the resulting dilation is $\delta_C(\bar{c}(-t), \bar{c}(T+t)) = (d_C(p,q) + 2t)/|\bar{c}(T+t) - \bar{c}(-t)|$.

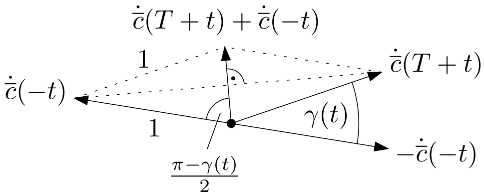

Fig. 12. The law of cosine yields $|\dot{\bar{c}}(T+t)+\dot{\bar{c}}(-t)| = 2\cos\frac{\pi-\gamma(t)}{2}$.

We want to find an upper bound to the denominator. Consider Fig. 12. Let $\gamma(t)$ denote the angle between the derivative vectors $\dot{\bar{c}}(T+t)$ and $-\dot{\bar{c}}(-t))$ for the t-values where the derivatives exist. Note that $\gamma(t)$ is positive for small t. Then, by convexity of C, we have $\gamma(t) \leqslant \alpha_2+\beta_2-\pi = 2\arccos(-1/\delta(C))-\pi$. It follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\big|\bar{c}(T+\tau)-\bar{c}(-\tau)\big|\Big|_{\tau=t} &\leqslant \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\big(\bar{c}(T+\tau)-\bar{c}(-\tau)\big)\Big|_{\tau=t}\right| \\
&= \big|\dot{\bar{c}}(T+t)+\dot{\bar{c}}(-t)\big| = 2\cos\frac{\pi-\gamma(t)}{2} \\
&\leqslant 2\cos\frac{\pi-(2\arccos(-\frac{1}{\delta(C)})-\pi)}{2} = \frac{2}{\delta(C)}.
\end{aligned}\tag{4}$$

Plugging everything together, we obtain

$$\delta_C\big(\bar{c}(T+t),\bar{c}(-t)\big) = \frac{d_C(p,q)+2t}{|\bar{c}(T+t)-\bar{c}(-t)|} \overset{(4)}{\geqslant} \frac{d_C(p,q)+2t}{|pq|+\frac{2}{\delta(C)}t} \overset{*}{\geqslant} \min\left(\frac{d_C(p,q)}{|pq|},\frac{2t}{\frac{2}{\delta(C)}t}\right) = \delta(C).$$

The inequality marked by a star is valid because for arbitrary $a,b,c,d>0$ holds $(a+c)/(b+d) \geqslant \min(a/b,c/d)$. This is an argument appearing quite often within dilation analysis. It completes the proof of Lemma 11. □

5. Centrally symmetric closed curves

Before we apply the central symmetrization which maps convex cycles to centrally symmetric convex cycles, we want to examine this latter class of closed curves.

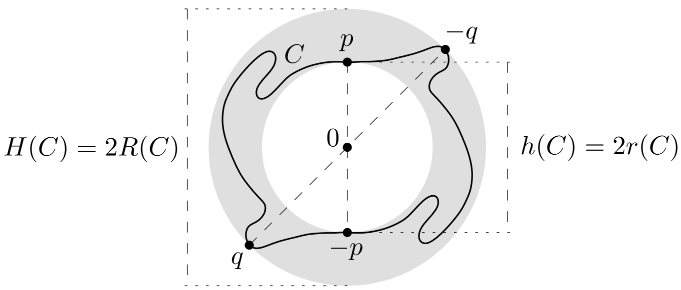

Fig. 13. Minimum and maximum halving distance of a centrally symmetric cycle C.

We will consider a closed curve which is centrally symmetric about the origin 0. Its halving pairs are the pairs $(-p,p)$ where p is any point of C, because these pairs are connected by two paths on C which are copies of each other mirrored by 0. This implies that the minimum halving distance equals twice the *inradius* $r(C)$, the radius of the smallest disk contained in $\overline{C^\circ}$, and the maximum halving distance equals twice the *circumradius* $R(C)$, the radius of the smallest disk containing $\overline{C^\circ}$. We have:

$$h(C) = 2r(C) = 2\min_{p\in C}|p|, \qquad H(C) = 2R(C) = 2\max_{p\in C}|p|. \tag{5}$$

If we assume that C is convex, we get some further properties:

Lemma 12. *Let C be a convex closed curve which is centrally symmetric about the origin* 0*. Then, the following holds*:

(1) *For every direction $v \in \mathbb{S}^1$ we have $h_C(v) = l_C(v)$.*
(2) *The length of C is given by*

$$|C| = \int_0^\pi \sqrt{\dot{h}_C^2(\alpha) + h_C^2(\alpha)}\,\mathrm{d}\alpha = \int_0^\pi \sqrt{\dot{l}_C^2(\alpha) + l_C^2(\alpha)}\,\mathrm{d}\alpha.$$

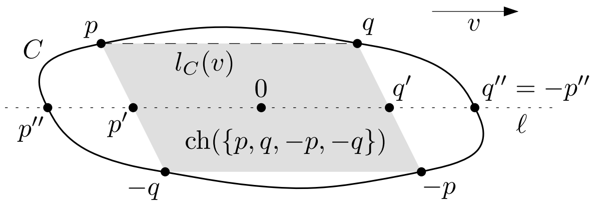

Fig. 14. For a convex, centrally symmetric cycle C we have $h_C(v) \geqslant l_C(v)$.

