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Interpolation on spherical geodesic grids: A comparative study

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Abstract

Most operational models in atmospheric physics, meteorology and climatology nowadays adopt spherical geodesic grids and require "ad hoc" developed interpolation procedures. The author does a comparison between chosen representatives of linear, distance-based and cubic interpolation schemes outlining their advantages and drawbacks in this specific application field. Numerical experiments on a standard test problem, while confirming a good performance of linear and distance-based schemes in a single interpolation step, also show their minor accuracy with respect to the cubic scheme in the more realistic simulation of advection of a meteorological field.

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1. Introduction

Numerical weather prediction (NWP) as well as climatology and environmental studies require a suitable representation of the Earth surface in order to model weather evolution or also to process geophysical and atmospherical data. Oceanic and limited area models often adopt uniform latitude–longitude grid for their ease of implementation even if their horizontal resolution varies systematically and anisotropically with latitude; on the contrary, geodesic grids, based on the refinements of a uniform triangulation of the sphere (where the edges of the spherical triangles are geodesic arcs) can offer quasi-uniform resolution on the whole Earth surface; nevertheless, their implementation often suffers from high computational overhead.

Spherical geodesic grids have been introduced in some early papers in NWP [21,24,10] but have been rediscovered in the last 10 years ([12,23,11] among the others). The German Weather Service has recently developed a new operational global NWP model based on such a grid [15].

As a matter of fact, most operational models adopt semi-Lagrangian dynamics [22]: the equations of motion are integrated in their Lagrangian formulation along the trajectories of the physical particles. This approach requires, at every time step in the model evolution, suitable interpolation procedures. Such procedures, while well consolidated on regular longitude–latitude grids, are still an open research field on geodesic grids. Indeed, starting from the early works of Lawson [13] and Renka [18], several interpolation methods on spherical triangles have been proposed ([3,5,1,19,9]

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among the others), in an attempt of balancing computational cost, accuracy, monotonicity; the common starting point is the use of barycentric coordinates in each triangle, even if different authors often do not agree on their definition on spherical triangles. Besides, interpolation methods for scattered data employed in other research fields, such as radial basis functions (RBFs), have recently been considered for use in meteorological applications [4,20].

Aim of this work is to perform a comparison among these methods, outlining their advantages and drawbacks in the specific application field of NWP, since the author had previously published works on solving advection and shallow water equations on the sphere [2,7,8] that used a regular longitude–latitude grid; she is now considering the solution of the same problems on a geodesic grid, that requires "ad hoc" developed interpolation procedures.

In the following section we describe the grid construction; then we introduce all the interpolation schemes found into the literature, grouped into the three classes: linear, distance-based and cubic. We choose three schemes as representatives of these classes and compare their results on a standard test problem in NWP.

2. Construction of a spherical geodesic grid

The spherical geodesic grid, also called icosahedral–hexagonal grid, first introduced in meteorological modeling by Sadourny et al. [21] and Williamson [24] about 50 years ago, has been gaining increasing interest in recent years. The approach described here closely follows the work of Baumgardner and Frederickson [3], who gave a detailed description of this grid in their seminal paper. To generate the grid, a regular icosahedron is constructed inside the sphere such that 2 of its 12 vertices coincide with the North Pole (NP) and South Pole. Next we connect the nearest neighbors among these 12 points with great circle arcs, so dividing the spherical surface into 20 equal spherical triangles. Beginning from this grid of icosahedral triangles (level $i = 0$), a new finer grid of triangles is generated by connecting midpoints of the spherical triangle sides by an additional set of great circle arcs (level $i = 1$). This process may be repeated until a grid of the desired resolution is obtained. This construction procedure yields, at level i, a grid consisting of $10\,n_i^2 + 2$ grid points (nodes) and $20\,n_i^2$ elementary spherical triangles, where $n_i = 2^i$ is the number of equal intervals into which each side of the 20 original triangles is divided. Each of these grid points is surrounded by 6 nearest neighbors except for the original 12 icosahedral vertices, which are surrounded by only 5. We therefore refer to these 12 special points as *pentagonal* points.

The number n_i is a natural parameter for specifying the resolution of the grid. The (minimum) spacing between grid points is then the length l_0 of the side of any of the original icosahedral triangles (about 7054 km for the Earth) divided by n_i. For example, at the refinement level 7, where $n_i = 128$, we obtain a minimum spacing l_i between grid points of about 55 km. The grid provides a nearly uniform coverage of the sphere even though the cells vary somewhat in their exact shape and size.

3. Interpolation

The choice of an interpolation procedure in operational NWP models is still an open problem: while in previous models, which adopted orthogonal longitude–latitude grids, cubic interpolation was a consolidated strategy [22], in the more recent models based on icosahedral grids different interpolation choices have been performed. Here, we analyze the most popular linear interpolation schemes and compare them with weighted distance and cubic interpolation schemes.

3.1. Linear interpolation

Several different formulations for linear triangle interpolation have been proposed. In the planar case, these different formulations can be proved to be equivalent and lead to identical interpolation weights. On the contrary, on the spherical surface, they are different and need to be compared.

In order to introduce the considered linear interpolation schemes, let us denote by P_1, P_2, P_3 and by $\alpha_1, \alpha_2, \alpha_3$ the vertices and angles of a spherical triangle T on a unit sphere. Moreover, we denote by a_1, a_2, a_3 the triangle edges, i.e., the geodesic arcs joining two of its vertices: a_1 joins P_2 to P_3 and so on.

