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Inversion of Toeplitz Operators, Levinson Equations, and Gohberg–Krein Factorization—A Simple and Unified Approach for the Rational Case

E. JONCKHEERE* AND P. DELSARTE

Philips Research Laboratory, Brussels, Belgium

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The problem of the inversion of the Toeplitz operator T_Φ, associated with the operator-valued function Φ defined on the unit circle, is known to involve the associated Levinson system of equations and the Gohberg–Krein factorization of Φ. A simplified and self-contained approach, making clear the connections between these three problems, is presented in the case where Φ is matrix-valued and rational. The key idea consists in looking at the Levinson system of equations associated with $\Phi^{-1}(z^{-1})$, rather than that associated with $\Phi(z)$. As a consequence, a new invertibility criterion for Toeplitz operators with rational matrix-valued symbols is derived.

1. INTRODUCTION

Let $L^2_{n\times n}$ and $L^\infty_{n\times n}$ be the classical Lebesgue spaces of functions from the unit circle into the space of $n \times n$ complex matrices [1]. Let $H^2_{n\times n}$ and $H^\infty_{n\times n}$ be the corresponding Hardy spaces [2] of functions with vanishing Fourier coefficients of negative index. Observe that, for a rational function, it is equivalent to say that it is in $H^2_{n\times n}$, in $H^\infty_{n\times n}$, or that it is analytic in the closed unit disk. Throughout the paper, unless stated otherwise, capital Greek letters will denote functions in $L^2_{n\times n}$, and capital Latin letters will denote functions in $H^2_{n\times n}$. A capital letter with an integer subscript denotes the corresponding Fourier coefficient. $l^2_{n\times n}$ ($h^2_{n\times n}$) denotes the set of square summable functions from $\mathbf{Z}(\mathbf{N})$ into the set of $n \times n$ matrices. In the sequel, we shall not make explicit distinction between, say, $\{R_k : k \in \mathbf{N}\} \in h^2_{n\times n}$ and the function $R(e^{j\theta}) = \sum_{k=0}^{\infty} R_k e^{jk\theta}$; this is allowed by a well-known isomorphism between the spaces $h^2_{n\times n}$ and $H^2_{n\times n}$.

Let $\Phi(z)$ be a discrete, rational $n \times n$ matrix without poles on the unit circle and with Fourier series

$$\Phi(z) = \sum_{k=-\infty}^{+\infty} \Phi_k z^k.$$

* Present address: Department of Electrical Engineering–Systems, University of Southern California, Los Angeles, California 90007.

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The Toeplitz operator T_Φ associated with Φ is defined by its semi-infinite matrix representation

$$T_\Phi = \left[\begin{matrix} \Phi_0 & \Phi_{-1} & \Phi_{-2} & \\ \Phi_1 & \Phi_0 & \Phi_{-1} & \ddots \\ \Phi_2 & \Phi_1 & \Phi_0 & \ddots \\ & \ddots & \ddots & \ddots \end{matrix}\right.,$$

$T_\Phi : h^2_{n\times 1} \to h^2_{n\times 1}$.

The Levinson system of equations associated with Φ is defined by

$$T_\Phi \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \tag{1a}$$

$$\begin{bmatrix} M_0 & M_1 & M_2 \cdots \end{bmatrix} T_\Phi = \begin{bmatrix} I & 0 & 0 \cdots \end{bmatrix}, \tag{1b}$$

where $\{S_k:\ k \in \mathbf{N}\} \in h^2_{n\times n}$ and $\{M_k : k \in \mathbf{N}\} \in h^2_{n\times n}$ are the unknown quantities. If T_Φ is invertible, this system of equations obviously admits a (unique) solution; it turns out that the converse is also true. We consider in this paper the Levinson system of equations as a first step towards another related problem—the ability to factor $\Phi(z)$ as $N(z^{-1})\, T(z)$, where both $N(z)$ and $T(z)$ are $n \times n$ matrices, invertible, and lying together with their inverses in $H^\infty_{n\times n}$. Observe that $T(z)$ and $T^{-1}(z)$ having vanishing Fourier coefficients of negative index require T_0 to be nonsingular. Hence, for notational reasons that will become clear later, we prefer to rewrite the factorization of $\Phi(z)$ in the normalized form

$$\Phi(z) = N(z^{-1})\, T_0^{-1} T(z), \tag{2a}$$

$$N, N^{-1}, T, T^{-1} \in H^\infty_{n\times n}. \tag{2b}$$

The linkage between the inversion problem and the factorization problem seems to have been shown for the first time by Gohberg and Krein in a famous series of papers [3–5]. We shall hence refer to (2) as the *(normalized) Gohberg–Krein factorization of* Φ. The ideas of Gohberg and Krein have been subsequently developed in several places; see, for example, [6–17]. Systems of equations of the Levinson type have also been extensively studied; see, for example, [18–20]. These lists of references do not claim to be exhaustive; they merely provide important contributions close to our point of view.

In this paper, we consider the inversion of T_Φ and the factorization of Φ as two different facets of the same problem, with the Levinson system of equations somewhere half-way between the inversion problem and the factorization problem. The aim of the paper is to set up a simplified and self-contained approach to this triple problem—namely, the inversion of T_Φ, the associated Levinson system of equations, and the factorization of Φ—for the special case where Φ is matrix-valued and rational. The approach avoids hard analysis as much as possible by trying to fully exploit the rationality of Φ. The key idea of the paper consists of looking at the Levinson system associated with $\Phi^{-1}(z^{-1})$, rather than that associated with $\Phi(z)$. This idea was successfully exploited by Prabhakar Murthy [21] and Delsarte *et al.* [22] to work out a simple approach to the spectral factorization problem. Here, somehow, we extend this idea to the more general problem of the Gohberg–Krein factorization. This approach leads to a new invertibility criterion for Toeplitz operators with rational symbols. This criterion takes the form of two finite systems of linear equations to be solvable and is believed to be more transparent than that of Pattanayak [14].