Proof. (1) The basic inequality of Lemma 6 gives $l_C(v) \geqslant h_C(v)$. Consider Fig. 14. Let $(p, q) \in C \times C$ be a pair of points of C with direction v having maximum distance, i.e. $q - p = |q - p|v = l_C(v)v$. Then, by symmetry, the central-symmetric copy $(-p, -q)$ is also a pair of points of C.

Due to convexity the closure $\overline{C^\circ}$ of the interior domain of C must contain the convex hull $\mathrm{ch}(\{p, q, -p, -q\})$. And this parallelogram contains a pair of points (p', q') of the same distance and direction as (p, q) and centrally symmetric about the origin, i.e. $q' = -p'$.

By possibly extending the line segment $\overline{p'q'}$, which contains 0, we find intersection points p'', q'' with C. They are unique because of the convexity of C. The arguments of the beginning of this section imply that (p'', q'') is the unique halving pair with direction v. The equation $h_C(v) = |p''q''| \geqslant |p'q'| = l_C(v)$ holds.

(2) Let $c : [0, 2\pi) \to C$ be the halving pair parametrization of C, i.e.

$$c(\alpha) := \frac{1}{2} h(\alpha)(\cos\alpha, \sin\alpha).$$

The derivative of c is given by

$$\dot{c}(\alpha) := \frac{1}{2}\dot{h}(\alpha)(\cos\alpha, \sin\alpha) + \frac{1}{2}h(\alpha)(-\sin\alpha, \cos\alpha).$$

As the two occurring vectors are orthogonal, the norm of $\dot{c}$ can be calculated by $|\dot{c}(\alpha)| = \frac{1}{2}\sqrt{h^2(\alpha) + \dot{h}^2(\alpha)}$, and we get:

$$|C| = \int_0^{2\pi} \left|\dot{c}(\alpha)\right| \mathrm{d}\alpha = \int_0^{2\pi} \frac{1}{2}\sqrt{h^2(\alpha) + \dot{h}^2(\alpha)}\,\mathrm{d}\alpha \overset{h(\alpha)=h(\alpha+\pi)}{=} \int_0^{\pi} \sqrt{h^2(\alpha) + \dot{h}^2(\alpha)}\,\mathrm{d}\alpha.$$

The second equation follows from (1). □

Remark 13.

(1) If we plug the equation of Lemma 12(2) into our dilation formula for convex cycles from Lemma 11, we see that the dilation of a centrally symmetric, convex cycle is determined by its halving distances. By ignoring the

influence of $\dot{h}(\alpha)$, we get a lower bound depending only on the ratio of its mean halving distance and its smallest halving distance:

$$\delta(C) \overset{\text{Lem. 11(2)}}{=} \frac{\int_0^\pi \sqrt{\dot{h}_C^2(\alpha)+h_C^2(\alpha)}\,\mathrm{d}\alpha}{2h(C)} \geqslant \frac{\int_0^\pi h_C(\alpha)\,\mathrm{d}\alpha}{2h(C)} = \frac{\pi}{2}\,\frac{\frac{1}{\pi}\int_0^\pi h_C(\alpha)\,\mathrm{d}\alpha}{\min_\alpha h_C(\alpha)}.$$

(2) Lemma 12(2) shows that

$$\int\limits_0^\pi b_C(\alpha)\,\mathrm{d}\alpha \overset{\text{Lem. 7}}{=} |C| \overset{\text{Lem. 12(2)}}{=} \int\limits_0^\pi \sqrt{\dot{h}^2(\alpha)+h^2(\alpha)}\,\mathrm{d}\alpha.$$

However, in general it is NOT true that $b_C(\alpha) = \sqrt{\dot{h}^2(\alpha)+h^2(\alpha)}$, since the right-hand side depends only on the behavior of C at the halving pair in direction α, which is usually different from the points determining $b_c(\alpha)$, see Fig. 15.

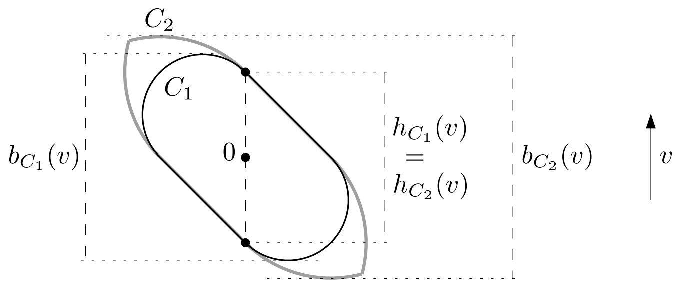

Fig. 15. The cycles C_1 and C_2 have equal halving distances $h_{C_1}(v) = h_{C_2}(v)$ and halving distance derivatives $\dot{h}_{C_1}(v) = \dot{h}_{C_2}(v)$ but a different breadth $b_{C_1}(v) \neq b_{C_2}(v)$ in direction v.

6. Central symmetrization

Because of the inequality $\delta(C) \geqslant \delta(\partial\mathrm{ch}(C))$ proved in Lemma 9 we can restrict our search for lower dilation bounds to the case of convex cycles. This allows us to apply the well-known central symmetrization (see e.g. [11, p. 101] and [17, pp. 50–52]) which maps any convex closed curve to a centrally symmetric, convex cycle.

For the convenience of the reader we review, in this section, the basic properties of this transformation, that will be needed later on. Then, in Section 7, we introduce a new transformation technique, that will be needed, too, in establishing our lower bounds.

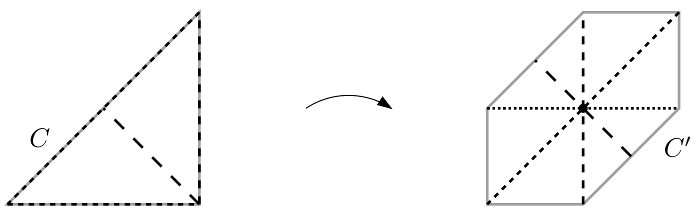

Fig. 16. The central symmetrization C' of an isosceles right-angled triangle C.

Consider Fig. 16. The central symmetrization of a convex closed curve C can be constructed by translating the centers of the pairs of maximum length to the origin. Then, the translated pairs build the central symmetrization C'. We can define this more formally:

Definition 14. Let C be a convex closed curve. Then, the *central symmetrization* of C is the cycle C' given by the parametrization $c' : \mathbb{S}^1 \to \mathbb{R}^2$, $c'(v) := (l_C(v)/2)v$.

Remark 15. It is easy to prove that the following gives an equivalent definition. Let X be the *arithmetic mean* [17] of the interior domain C° and its negative $-C^\circ$. It is the Minkowski sum $C^\circ \oplus -C^\circ$ scaled by $\frac{1}{2}$. Then, the central symmetrization C' of C is the boundary of X:

$$C' = \partial\left(\frac{1}{2}(C^\circ \oplus -C^\circ)\right) = \partial\left\{\frac{1}{2}(p-q) \;\middle|\; p, q \in C^\circ\right\}. \tag{6}$$

The following lemma lists some properties of the central symmetrization. The ones related neither to halving pairs nor to dilation are all well-known. Still, we give easy proofs to make this paper self-contained.

Lemma 16. *Let C be a convex closed curve. Then, its central symmetrization C' has the following properties*:

(1) *The curve C' is simple and centrally symmetric about the origin.*
(2) *The cycle C' is convex.*
(3) *If C is a polygonal closed curve of n edges, then C' is also a polygonal closed curve and has at least n and at most $2n$ edges.*
(4) *For every direction $v \in \mathbb{S}^1$, $h_{C'}(v) = l_{C'}(v) = l_C(v) \geqslant h_C(v)$, and $b_{C'}(v) = b_C(v)$.*
(5) *The width, the diameter and the perimeter are preserved by central symmetrization, i.e. $w(C') = w(C)$, $D(C') = D(C)$ and $|C'| = |C|$.*
(6) *The dilation of the central symmetrization C' is not bigger than the original dilation, i.e. $\delta(C') \leqslant \delta(C)$.*

Proof. (1) This follows immediately from Definition 14.

(2) The convexity of C' can best be shown by taking advantage of the arithmetic mean definition of Remark 15. The interior domain $(1/2)(C^\circ \oplus -C^\circ)$ of C' is convex because the Minkowski sum of two convex bodies is convex.

(3) It is well-known (see e.g. de Berg et al. [4, p. 276]) that the Minkowski sum $X := A \oplus B$ of two convex polygons A and B having n_A and n_B edges is a convex polygon having at most $n_A + n_B$ edges. The proof in [4] also shows that usually X' has the maximal amount of $2n$ edges. However, for every pair of parallel edges of C, the number is reduced by one. Thus, the minimal number of edges is n.

(4) As proved in Lemma 12(1), the equation $l_{C'}(v) = h_{C'}(v)$ holds for every $v \in \mathbb{S}^1$ because C' is convex and centrally symmetric. Obviously, the halving distance $h_{C'}(v)$ in direction v is attained by the pair of points $(l_C(v)/2)v$, $-(l_C(v)/2)v$. Hence, $h_{C'}(v) = l_C(v)$. And $l_C(v) \geqslant h_C(v)$ is one of the basic inequalities from Lemma 6.

It remains to show that $b_{C'}(v) = b_C(v)$. We first prove that $b_{C'}(v) \leqslant b_C(v)$. Consider the left-hand side of Fig. 17. Let $(p, q) \in C' \times C'$ be a pair of points lying on opposite lines of support of C' perpendicular to v. Let u be the direction of (p, q), i.e. $q - p = |q - p|u$. Then, $b_{C'}(v) = (|q - p|u) \cdot v$.

And, by definition, $l_{C'}(u) \geqslant |q - p|$. Above we have shown that $l_{C'}(u) = l_C(u)$. It follows:

$$l_C(u) \geqslant |q - p|. \tag{7}$$

By definition of $l_C(u)$, a pair of points with direction u and distance $l_C(u)$ has to fit between the two supporting lines of C perpendicular to v (see right-hand side of Fig. 17). Thus, $b_C(v) \geqslant (l_C(u)u) \cdot v$.

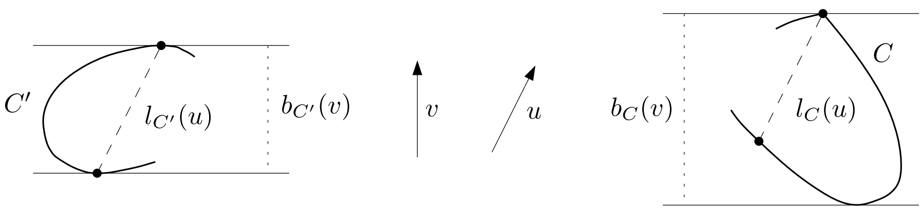

Fig. 17. Proof of $b_{C'}(v) \leqslant b_C(v)$.

Putting everything together, we get:

$$b_{C'}(v) \overset{\text{Fig. 17, left}}{=} \big(|q-p|u\big)\cdot v \overset{(7)}{\leqslant} \big(l_C(u)u\big)\cdot v \overset{\text{Fig. 17, right}}{\leqslant} b_C(v).$$

We can use the same arguments to prove $b_C(v) \leqslant b_{C'}(v)$ by simply reversing the roles of C and C'. This is possible because the main argument $l_{C'}(u) = l_C(u)$ is symmetric.