The importance of Toeplitz operators is not to be proved; they have become increasingly important in systems and engineering. Toeplitz operators naturally emerge from inifinite systems of convolution equations [3, 25]; also, the spectral theory of the linear-quadratic optimal control problem [23, 24] relies almost completely on Toeplitz operators. All of these things are the background of this paper.

2. Preliminaries

Let $\Phi(z)$ be an $n \times n$ discrete, rational matrix. For the problem of the inversion of T_Φ to make some sense, some restrictions on the pole/zero configurations of $\Phi(z)$ are needed.

PROPOSITION 1. *Let $\Phi(z)$ be an $n \times n$ discrete, rational matrix. Then T_Φ is a bounded operator if and only if Φ has no poles on the unit circle. Moreover, for T_Φ to be invertible it is necessary that Φ have no zeros on the unit circle.*

Proof. These are direct consequences of known and easily proved results; see, for example, the monograph of Douglas [13, Introduction and Corollary 1.4].

Since our goal is not to handle unbounded operators, we shall thus assume in the sequel that $\Phi(z)$ has no poles on the unit circle. Also, most of the time, it will be assumed that $\Phi(z)$ is devoid of zeros on the unit circle.

The fact that $\Phi(z)$ has no poles on the unit circle allows us to eliminate all poles by easy manipulation and to come up with a trigonometric polynomial.

LEMMA 1. *Let $\Phi(z)$ be an $n \times n$ rational matrix without poles on the unit circle. Then there exist polynomial factors $J(z)$ and $P(z)$, enjoying the properties*

$$J, J^{-1}, P, P^{-1} \in H^{\infty}_{n \times n}, \tag{3a}$$

that reduce $\Phi(z)$ to a trigonometric polynomial

$$J(z^{-1})\, \Phi(z)\, P(z) = \Xi(z) = \sum_{k=-\alpha}^{\beta} \Xi_k z^k. \tag{3b}$$

Moreover, T_Φ is invertible if and only if T_Ξ is. Furthermore, the Levinson system associated with Φ has a solution if and only if the Levinson system associated with Ξ has. Finally, Φ has a Gohberg–Krein factorization if and only if Ξ has.

Proof. Write the (k, l) entry of $\Phi(z)$ as

$$n_{kl}(z)/z^{m_{kl}} d^{+}_{kl}(z)\, d^{-}_{kl}(z^{-1}),$$

where $n_{kl}(z)$, $d^{+}_{kl}(z)$, and $d^{-}_{kl}(z)$ are polynomials, with $d^{+}_{kl}(z)$ and $d^{-}_{kl}(z)$ having all their zeros outside the closed unit disk, and where m_{kl} is an integer. Define $P(z) = \mathrm{diag}\{p_{11}(z),..., p_{nn}(z)\}$, where $p_{ll}(z)$ is the least common multiple of $\{d^{+}_{kl}(z): k = 1,..., n\}$. Similarly, define $J(z) = \mathrm{diag}\{j_{11}(z),..., j_{nn}(z)\}$, where $j_{kk}(z)$ is the least common multiple of $\{d^{-}_{kl}(z): l = 1,..., n\}$. It is a simple exercice to verify that the polynomial matrices $J(z)$ and $P(z)$, so defined, satisfy (3).

Let $\bar{J}(z) = J(z^{-1})$. From (3), it is readily seen that

$$T_{\bar{J}} T_\Phi T_P = T_\Xi. \tag{4}$$

Moreover, because J and P are with their inverses in $H^{\infty}_{n \times n}$, it follows that $T_{\bar{J}}$ and T_P are invertible (in fact, their inverses are $T_{\bar{J}^{-1}}$ and $T_{P^{-1}}$, respectively). Hence T_Φ is invertible if and only if T_Ξ is invertible.

From (4), the invertibility of $T_{\bar{J}}$ and T_P, and the special structure of these Toeplitz operators and their inverses, it follows that the Levinson system associated with Φ has a solution if and only if the Levinson system associated with Ξ has.

Finally, owing to (3), Φ has a Gohberg–Krein factorization if and only if Ξ has. This completes the proof.

In view of this theorem, one can work on the trigonometric polynomial $\Xi(z)$, rather than on the rational function $\Phi(z)$. In the sequel, we shall look at the invertibility of T_{Ξ}, the Levinson system of equations associated with $\Xi(z)$, and the factorization of $\Xi(z)$. The Levinson system of equations associated with $\Xi(z)$ will be written

$$T_{\Xi}\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \end{bmatrix}. \tag{5a}$$

$$\begin{bmatrix} K_0 & K_1 & K_2 & \cdots \end{bmatrix} T_{\Xi} = \begin{bmatrix} I & 0 & 0 & \cdots \end{bmatrix}, \tag{5b}$$

with $\{Q_k : k \in \mathbf{N}\} \in h^2_{n\times n}$ and $\{K_k : k \in \mathbf{N}\} \in h^2_{n\times n}$ the unknown quantities. The normalized Gohberg–Krein factorization of $\Xi(z)$ will be written

$$\Xi(z) = L(z^{-1})\, R_0^{-1} R(z), \tag{6a}$$

$$L, L^{-1}, R, R^{-1} \in H^{\infty}_{n\times n}. \tag{6b}$$

Finally, in view of Proposition 1, we shall assume in the sequel that $\Xi(z)$ has no zeros on the unit circle.

3. Basic Results

Throughout this section, it is assumed that $\Xi(z)$ is the trigonometric polynomial defined by (3), and that the condition $\det \Xi(e^{j\theta}) \neq 0$ for all $\theta \in [0, 2\pi)$, necessary for T_{Ξ} to be invertible, is satisfied. This allows us to define

$$\Xi^{-1}(z^{-1}) = \Psi(z) = \sum_{k=-\infty}^{+\infty} \Psi_k z^k, \tag{7}$$

a rational function without poles and zeros on the unit circle. As said in the introduction, this function will play a crucial role. The Levinson system of equations associated with Ψ is defined as

$$T_{\Psi}\begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \tag{8a}$$

$$\begin{bmatrix} R_0 & R_1 & R_2 & \cdots \end{bmatrix} T_{\Psi} = \begin{bmatrix} I & 0 & 0 & \cdots \end{bmatrix}, \tag{8b}$$

where $\{L_k : k \in \mathbf{N}\} \in h^2_{n\times n}$ and $\{R_k : k \in \mathbf{N}\} \in h^2_{n\times n}$ are the unknown quantities. The normalized Gohberg–Krein factorization of Ψ is defined as

$$\Psi(z) = Q(z^{-1})\, K_0^{-1} K(z), \tag{9a}$$

$$K, K^{-1}, Q, Q^{-1} \in H^\infty_{n\times n}. \tag{9b}$$

The fact that the notations $R(z) = \sum_{k=0}^{\infty} R_k z^k$ and $L(z) = \sum_{k=0}^{\infty} L_k z^k$ are used to denote both the solution of the Levinson system (8) and the solution of the normalized Gohberg–Krein factorization problem (6) is not accidental. As we shall see, they coincide. Likewise, the solution of the Levinson system (5) provides the solution to the normalized Gohberg–Krein factorization problem (9), and conversely.

We now come to the basic result.

THEOREM 1. *Consider the trigonometric polynomial* $\Xi(z) = \sum_{k=-\alpha}^{\beta} \Xi_k z^k$, *with* $\det \Xi(e^{j\theta}) \neq 0$ *for all* $\theta \in [0, 2\pi)$. *Let* $\Xi^{-1}(z^{-1}) = \Psi(z)$. *Then the following six statements are equivalent*:

(a) T_Ξ *is invertible.*

(b) *There exists a solution to the Levinson system* (5) *associated with* Ξ.

(c) Ξ *has a normalized Gohberg–Krein factorization* (6).

(a′) T_Ψ *is invertible.*

(b′) *There exists a solution to the Levinson system* (8) *associated with* Ψ.

(c′) Ψ *has a normalized Gohberg–Krein factorization* (9).

Moreover, if any of these statements holds, then $L_k = 0$ *for* $k \geqslant \alpha + 1$ *and* $R_k = 0$ *for* $k \geqslant \beta + 1$. *Furthermore,* $Q^{-1}(z) = R(z)$, $K^{-1}(z) = L(z)$, $\Psi(z)\, L(z) = Q(z^{-1})\, L_0$, *and* $R(z^{-1})\, \Psi(z) = R_0 K(z)$. *Finally, the solutions to the Levinson systems associated with* Ξ *and* Ψ *are unique, and the normalized Gohberg–Krein factorizations of* Ξ *and* Ψ *are unique.*

Proof.

(a′) $\Rightarrow$ (b′). Trivial.

(b′) $\Rightarrow$ (b). Levinson equation (8a) can be rewritten

$$\Psi(z)\, L(z) = I + \sum_{k=1}^{\infty} X_k z^{-k} = X(z^{-1}). \tag{10}$$

$\Psi \in L^\infty_{n\times n}$ and $L \in H^2_{n\times n}$ yield $X \in H^2_{n\times n}$. From (10), it also follows that $L(z) = \Xi(z^{-1})\, X(z^{-1})$, from which we deduce $L(z) = \sum_{k=0}^{\alpha} L_k z^k$; it further follows that X is rational, and hence in $H^{\infty}_{n\times n}$. Premultiplying both sides of (10) by $R(z^{-1})$ yields

$$R(z^{-1})\,\Psi(z)\,L(z) = R(z^{-1})\left(I + \sum_{k=1}^{\infty} X_k z^{-k}\right). \tag{11}$$

Likewise, from Levinson equation (8b), one obtains

$$R(z^{-1})\,\Psi(z) = I + \sum_{k=1}^{\infty} Y_k z^k = Y(z), \tag{12}$$

with, successively, $Y \in H^2_{n\times n}$, $R(z) = \sum_{k=0}^{\beta} R_k z^k$, and Y rational and in $H^{\infty}_{n\times n}$. Further, (12) implies

$$R(z^{-1})\,\Psi(z)\,L(z) = \left(I + \sum_{k=1}^{\infty} Y_k z^k\right) L(z). \tag{13}$$

Comparison of (11) and (13) yields

$$R(z^{-1})\,\Psi(z)\,L(z) = R_0 = L_0. \tag{14}$$

It is claimed that $R_0 = L_0$ is nonsingular. Indeed, assume it is singular. Then (14) yields $\det R(z^{-1}) \det \Psi(z) \det L(z) \equiv 0$. Since $\det \Psi(e^{j\theta}) \neq 0$, $\forall \theta \in [0, 2\pi)$, it follows that $\det R(e^{-j\theta}) \det L(e^{j\theta}) = 0$, $\forall \theta \in [0, 2\pi)$. R and L being polynomial, this implies either $\det R(z) \equiv 0$ or $\det L(z) \equiv 0$. The first alternative, by (12), yields $\det Y(z) \equiv 0$, which is impossible, because $Y(z) = I + \sum_{k=1}^{\infty} Y_k z^k \in H^{\infty}_{n\times n}$. The second alternative leads to the same kind of contradiction. Hence $R_0 = L_0$ is nonsingular.