(5) It follows immediately from (4) and from the definitions that width and diameters are preserved. The perimeter is preserved because all the v-breadth values are equal by (4) and Cauchy's surface area formula (Lemma 7) yields $|C'| = \int_0^{2\pi} b_{C'}(\alpha)\,\mathrm{d}\alpha = \int_0^{2\pi} b_C(\alpha)\,\mathrm{d}\alpha = |C|$.

(6) It follows easily from the results shown above that the dilation of the transformed cycle cannot be bigger then the original one:

$$\delta(C') \overset{\text{Lem. 11}}{=} \frac{|C'|}{2\min_{v\in\mathbb{S}^1} h_{C'}(v)} \overset{(4),(5)}{\leqslant} \frac{|C|}{2\min_{v\in\mathbb{S}^1} h_C(v)} \overset{\text{Lem. 11}}{=} \delta(C).$$

This completes the proof. □

Remark 17. The dilation ratio between the central symmetrization and the original cycle is given by:

$$\frac{\delta(C')}{\delta(C)} \overset{\text{Lem. 11}}{=} \frac{\frac{|C'|}{2\min_{v\in\mathbb{S}^1} h_{C'}(v)}}{\frac{|C|}{2\min_{v\in\mathbb{S}^1} h_C(v)}} \overset{\text{Lem. 16(5)}}{=} \frac{h(C)}{h(C')} \overset{\text{Lem. 16(4)}}{=} \frac{h(C)}{w(C)}. \tag{8}$$

The ratio has a value in $(0, 1]$ (see Lemma 6). Note that the ratio is NOT a measure of the symmetry of C. If C is centrally symmetric, then the ratio equals 1 because of Lemma 12(1). But if the ratio equals 1, C does not have to be centrally symmetric, see Fig. 18.

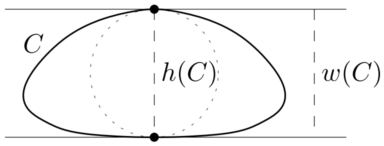

Fig. 18. The condition $h(C) = w(C)$ is not sufficient for C being centrally symmetric.

We now know that every convex cycle can be transformed into a centrally symmetric, convex cycle without increasing its dilation. By Remark 17 we even know how much the dilation has to decrease. Of course, this will help us in finding lower dilation bounds because we can restrict our search to the case of convex, centrally symmetric cycles. Before taking advantage of this knowledge we introduce the new halving pair transformation which will offer new insight and is of independent interest.

7. Halving pair transformation

The halving pair transformation translates the midpoints of the halving pairs to the origin (see Fig. 19). We will show that it also maps convex closed curves to centrally symmetric closed curves. However, it can easily be defined even for non-convex closed curves.

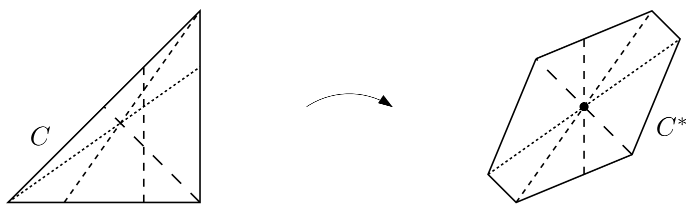

Fig. 19. The halving pair transformation C^* of an isosceles right-angled triangle C.

Definition 18. Let C be an arbitrary closed curve. Then, the *halving pair transformation* of C is the closed curve C^* given by the parametrization $c^*(v) := \frac{1}{2}(\bar{c}(t) - \bar{c}(t + |C|/2))$ where $\bar{c}(.)$ is an arc-length parametrization of C and $t + |C|/2$ is calculated modulo $|C|$. If C is convex, the parametrization $c^* : \mathbb{S}^1 \to \mathbb{R}^2$, $c^*(v) := \frac{h_C(v)}{2} v$, defines the same curve C^*.

The halving pair transformation has properties similar to the central symmetrization. However, it preserves the halving distance rather than the length- and the breadth-values.

Lemma 19. *Let C be a closed curve. Then, the result, C^*, of applying the halving pair transformation to C^* has the following properties*:

(1) *The transformed curve C^* is centrally symmetric about the origin.*
(2) *If C is convex, then C^* is simple and convex.*
(3) *If C is a polygonal cycle of n edges, then C^* is also a polygonal cycle and has at most $2n$ edges.*
(4) *The halving distances are preserved.*[4] *The breadth values cannot increase, i.e. $b_{C^*}(v) \leqslant b_C(v)$. If C is convex, the length-values do not become bigger, i.e. $l_{C^*}(v) \leqslant l_C(v)$.*
(5) *The width, the diameter and the length of C^* are not bigger than the original values, i.e. $w(C^*) \leqslant w(C)$, $D(C^*) \leqslant D(C)$ and $|C^*| \leqslant |C|$.*
(6) *If C is convex, the dilation of C^* is not bigger than the original dilation, i.e. $\delta(C^*) \leqslant \delta(C)$.*

If C is not convex, length values and the dilation can increase (see Figs. 20, 21).

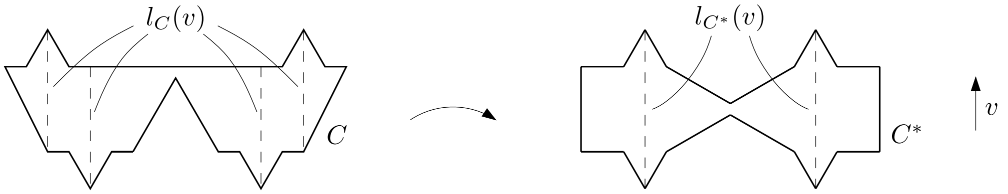

Fig. 20. If C is not convex, the v-length can increase by halving pair transformation.

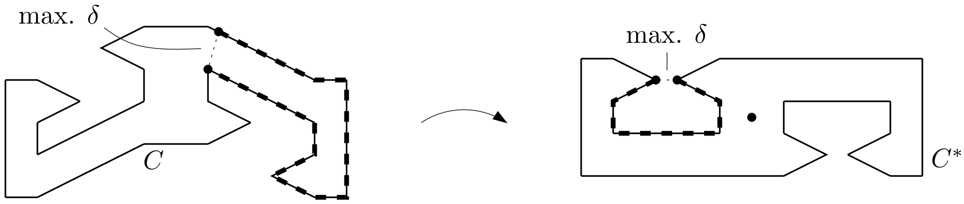

Fig. 21. If C is not convex, the dilation can increase by halving pair transformation.

Proof. (1) The curve C^* is centrally symmetric about the origin because by definition $c^*(t) = -c^*(t + |C|/2)$ holds.

(2) Let C be a convex closed curve, and let $\bar{c} : [0, |C|) \to C$ be an arc-length parametrization of C. We first assume that C is continuously differentiable. Consider Fig. 22. Because of the assumption, the derivative $\dot{\bar{c}}(.)$ is a continuous function mapping $[0, |C|)$ to the unit circle $\mathbb{S}^1$. Due to convexity, it always turns into the same direction, i.e., the mapping is bijective. The properties carry over to $-\dot{\bar{c}}(. + |C|/2)$, and this derivative turns into the same direction as $\dot{\bar{c}}(.)$.

[4] For every halving pair of C with direction $v \in \mathbb{S}^1$ there is exactly one halving pair of C^* with direction v having the same distance. For convex closed curves we can simply write $h_{C^*}(v) = h_C(v)$.

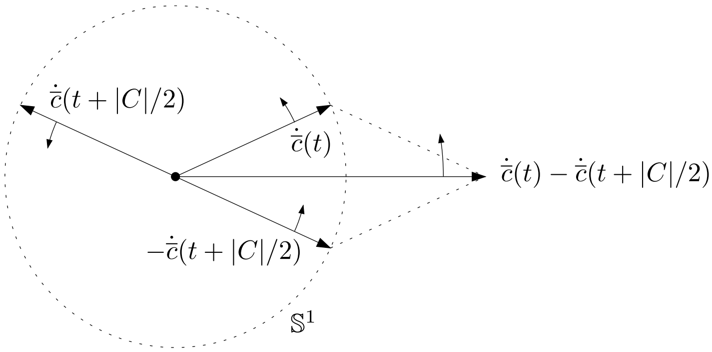

Fig. 22. The derivative vector $\dot{\bar{c}}(t)-\dot{\bar{c}}(t+|C|/2)$ always turns into the same direction.

Furthermore, the case $\dot{\bar{c}}(t)-\dot{\bar{c}}(t+|C|/2)=0$ cannot occur. Because of the convexity, this would imply that $\dot{\bar{c}}(\tau)=\dot{\bar{c}}(t)=\dot{\bar{c}}(t+|C|/2)$ for all τ-values in $[t,t+|C|/2]$ or in $[t+|C|/2,t+|C|]$. But then, C would contain a line segment of length $|C|/2$. It would have to be a line segment and could not be simple.

Moreover, the angles $\angle(\dot{\bar{c}}(t),\dot{\bar{c}}(t)-\dot{\bar{c}}(t+|C|/2))$ and $\angle(-\dot{\bar{c}}(t+|C|/2)),\dot{\bar{c}}(t)-\dot{\bar{c}}(t+|C|/2))$ cannot exceed 90°. Hence, the sum $\dot{\bar{c}}(t)-\dot{\bar{c}}(t+|C|/2))$ defines a vector continuously turning into the same direction as $\dot{\bar{c}}(t)$ and $-\dot{\bar{c}}(t+|C|/2)$, which shows that C^* is convex. The result can be generalized to piecewise continuously differentiable curves by approximating them with continuously differentiable curves. The simplicity of C^* follows from Lemma 3.

(3) If C is a polygonal cycle, then $\bar{c}(.)$ and $\bar{c}(.+|C|/2)$ are piecewise affine. Hence, the halving pair transformation $c^*(t)=\frac{1}{2}(\bar{c}(t)-\bar{c}(t+|C|/2))$ is also piecewise affine. And there can only be a corner at $c^*(t)$ if there is a corner at $\bar{c}(t)$ or $\bar{c}(t+|C|/2)$.

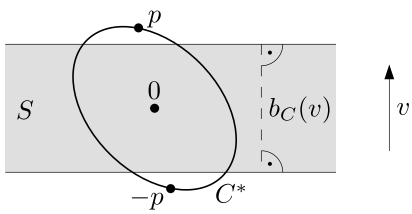

Fig. 23. This situation is impossible: C^* cannot leave S.

(4) The halving distances are preserved because of the definition of the halving pair transformation and the fact that for centrally symmetric cycles the halving pairs are the pairs $(p,-p)$, $p\in C$.