With L_0 nonsingular, (10) yields $L(z^{-1}) L_0^{-1} = \Xi(z) X(z) L_0^{-1}$. Setting $Q = X L_0^{-1} \in H^2_{n\times n}$ yields Levinson equation (5a). Likewise, with R_0 nonsingular, (12) yields $R_0^{-1} R(z) = R_0^{-1} Y(z^{-1})\,\Xi(z)$. Setting $K = R_0^{-1} Y \in H^2_{n\times n}$ yields Levinson equation (5b).

(b) $\Rightarrow$ (c). Levinson equation (5a) can be rewritten

$$\Xi(z)\,Q(z) = I + \sum_{k=1}^{\alpha} U_k z^{-k}, \tag{15}$$

which implies that Q is rational. This and $Q \in H^2_{n\times n}$ yield $Q \in H^{\infty}_{n\times n}$. Premultiplying both sides of (15) by $K(z^{-1})$ yields

$$K(z^{-1})\,\Xi(z)\,Q(z) = K(z^{-1})\left(I + \sum_{k=1}^{\alpha} U_k z^{-k}\right). \tag{16}$$

Likewise, Levinson equation (5b) can be rewritten

$$K(z^{-1})\,\Xi(z) = I + \sum_{k=1}^{\beta} V_k z^k, \tag{17}$$

with, successively, K rational and in $H^{\infty}_{n\times n}$. Further, (17) also implies

$$K(z^{-1})\,\Xi(z)\,Q(z) = \left(I + \sum_{k=1}^{\beta} V_k z^k\right) Q(z). \tag{18}$$

Comparison of (16) and (18) yields

$$K(z^{-1})\,\Xi(z)\,Q(z) = K_0 = Q_0. \tag{19}$$

It is claimed that $Q_0 = K_0$ is nonsingular. Indeed, assume it is singular. From (19), we then have $\det K(z^{-1}) \det \Xi(z) \det Q(z) \equiv 0$. Since $\det \Xi(e^{j\theta}) \neq 0$, $\forall \theta \in [0, 2\pi)$, and since K and Q are rational and in $H^{\infty}_{n\times n}$, it follows that either $\det K(z^{-1}) \equiv 0$ or $\det Q(z) \equiv 0$. The first alternative contradicts (17), and the second (15). Hence $Q_0 = K_0$ is nonsingular.

From (19), we deduce

$$\det Q(z) = \det Q_0 / \det[K(z^{-1})\,\Xi(z)].$$

By (17), $\det[K(z^{-1})\,\Xi(z)]$ is a polynomial which cannot vanish on the unit circle, because $Q \in H^{\infty}_{n\times n}$. Hence Q^{-1} exists and is in $H^{\infty}_{n\times n}$. A similar argument yields the existence of $K^{-1} \in H^{\infty}_{n\times n}$. From (19), it follows that $\Psi(z)\,K^{-1}(z) = Q(z^{-1})\,Q_0^{-1}$. Comparing this with (8a) yields $K^{-1} = L$, where L is defined as the solution of Levinson equation (8a). A similar argument yields $Q^{-1} = R$. Setting $Q^{-1} = R$ and $K^{-1} = L$ in (19) yields the normalized Gohberg–Krein factorization (6) of Ξ.

(c) $\Leftrightarrow$ (c′). Trivial.

(c′) $\Rightarrow$ (a′). Define $\bar{Q}(z) = Q(z^{-1})$. From the normalized Gohberg–Krein factorization (9) of Ψ, it follows that $T_{\Psi} = T_{\bar{Q}} T_{K_0^{-1}K}$. Moreover, because Q, Q^{-1}, K, and K^{-1} are in $H^{\infty}_{n\times n}$, it follows that $T_{\bar{Q}}$ and $T_{K_0^{-1}K}$ are invertible (in fact, their inverses are $T_{\bar{Q}^{-1}}$ and $T_{K^{-1}K_0}$, respectively). Hence T_{Ψ} is invertible.

(c) $\Rightarrow$ (a). Essentially the same as the proof of (c′) $\Rightarrow$ (a′).

(a) $\Rightarrow$ (b). Trivial.

This has finished closing up the proof cycle. It remains to prove the additional claims. In the proof of (b′) $\Rightarrow$ (b), we have seen that $L(z) = \sum_{k=0}^{\alpha} L_k z^k$ and $R(z) = \sum_{k=0}^{\beta} R_k z^k$. Still in the proof of (b′) $\Rightarrow$ (b), we have seen that $\Psi(z)\,L(z) = X(z^{-1})$, with $Q(z) = X(z)\,L_0^{-1}$; this yields $\Psi(z)\,L(z) =$

$Q(z^{-1})L_0$. The proof of $R(z^{-1})\Psi(z) = R_0 K(z)$ goes similarly and is omitted. In the proof of (b) $\Rightarrow$ (c), we have seen that $Q^{-1} = R$ and $K^{-1} = L$. The uniqueness of the solution to the Levinson systems (5) and (8) is trivial from the invertibility of T_{Ξ} and T_{Ψ}. Since the solutions to the Levinson systems provide the solutions to the factorization problems, and conversely, the uniqueness of the former implies the uniqueness of the latter. This completes the proof.

It is clear that the existence of a solution to either the Levinson system (5) or (8) is close to a potential invertibility criterion for Toeplitz operators with rational symbols. While the Levinson system associated with Ξ has a solution with infinitely many nonzero terms, the Levinson system associated with Ψ has only finitely many nonzero terms in its solution. Hence we prefer working with the Levinson system associated with Ψ to derive an invertibility criterion for Toeplitz operators with rational symbols. This is the goal of the next section.

4. New Invertibility Criterion

The big advantage of looking at the Levinson system associated with Ψ is that, if it admits a solution, then this solution has only finitely many nonzero terms. With a little extra effort, this problem will be reduced to a purely finite-dimensional one. For notational convenience, let $T_{\Psi}^{a,b}$ be the top left-hand corner submatrix of T_{Ψ} consisting of an rows and bn columns. We have

Lemma 2. *Consider the $n \times n$ trigonometric polynomial $\Xi(z) = \sum_{k=-\alpha}^{\beta} \Xi_k z^k$, with $\det \Xi(e^{j\theta}) \neq 0$ for all $\theta \in [0, 2\pi)$. Let $\Psi(z) = \Xi^{-1}(z^{-1})$. There exists a triple (A, B, C), with A an $m \times m$ ($m \leqslant (\alpha+\beta+1)n$) matrix, B an $m \times n$ matrix, and C an $n \times m$ matrix, defining a linear system of state space dimension m, observable with observability index less than or equal to $\alpha+\beta+1$, and such that*

$$T_{\Psi}^{\infty,\alpha+1} = \left[\begin{array}{cccc} \Psi_0 & \Psi_{-1} & \cdots & \Psi_{-\alpha} \\ \Psi_1 & \Psi_0 & \cdots & \Psi_{-\alpha+1} \\ \Psi_2 & \Psi_1 & \cdots & \Psi_{-\alpha+2} \\ \vdots & \vdots & & \vdots \end{array}\right] = \left[\begin{array}{c} C \\ CA \\ CA^2 \\ \vdots \end{array}\right] \left[\begin{array}{cccc} A^{\alpha}B & \cdots & AB & B \end{array}\right]. \tag{20}$$

Moreover, a similar statement holds for the matrix $T_{\Psi}^{\beta+1,\infty}$.

Proof. The proof of this result proceeds along the lines of finite-dimensional realization theory. For the details, see Appendix A.

We can now state the final result—a new invertibility criterion for Toeplitz operators with trigonometric polynomial symbols.

THEOREM 2. *Consider the* $n \times n$ *trigonometric polynomial* $\Xi(z) = \sum_{k=-\alpha}^{\beta} \Xi_k z^k$, *with* $\det \Xi(e^{j\theta}) \neq 0$ *for all* $\theta \in [0, 2\pi)$. *Let* $\Xi^{-1}(z^{-1}) = \Psi(z)$. *Then the Toeplitz operator* T_Ξ *is invertible if and only if there exists a solution* $(\{L_k: k = 0,..., \alpha\}, \{R_k: k = 0,..., \beta\})$ *to the system of equation*

$$T_\Psi^{\alpha+\beta+2,\alpha+1} \begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ L_\alpha \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{21a}$$

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_\beta \end{bmatrix} T_\Psi^{\beta+1,\alpha+\beta+2} = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}. \tag{21b}$$

Moreover, if it exists, this solution provides the nonzero terms of the solution to the Levinson system (8), *and is unique.*

Proof. Assume T_Ξ is invertible. Hence, by Theorem 1, there exists a solution to the Levinson system (8), with $L_k = 0$ for $k \geqslant \alpha + 1$ and $R_k = 0$ for $k \geqslant \beta + 1$. This directly yields (21).

Conversely, assume (21) has a solution. By making use of Lemma 2, Eq. (21a) can be rewritten

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha+\beta+1} \end{bmatrix} X = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{22}$$

with $X = A^\alpha BL_0 + \cdots + BL_\alpha$, an $m \times n$ matrix. Still by Lemma 2, the pair (A, C) is observable with observability index less than or equal to $\alpha + \beta + 1$. Hence the $(\alpha + \beta + 1)n$ last equations of (22) imply $AX = 0$. It follows that $A^{\alpha+\beta+2}X = A^{\alpha+\beta+3}X = \cdots = 0$. This and (22) yield

$$T_\Psi^{\infty,\alpha+1} \begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ L_\alpha \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

Setting $L_{\alpha+1} = L_{\alpha+2} = \cdots = 0$ yields a solution to Levinson equation (8a). Likewise, it is proved that a solution to (21b) yields a solution to Levinson equation (8b). Since there exists a solution to the Levinson system (8), by Theorem 1, T_Ξ is invertible.

In the above paragraphs, it has been proved that the solution to (21) provides the nonzero terms of the solution to the Levinson system (8), and conversely. Since the solution to (8) is unique by Theorem 1, so is the solution to (21). This completes the proof.

Remark. Pattanayak [14] also provided an invertibility criterion for Toeplitz operators with rational symbols; this criterion takes the form of a winding number condition together with a finite-dimensional matrix having maximal rank. The latter part of this criterion involves, in an explicit and very intricated way, the zeros of the symbol; also, this criterion is neatly related to neither the inverse of the Toeplitz operator nor the solution to the Gohberg–Krein factorization problem. (It seems that the zeros of the symbol are intrinsic in the invertibility of a Toeplitz operator and in the factorability of its symbol; see [24, Conclusion].) Our criterion does not involve any winding number condition. Also, in our criterion, the zeros of $\Xi(z)$ appear only implicitly in the Fourier expansion of $\Xi^{-1}(z^{-1}) = \Psi(z)$. Finally, as stated by Theorem 2, our criterion readily provides the solution to the factorization problem and hence to the inversion problem.

5. Conclusion

A simple and self-contained approach to a triple problem—the invertibility of the Toeplitz matrix T_Ξ associated with a trigonometric polynomial Ξ, the existence of a solution to the associated Levinson system of equations, and the Gohberg–Krein factorability of Ξ—has been presented. As a main result, the usefulness of looking at the Toeplitz matrix T_Ψ with $\Psi(z) = \Xi^{-1}(z^{-1})$ has been shown, all the relevant information related to this triple problem being contained in two *finite* submatrices extracted from T_Ψ.