We use a proof by contradiction to show $b_{C^*}(v)\leqslant b_C(v)$. Consider the strip $S:=\{p\in\mathbb{R}^2 \mid -b_C(v)/2\leqslant p\cdot v\leqslant b_C(v)/2\}$ as depicted in Fig. 23. If there was a point $p\in C^*$ outside of S, also its halving partner $-p$ would be in $C^*\setminus S$. But then, there would be a halving pair $(q,\hat{q})$ of C of identical direction and distance. Hence, $(q-\hat{q})\cdot v=(p-\hat{p})\cdot v>b_C(v)$ which contradicts the definition of $b_C(v)$.

Next, we prove $l_{C^*}(v)\leqslant l_C(v)$ for a convex closed curve C. In this case C^* is convex and centrally symmetric. Hence, we can apply Lemma 12(1) and get $l_{C^*}(v)=h_{C^*}(v)$. Because of the arguments above, $h_{C^*}(v)=h_C(v)$ holds. And $h_C(v)\leqslant l_C(v)$ is one of the basic inequalities in Lemma 6.

(5) The inequality $b_{C^*}(v)\leqslant b_C(v)$ of (4) implies $w(C^*)\leqslant w(C)$ and $D(C^*)\leqslant D(C)$. The length relation can be shown considering an arc-length parametrization $\bar{c}:[0,|C|)\to C$. We have

$$|C|=\int_0^{\frac{|C|}{2}}\left|\dot{\bar{c}}(t)\right|+\left|\dot{\bar{c}}\big(t+|C|/2\big)\right|\mathrm{d}t\overset{\triangle\text{-inequ.}}{\geqslant}\int_0^{\frac{|C|}{2}}\left|\dot{\bar{c}}(t)-\dot{\bar{c}}\big(t+|C|/2\big)\right|\mathrm{d}t$$

$$\overset{\text{(Def. 18)}}{=}\int_0^{\frac{|C|}{2}}2\left|\dot{c}^*(t)\right|\mathrm{d}t\overset{c^*(t)=-c^*(t+|C|/2)}{=}\int_0^{|C|}\left|\dot{c}^*(t)\right|\mathrm{d}t=|C^*|.$$

(6) The inequality between the dilation values follows immediately.

$$\delta(C^*) \overset{\text{Lem. 11}}{=} \frac{|C^*|}{2\min_{v\in\mathbb{S}^1} h_{C^*}(v)} \overset{(4),\,(5)}{\leqslant} \frac{|C|}{2\min_{v\in\mathbb{S}^1} h_C(v)} \overset{\text{Lem. 11}}{=} \delta(C). \tag{9}$$

This concludes the proof of the properties. □

Lemma 20. *Let C be an arbitrary convex closed curve. Then, the dilation ratio between the transformed and the original cycle is given by*:

$$\frac{\delta(C^*)}{\delta(C)} \overset{\text{Lem. 11, Lem. 19(4)}}{=} \frac{|C^*|}{|C|} \overset{\text{Lem. 7, Lem. 12(2)}}{=} \frac{\int_0^\pi \sqrt{h_C^2(\alpha) + \dot{h_C}^2(\alpha)}\,\mathrm{d}\alpha}{\int_0^\pi b_C(\alpha)\,\mathrm{d}\alpha}. \tag{10}$$

The dilation ratio has a value in $(0, 1]$ *and is a measure of the symmetry of C. The value equals* 1, *if and only if C is centrally symmetric, which is equivalent to $C^* = C$ up to translation.*

Proof. The equations follow from the lemmata cited above the corresponding "="-sign. We still have to prove that C is centrally symmetric iff the ratio equals 1. We consider an arc-length parametrization $\bar{c}: [0, |C|) \to C$.

Let C be centrally symmetric. W.l.o.g. we can assume that C is centrally symmetric about the origin. Then, the halving pairs of C are the pairs $(p, -p)$ for every $p \in C$. Hence, we have $\bar{c}(t + |C|/2) = -\bar{c}(t)$ for every $t \in [0, |C|)$, which implies $C^* = C$, and the ratio equals 1.

If, on the other hand, the ratio equals 1, we have

$$\int_0^{\frac{|C|}{2}} \left|\dot{\bar{c}}(t) - \dot{\bar{c}}\big(t + |C|/2\big)\right| \mathrm{d}t = |C^*| = |C| = \int_0^{\frac{|C|}{2}} \left|\dot{\bar{c}}(t)\right| + \left|\dot{\bar{c}}\big(t + |C|/2\big)\right| \mathrm{d}t.$$

As used in the proof of Lemma 19(5), the triangle inequality implies that the right integrand is never smaller than the left one. Hence, $|\dot{\bar{c}}(t + |C|/2) - \dot{\bar{c}}(t)| = |\dot{\bar{c}}(t + |C|/2)| + |\dot{\bar{c}}(t)|$ must hold for almost every $t \in [0, |C|/2)$. In this case the direction and orientation of $\dot{\bar{c}}(t)$ and $-\dot{\bar{c}}(t + |C|/2)$ must be the same. Because both their absolute values equal 1, we obtain $\dot{\bar{c}}(t) = -\dot{\bar{c}}(t + |C|/2)$.