APPENDIX A

Lemma A.1. *Let $P(z) = \sum_{k=0}^{d} P_k z^k$ be an $n \times n$ polynomial matrix of formal degree $d (P_d \neq 0)$ and normal rank n. Then there exist an $n \times n$ polynomial matrix $D(z) = \sum_{k=0}^{d-1} D_k z^k + Iz^d$ and an $n \times n$ polynomial matrix $M(z) = \sum_{k=0}^{a} M_k z^k$ such that $P^{-1}(z) = D^{-1}(z) M(z)$.*

Proof. This result is implicitly contained in a paper by Silverman and Payne [27, Corollaries 4.3 and 4.4, Theorem 4.3]. It is also contained in a yet more implicit form in a paper by Forney [29, Theorems 3 and 5]. To be self-contained, we sketch a simple proof.

If the polynomial matrix $P(z)$ has nonsingular leading coefficient, that is, if rank $(P_d) = n$, then the theorem is proved. It suffices to take $M(z) = P_d^{-1}$ and $D(z) = P_d^{-1} P(z)$.

If $\operatorname{rank}(P_d) < n$, a recursive algorithm defines a sequence $\{P^l(z) = \sum_{k=0}^{d} P_k^l z^k : l = 0, 1, 2,...\}$ of $n \times n$ polynomial matrices of full normal rank and such that $P^l(z) = M^l(z) P(z)$, where $M^l(z)$ is an $n \times n$ polynomial matrix of full normal rank.

The sequence of polynomial matrices is defined recursively as follows: For $l = 0$, $P^0(z) = P(z)$. Now, assume $P^l(z)$ is known. If rank $(P_d) = n$, the theorem is proved; it suffices to take $D(z) = (P_d^l)^{-1} P^l(z)$ and $M(z) = (P_d^l)^{-1} M^l(z)$. If rank $(P_d^l) = r_l < n$, we define an $n \times n$ nonsingular row operation matrix S^l such that $S^l P_d^l$ has r_l linearly independent rows and $n - r_l$ zero rows. Obviously, there is a great deal of freedom in choosing S^l [28, Remarks 1 and 2]. It is rather easily seen that one can take S^l upper-triangular with 1's on the diagonal. Let $i_1,..., i_\nu$ be the indices of the vanishing rows of $S^l P_d^l$, with $\nu = n - r_l$. The rows $i_1,..., i_\nu$ of $P^l(z)$ must all be nonvanishing, for otherwise $P^l(z)$ would not have full normal rank. Let $p_{i_1},..., p_{i_\nu}$ be the largest powers of z appearing in rows $i_1,..., i_\nu$, respectively, of $P^l(z)$. Define

$$T^{l+1}(z) = \operatorname{diag}\{1,..., 1, \underset{\substack{\uparrow \\ i_1\text{th position}}}{z^{d-p_{i_1}}}, 1,..., 1, \underset{\substack{\uparrow \\ i_\nu\text{th position}}}{z^{d-p_{i_\nu}}}, 1,..., 1\}.$$

Obviously, $T^{l+1}(z) S^l P^l(z)$ has formal degree d with the rows of the leading coefficient nonvanishing. Let U^{l+1} be an $n \times n$ row permutation matrix such that the rows of $U^{l+1} T^{l+1}(z) S^l P^l(z)$ have increasing delays

$$U^{l+1} T^{l+1}(z) S^l P^l(z) = \begin{bmatrix} \sum_{k=\delta_1^{l+1}}^{d} P_{1,k}^{l+1} z^k \\ \vdots \\ \sum_{k=\delta_n^{l+1}}^{d} P_{n,k}^{l+1} z^k \end{bmatrix},$$

$$\delta_1^{l+1} \leqslant \delta_2^{l+1} \leqslant \cdots \leqslant \delta_n^{l+1}.$$

Then define $P^{l+1}(z) = U^{l+1} T^{l+1}(z) S^l P^l(z)$.

The key for understanding how the algorithm terminates is the following: For $l = 1, 2,...$, because S^l has been chosen as upper-triangular, the delays of $P^{l+1}(z)$ are greater than those of $P^l(z)$; more precisely,

$$\delta_i^{l+1} \geqslant \delta_i^l; \qquad i = 1, 2,..., n; \quad l = 1, 2,...;$$

$$\sum_{i=1}^{n} (\delta_i^{l+1} - \delta_i^l) > 0.$$

It is now clear that the algorithm must terminate by a matrix $P^t(Z)$ with nonsingular leading coefficient, for otherwise $P(z)$ would not have full normal rank. Hence taking $D(z) = (P_d^t)^{-1} P^t(z)$ and $M(z) = (P_d^t)^{-1} M^t(z)$ completes the proof.

COROLLARY A.1. *Under the same hypothesis as in Lemma* A.1, *there exist an* $n \times n$ *polynomial matrix* $D(z) = \sum_{k=0}^{d-1} D_k z^k + Iz^d$, *an* $n \times n$ *polynomial matrix* $N(z) = \sum_{k=0}^{d-1} N_k z^k$, *and an* $n \times n$ *polynomial matrix* $Q(z) = \sum_{k=0}^{a-d} Q_k z^k$ *such that* $P^{-1}(z) = D^{-1}(z)\, N(z) + Q(z)$.

Proof. By Lemma A.1, there exist an $n \times n$ polynomial matrix $D(z)$ of formal degree d with unit leading coefficient and an $n \times n$ polynomial matrix $M(z)$ of formal degree a such that $P^{-1}(z) = D^{-1}(z)\, M(z)$.

If $a \leqslant d-1$, define $N(z) = M(z)$, and the theorem is proved.

For the case $a \geqslant d$, observe that we have $Iz^d = D(z) - D_{d-1} z^{d-1} - \cdots - D_0$. It readily follows that $P^{-1}(z) = D^{-1}(z)[-D_{d-1} M_a z^{a-1} - \cdots - D_0 M_a z^{a-d} + M_{a-1} z^{a-1} + \cdots + M_0] + M_a z^{a-d}$. Hence the formal degree of the polynomial matrix postmultiplying $D^{-1}(z)$ has dropped from a to $a-1$. Continuing this procedure yields $P^{-1}(z) = D^{-1}(z)\, N(z) + Q(z)$, with $N(z) = \sum_{k=0}^{d-1} N_k z^k$ and $Q(z) = \sum_{k=0}^{a-d} Q_k z^k$. This completes the proof.