Let w.l.o.g. $\bar{c}(0) = -\bar{c}(|C|/2)$. This can be reached by translating C. Then, the following equation holds.

$$\bar{c}(t) = \bar{c}(0) + \int_0^t \dot{\bar{c}}(\tau)\,\mathrm{d}\tau = -\bar{c}\left(\frac{|C|}{2}\right) + \int_0^t \left(-\dot{\bar{c}}\left(\tau + \frac{|C|}{2}\right)\right)\mathrm{d}\tau = -\bar{c}\left(t + \frac{|C|}{2}\right).$$

The closed curve C is centrally symmetric. □

8. Lower dilation bounds

We have shown that the dilation of the boundary of the convex hull of any closed curve is not bigger than the original dilation (Lemma 9). Furthermore, for any convex closed curve we can find a centrally symmetric closed curve not having bigger dilation, by applying central symmetrization or halving pair transformation. We even obtained closed formulas telling us how much the dilation decreases by these transformations (see Remark 17, Lemma 20). We can apply this knowledge directly to get lower dilation bounds which are more specific than the general lower bound $\pi/2$ of Corollary 8.

Theorem 21. *Let $C \subset \mathbb{R}^2$ be a convex closed curve. Furthermore, let $\overline{l_C} := \frac{1}{\pi}\int_0^\pi l_c(\alpha)\,\mathrm{d}\alpha$ be the mean length of C. And let $\overline{h_C} := \frac{1}{\pi}\int_0^\pi h_c(\alpha)\,\mathrm{d}\alpha$ be the mean halving distance of C. As in Section* 7, *the halving pair transformation of C is denoted by C^*. Then, the following lower bounds hold.*

$$\delta(C) = \frac{|C|}{|C^*|} \frac{\frac{1}{\pi}\int_0^\pi \sqrt{\dot{h}^2(\alpha) + h^2(\alpha)}\,\mathrm{d}\alpha}{h(C)} \frac{\pi}{2} \geqslant \frac{|C|}{|C^*|} \frac{\overline{h_C}}{h(C)} \frac{\pi}{2} \quad \textit{and} \tag{11}$$

$$\delta(C) = \frac{w(C)}{h(C)} \frac{\frac{1}{\pi}\int_0^\pi \sqrt{\dot{l}^2(\alpha) + l^2(\alpha)}\,\mathrm{d}\alpha}{l(C)} \frac{\pi}{2} \geqslant \frac{w(C)}{h(C)} \frac{\overline{l_C}}{l(C)} \frac{\pi}{2}. \tag{12}$$

Note that all the ratios occurring in the lower bounds of Theorem 21 have values in $[1, \infty)$. As described in Lemma 20, the ratio $|C|/|C^*|$ is a measure of the symmetry of C. Its value equals 1 iff C is centrally symmetric.

The ratio $\overline{h_C}/h(C)$ (or $\frac{1}{\pi}\int_0^\pi \sqrt{\dot{h}^2(\alpha)+h^2(\alpha)}\,\mathrm{d}\alpha/h(C)$) is a measure of the oscillation of the halving distance $h_C(.)$. Its value equals 1 iff C is a cycle of constant halving distance.

As described in Remark 17, the ratio $w(C)/h(C)$ equals 1 if C is centrally symmetric. However, this is not a sufficient condition.

The ratio $\overline{l_C}/l(C)$ (or $\frac{1}{\pi}\int_0^\pi \sqrt{\dot{l}^2(\alpha)+l^2(\alpha)}\,\mathrm{d}\alpha/l(C)$) is a measure of the oscillation of the length values $l_C(.)$. Its value equals 1 iff all the length-values are equal, i.e. C is a curve of constant length and breadth (remember Lemma 5).

By considering the integrals one can easily see that the first lower bound becomes an equality iff C is a curve of constant halving distance, and the second bound turns into an equality iff C is a curve of constant breadth.

Proof. (11) By Lemma 20 we know that the dilation of C satisfies $\delta(C) = (|C|/|C^*|)\delta(C^*)$. Remark 13(1) yields that the centrally symmetric result C^* of the halving pair transformation has a dilation value of $\frac{\frac{1}{\pi}\int_0^\pi \sqrt{\dot{h}^2_{C^*}(\alpha)+h^2_{C^*}(\alpha)}\,\mathrm{d}\alpha}{h(C^*)}\frac{\pi}{2}$. And finally, Lemma 19(4) tells us that $h_{C^*}(\alpha) = h_C(\alpha)$.

(12) Remark 17 shows that the dilation of C satisfies $\delta(C) = \frac{w(C)}{h(C)}\delta(C')$ where C' denotes the result of the central symmetrization of C. Again, Remark 13(1) yields that the centrally symmetric cycle C' has a dilation value of $\frac{\frac{1}{\pi}\int_0^\pi \sqrt{\dot{h}^2_{C'}(\alpha)+h^2_{C'}(\alpha)}\,\mathrm{d}\alpha}{h(C')}\frac{\pi}{2}$. And finally, Lemma 16(4) tells us that $h_{C'}(\alpha) = l_C(\alpha)$. $\square$

Theorem 21 contains tight lower dilation bounds. Still, if we want to apply these bounds to certain closed curves, we have to know something about their mean length, their mean halving distance or about the ratios $w(C)/h(C)$, $|C|/|C^*|$. The following theorem offers an alternative which is easier to apply.

Theorem 22. *Let $C \subset \mathbb{R}^2$ be a convex closed curve. Then the dilation of C is bounded by*[5]

$$\delta(C) \geqslant \frac{|C|}{|C^*|}\left(\arcsin\left(\frac{h}{H}\right) + \sqrt{\left(\frac{H}{h}\right)^2 - 1}\right) \quad \textit{and} \tag{13}$$

$$\delta(C) \geqslant \frac{w}{h}\left(\arcsin\left(\frac{w}{D}\right) + \sqrt{\left(\frac{D}{w}\right)^2 - 1}\right). \tag{14}$$

Note once again that the ratios $|C|/|C^*|$ and w/h can attain values in $[1, \infty)$ depending on the symmetry of C (see Remark 17 and Lemma 20). The function $f : [1, \infty) \to [\pi/2, \infty)$, $f(x) := \arcsin(1/x) + \sqrt{x^2 - 1}$ appearing on the right-hand side is continuous, bijective and strictly increasing. In particular it attains the minimum value $\pi/2$ only at $x = 1$.

Proof. Let us first assume that C is centrally symmetric about the origin. Then, as mentioned before, the halving pairs are the pairs $(p, -p)$ for every $p \in C$. Let $(p, \hat{p})$ be a halving pair of maximum distance $|p\hat{p}| = H$.[5]

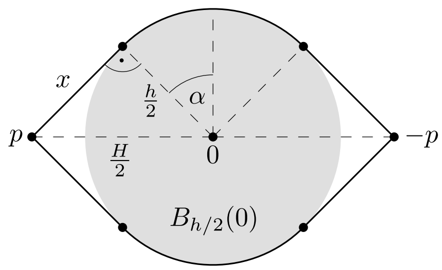

Fig. 24. The cap curve C is the shortest cycle connecting p and $-p$ while staying outside of $B_{h/2}(0)$.

[5] It should be understood that h means $h(C)$ and so forth.

Consider Fig. 24. The centrally symmetric cycle C cannot intersect with the disc of radius $h/2$ centered at 0, $B_{h/2}(0) := \{q \in \mathbb{R}^2 \mid |q| < h/2\}$. Otherwise there would be a halving pair $(q, -q)$ having a distance smaller than h.

Any convex closed curve, which contains the points p and $\hat{p}$ and does not intersect with $B_{h/2}(0)$, must have at least the length of the cap curve $\hat{C}$ shown in Fig. 24. Its length is $|\hat{C}| = 4x + 2h\alpha$.

Using Pythagoras' formula, we get $x = \frac{1}{2}\sqrt{H^2 - h^2}$. And by considering the angles in the rectangular triangle, we obtain $\sin(\alpha) = \cos(\frac{\pi}{2} - \alpha) = h/H$. Putting everything together yields:

$$\delta(C) \overset{\text{Lem. 11}}{=} \frac{|C|}{2h} \geqslant \frac{|\hat{C}|}{2h} = \frac{4x + 2h\alpha}{2h} = \sqrt{\left(\frac{H}{h}\right)^2 - 1} + \arcsin\left(\frac{h}{H}\right). \tag{15}$$

As stated in Lemma 12(2), for the centrally symmetric, convex cycle C the v-lengths and v-halving distances are equal, implying $w = h$ and $D = H$. We plug this into (15) and get

$$\delta(C) \geqslant \sqrt{\left(\frac{D}{w}\right)^2 - 1} + \arcsin\left(\frac{w}{D}\right). \tag{16}$$

Now, let C be an arbitrary convex cycle. And let C' denote the corresponding result of central symmetrization. Then, Remark 17 yields $\delta(C) = (w(C)/h(C))\delta(C')$. And by Lemma 16(4) we know that width and diameter are preserved, i.e. $w(C') = w(C)$ and $D(C') = D(C)$. Hence, applying (16) to C' completes the proof of the first lower bound.

Let C^* be the halving pair transformation of an arbitrary convex closed curve C. Then, by Lemma 19(4) we know that the halving distances are preserved, implying $h(C^*) = h(C)$ and $H(C^*) = H(C)$. And Lemma 20 gives $\delta(C) = (|C|/|C^*|)\delta(C^*)$. Applying (15) to C^* completes the proof of the second lower bound. □

After proving these results we learned that, in 1923, Kubota [18] used analogous arguments to prove that the length of any convex closed curve C of diameter D and width w satisfies $|C| \geqslant 2w\arcsin(w/D) + 2\sqrt{D^2 - w^2}$. Kubota also notes that the cap curves like the one shown in Fig. 24 are the extremal bodies of this inequality. However, he did not address the issue of halving pairs.

Our dilation bounds also get tight if C is such a cap curve. And by the arguments in the proof it is easy to see that these curves are the only centrally symmetric cycles where the inequalities become equalities. Since $|C^*|/|C|$ is a measure of the symmetry of C, this statement carries over to the case of arbitrary convex curves for the first bound.

Since the completion of this paper a generalized version of Theorem 22 was very recently obtained by Denne et al. [7], calculating new lower bounds on the ropelength of knots.

Theorems 21 or 22 can be used to answer the question whether circles are the only closed curves attaining dilation $\pi/2$, see also Gromov [14].

Corollary 23. *A closed curve $C \subset \mathbb{R}^2$ has dilation $\delta(C) = \frac{\pi}{2}$ if and only if it is a circle.*

Proof. We can easily calculate the dilation of a circle C of radius r:

$$\delta(C) \overset{\text{Lem. 11}}{=} \frac{|C|}{2h(C)} \overset{\text{Eq. (5)}}{=} \frac{|C|}{4r} = \frac{2\pi r}{4r} = \frac{\pi}{2}.$$

Now, let C be an arbitrary convex closed curve of dilation $\delta(C) = \pi/2$. Then, by (11) or (13), C has to be centrally symmetric. By the same inequalities, C has to be a closed curve of constant halving distance, i.e. $h(C) = H(C)$. Eq. (5) shows that in the central-symmetric case, this implies that the inradius $r(C)$ equals the circumradius $R(C)$, C is a circle.

If C is not convex, the arguments above and Lemma 9 imply that $\partial\mathrm{ch}(C)$ is a circle. But then, C itself has to be a circle. □

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