Remark. As defined in Corollary A.1, $D^{-1}(z)\, N(z)$ is a matrix fraction representation of a transfer matrix [32, Section II]. The coefficient matrices of $D(z)$ and $N(z)$ give, directly, the so-called observable canonical state space representation. This state space representation is controllable if and only if $D(z)$ and $N(z)$ are left coprime [32, Section II], [31, Chapter 2, Section 6]. From the construction outlined in the proof of Lemma A.1, it follows that $M(z)$ and $D(z)$ are not left coprime, unless $P(z)$ has already nonsingular leading coefficient. If $M(z)$ and $D(z)$ are not left coprime, it follows that $N(z)$ and $D(z)$ may not be left coprime and that the observable canonical state space representation of $D^{-1}(z)\, N(z)$ may not be controllable.

Proof of Lemma 2. With $\Xi(z)$ defined by (3b), $z^{\alpha+1}\Xi(z)$ is a polynomial. Application of Lemma A.1 and Corollary A.1 to the polynomial matrix $z^{\alpha+1}\Xi(z)$ yields polynomial matrices

$$D(z) = \sum_{k=1}^{\alpha+\beta} D_k z^k + Iz^{\alpha+\beta+1}, \tag{A.1a}$$

$$N(z) = \sum_{k=0}^{\alpha+\beta} N_k z^k, \tag{A.1b}$$

$$Q(z) = \sum_{k=0}^{\gamma} Q_k z^k, \tag{A.1c}$$

such that

$$Q(z) + D^{-1}(z)\,N(z) = [z^{\alpha+1}\Xi(z)]^{-1} = \sum_{k=-\infty}^{+\infty} \Psi_k z^{-k-\alpha-1}. \tag{A.1d}$$

The proper rational transfer matrix $D^{-1}(z)\,N(z)$ admits the "companion" realization

$$D^{-1}(z)\,N(z) = H(zI - F)^{-1}G, \tag{A.2a}$$

$$H = [0 \quad 0 \quad \cdots \quad 0 \quad I], \tag{A.2b}$$

$$F = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ I & 0 & 0 & \cdots & 0 & -D_1 \\ 0 & I & 0 & \cdots & 0 & -D_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -D_{\alpha+\beta-1} \\ 0 & 0 & 0 & \cdots & I & -D_{\alpha+\beta} \end{bmatrix}, \tag{A.2c}$$

$$G = \begin{bmatrix} N_0 \\ N_1 \\ \vdots \\ N_{\alpha+\beta} \end{bmatrix}. \tag{A.2d}$$

This realization has state space dimension $(\alpha+\beta+1)n$, and is observable with observability index $\alpha+\beta+1$. Note that it is not necessarily controllable, because, as explained in the above remark, $N(z)$ and $D(z)$ are not necessarily left coprime. Without loss of generality, assume that by a similarity transformation the state space realization (A.2) has been brought into the form

$$D^{-1}(z)\,N(z) = H(zI - F)^{-1}G, \tag{A.3a}$$

$$H = [H_1 \quad H_2], \tag{A.3b}$$

$$F = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix}, \qquad |\lambda_i(F_{11})| < 1, \qquad |\lambda_i(F_{22})| > 1, \tag{A.3c}$$

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}; \tag{A.3d}$$

the partitionings of H and G are consistent with that of F.

From (A.1) and (A.3), it follows that

$$\sum_{k=-\infty}^{+\infty} \Psi_k z^{-k-\alpha-1} = H_1(zI - F_{11})^{-1}G_1 + H_2(zI - F_{22})^{-1}G_2 + Q(z).$$

This and (A.3c) yield

$$\sum_{k=-\alpha}^{\infty} \Psi_k z^{-k-\alpha-1} = H_1(zI - F_{11})^{-1} G_1. \tag{A.4}$$

Recall that (F, H) is observable with observability index $\alpha + \beta + 1$, and notice that

$$\begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_1 F_{11} & H_2 F_{22} \\ H_1 F_{11}^2 & H_2 F_{22}^2 \\ \vdots & \vdots \end{bmatrix}.$$

Hence (F_{11}, H_1) is observable. Furthermore, the row range of $(F^{(\alpha+\beta+1)\mathrm{T}} H^{\mathrm{T}}, F^{(\alpha+\beta+2)\mathrm{T}} H^{\mathrm{T}} \cdots)^{\mathrm{T}}$ is included in the row range of $(H^{\mathrm{T}}, F^{\mathrm{T}} H^{\mathrm{T}}, \ldots, F^{(\alpha+\beta)\mathrm{T}} H^{\mathrm{T}})^{\mathrm{T}}$. Hence the row range of $(F_{11}^{(\alpha+\beta+1)\mathrm{T}} H_1^{\mathrm{T}}, F_{11}^{(\alpha+\beta+2)\mathrm{T}} H_1^{\mathrm{T}}, \ldots)^{\mathrm{T}}$ is included in the row range of $(H_1^{\mathrm{T}}, F_{11}^{\mathrm{T}} H_1^{\mathrm{T}}, \ldots, F_{11}^{(\alpha+\beta)\mathrm{T}} H_1^{\mathrm{T}})^{\mathrm{T}}$. Hence (F_{11}, H_1) is observable with observability index less than or equal to $\alpha + \beta + 1$.

Finally, from (A.4) it follows that

$$\begin{bmatrix} \Psi_{-\alpha} & \Psi_{-\alpha+1} & \Psi_{-\alpha+2} \\ \Psi_{-\alpha+1} & \Psi_{-\alpha+2} & \vdots \\ \Psi_{-\alpha+2} & \vdots & \\ \vdots & & \end{bmatrix} = \begin{bmatrix} H_1 \\ H_1 F_{11} \\ H_1 F_{11}^2 \\ \vdots \end{bmatrix} \begin{bmatrix} G_1 & F_{11} G_1 & F_{11}^2 G_1 & \cdots \end{bmatrix}.$$

Taking $(F_{11}, G_1, H_1) = (A, B, C)$ yields Lemma 2.

REFERENCES

1. K. HOFFMAN, "Banach Spaces of Analytic Functions," Prentice-Hall, Englewood Cliffs, N.J., 1962.
2. P. L. DÜREN, "Theory of H^p Spaces," Academic Press, New York, 1970.
3. I. C. GOHBERG AND M. G. KREIN, Systems of integral equations with kernels depending on the difference of arguments, *Amer. Math. Soc. Transl. Ser.* **14** (1960), 217–288.
4. M. G. KREIN, Integral equations on the half-line with kernels depending on the difference of arguments, *Amer. Math. Soc. Transl. Ser.* **22** (1962), 163–288.
5. I. C. GOHBERG AND M. G. KREIN, On the factorization of operators in Hilbert space, *Amer. Math. Soc. Transl. Ser.* **51** (1966), 155–188.
6. I. C. GOHBERG AND M. G. KREIN, The basic propositions on defect numbers, root numbers, and indices of linear operators, *Amer. Math. Soc. Transl. Ser.* **13** (1960), 185–264.
7. I. C. GOHBERG AND I. A. FEL'DMAN, "Convolution Equations and Projection Methods for their Solutions," Translations of Mathematical Monographs, Vol. 41, Amer. Math. Soc., Providence, R.I., 1974.

8. H. R. Pousson, Systems of Toeplitz operators on H^2, *Proc. Amer. Math. Soc.* **19** (1968), 603–608.
9. H. R. Pousson, Systems of Toeplitz operators on H^2, II, *Trans. Amer. Math. Soc.* **113** (1968), 527–536.
10. A. Devinatz, Toeplitz operators on H^2 spaces, *Trans. Amer. Math. Soc.* **112** (1964), 304–317.
11. H. Widom, Inversion of Toeplitz matrices, II, *Illinois J. Math.* **4** (1960), 88–99.
12. M. Rabindranathan, On the inversion of Toeplitz operators, *J. Math. Mech.* **19** (1969), 195–206.
13. R. G. Douglas, "Banach Algebra Techniques in the Theory of Toeplitz Operators," Regional Conference Series, Vol. 15, Amer. Math. Soc., Providence, R.I., 1972.
14. S. Pattanayak, "On Toeplitz Operators on Quarter Plane with Matrix Valued Symbols," Dissertation, State University of New York, Stony Brook, 1972.
15. A. McNabb and A. Schumitzky, Factorization of operators—I: Algebraic theory and examples, *J. Funct. Anal.* **9** (1972), 262–295.
16. R. Saeks, The factorization problem—A survey, *Proc. IEEE* **64** (1976), 90–95.
17. P. Dewilde and J. P. Vandewalle, On the factorization of a nonsingular rational matrix, *IEEE Trans. Circuits and Systems* **CAS-22** (1975), 637–645.
18. N. Levinson, The Wiener RMS error criterion in filter design and prediction, *J. Math. Phys.* **25** (1947), 261–278.
19. W. F. Trench, An algorithm for the inversion of finite Toeplitz matrices, *SIAM J. Appl. Math.* **12** (1974), 515–522
20. T. Kailath, A view of three decades in linear filtering theory, *IEEE Trans. Inform. Theory* **IT-20** (1974), 146–181.
21. D. N. Prabhakar Murthy, Factorization of discrete-process spectral matrices, *IEEE Trans. Inform. Theory* **IT-19** (1973), 693–696.
22. P. Delsarte, Y. Genin, and Y. Kamp, A simple approach to spectral factorization, *IEEE Trans. Circuits and Systems* **CAS-25** (1978), 943–946.
23. E. A. Jonckheere and L. M. Silverman, Spectral theory of the linear-quadratic optimal control problem: Discrete-time single-input case, *IEEE Trans. Circuits and Systems* **CAS-25** (1978), 810–825.
24. E. A. Jonckheere and L. M. Silverman, Spectral theory of the linear-quadratic optimal control problem: A new algorithm for spectral computations, *IEEE Trans. Automat. Control* **AC-25** (1980), 880–888.
25. F. M. Callier and C. A. Desoer, Matrix fraction representation theory for convolution systems, *in* "International Symposium on Mathematical Theory of Networks and Systems," pp. 297–302, Delft, The Netherlands, 1979.
26. L. M. Silverman, Realization of linear dynamical systems, *IEEE Trans. Automat. Control* **AC-16** (1971), 554–567.
27. L. M. Silverman and H. J. Payne, Input-output structure of linear systems with application to the decoupling problem, *SIAM J. Control* **9** (1971), 197–233.
28. B. P. Molinari, A strong controllability and observability in linear multivariable control, *IEEE Trans. Automat. Control* **AC-21** (1976), 761–764.
29. G. D. Forney, Convolutional codes I: Algebraic structure, *IEEE Trans. Inform. Theory* **IT-16** (1970), 720–738.
30. B. C. Moore and L. M. Silverman, A new characterization of feedforward delay-free inverses, *IEEE Trans. Inform Theory* **IT-19** (1973), 126–129.
31. H. H. Rosenbrock, "State Space and Multivariable Theory," Wiley, New York, 1970.
32. F. M. Callier and C. D. Nahum, Necessary and sufficient conditions for the complete controllability and observability of systems in series using the coprime factorization of a rational matrix, *IEEE Trans. Circuits and Systems* **CAS-22** (1975), 90–95.