(a) In [1] spherical barycentric coordinates (b_1, b_2, b_3) of a point P are introduced as the trihedral coordinates of its position vector v with respect to the trihedron generated by the position vectors $\{v_1, v_2, v_3\}$ of the vertices of the spherical triangle. As a consequence, these coordinates do not always sum to 1: $b_1(v) + b_2(v) + b_3(v) > 1$ if $v \in T$ and $v \neq v_1, v_2, v_3$. In the following, we use the notation w_i^A to refer to the interpolation weights b_i.

(b) In [13] the barycentric coordinates on a triangulation of the spherical surface are defined as normalized trihedral coordinates. It is straightforward to see that these coordinates are exactly the usual barycentric coordinates with respect to the planar triangle with vertices P_1, P_2, P_3: this choice corresponds then to project point P on this plane. We refer to these coordinates as w_i^L.

(c) In the planar case, the barycentric coordinates are proportional to the areas of the related subtriangles: more precisely, the barycentric coordinates are the normalized areas of these subtriangles. In a similar way, one can consider as linear interpolation weights on the spherical surface the ratios of spherical triangle areas. Then we introduce

$$w_1^{\mathrm{S}} = \mathrm{area}(P, P_2, P_3)/\mathrm{area}(P_1, P_2, P_3)$$

and corresponding formulas for w_2^{S} and w_3^{S}.

As a remark, we outline that for the evaluation of these areas we found quite inaccurate the classical formula relating the area A of a spherical triangle to the sum of its angles, $A = \alpha_1 + \alpha_2 + \alpha_3 - \pi$ since the evaluation of these angles by the spherical laws of cosines is susceptible to rounding errors when the distances are small. In this case, the alternative formulation given by l'Huilier theorem is preferable

$$\tan(A/4) = \sqrt{\tan\left(\frac{s}{2}\right)\tan\left(\frac{s-a_1}{2}\right)\tan\left(\frac{s-a_2}{2}\right)\tan\left(\frac{s-a_3}{2}\right)},$$

where s is the semi-perimeter of the spherical triangle. Again, the spherical edge lengths are evaluated by the more accurate formula

$$a_i = 2\arcsin\left(\frac{|P_j - P_k|}{2}\right)$$

instead of $a_i = \langle P_j, P_k\rangle/|P_j - P_k|$. Here a_i is the edge joining P_j to P_k.

(d) In [15] the weights for the bilinear interpolation are the barycentric coordinates evaluated in a local longitude–latitude coordinate system (η, χ): if a point $P(\eta, \chi)$ belongs to the spherical triangle, its barycentric coordinates $(\gamma_1, \gamma_2, \gamma_3)$ are given by the solution to the linear system

$$\mathbf{p} = \gamma_1\mathbf{p}_2 + \gamma_2\mathbf{p}_2 + \gamma_3\mathbf{p}_3$$

with $\gamma_1 + \gamma_2 + \gamma_3 = 1$. Here the symbol $\mathbf{p}_i$ denotes the position vector (η_i, χ_i) of point P_i in the (η, χ) plane. In the following, we use the notation w_i^M to refer to the interpolation weights γ_i.

Even if, in principle, all these weights are different, when they are used at the resolution of a typical operational model they give almost equivalent results. This empirical consideration can be motivated by a theorem due to Legendre that states that the area of a spherical triangle can be approximated by the area of a planar triangle with the same edge lengths with an error of the order $\mathrm{O}(l/R)^4$, where l is the edge length and R the sphere radius. Now, if we refer to the edge length l of a generic spherical triangle in the triangulation at refinement level k, it is $l/R \simeq (l_0/R) * 2^{-k}$ where $l_0 = 1.107 * R$ is the edge length of any of the 20 starting spherical triangles; it follows that, at the typical refinement level of the considered grid in operational models ($k = 7$), we have $l/R = \mathrm{O}(10^{-2})$. Then, the choice among the considered linear interpolation schemes has to be motivated only by computational efficiency and conservation properties. Indeed, the lack of normalization for the weights w^A represents a potential risk for mass increase during computations.

3.2. Distance-based interpolation

We group under this name several methods, all having in common that a neighborhood about the interpolated point is identified and a weighted average is taken of the observation values within this neighborhood. The weights are a decreasing function of distance.

(e) The simplest form of distance weighted interpolation is sometimes called "Shepard's method": the weights are $w_i = h_i^{-p}/\left(\sum_{j=1}^N h_j^{-p}\right)$ where p is an arbitrary positive real number called the power parameter (typically, $p = 2$) and h_i is the distance from the grid point P_i to the interpolation point P. A disadvantage of this inverse weighted distance method is that the function is forced to have a maximum or minimum at the data points.

(f) RBFs [6] are often the method of choice for interpolation of multivariate scattered data; they have recently been used in semi–Lagrangian models [4,20]. If $P_1, \ldots, P_N$ are given points in R^k, the approximant $s(x)$ at a point P is sought as

$$s(x) = \sum_{i=1}^{N} w_i \Phi(\|x - x_i\|),$$

where Φ is the RBF to be specified and the weights w_i have to be computed so that the estimated function agrees with the observations at points P_i. This requires the solution of a linear system. Probably the best known RBF are the thin-plate splines, for $\Phi(r) = r^2 \log r$, and Hardy's multiquadrics, for $\Phi(r) = (r^2 + d^2)^{1/2}$, with d being a tension parameter to be chosen: a general rule is to fix d of the order of the typical scale length of the domain. The generalization of multiquadrics to the sphere has been formulated by Pottmann and Eck [17], who also introduce a localized method: in this case the number N of considered neighbors has to be chosen to balance accuracy against computational cost.

(g) Several other interpolation techniques arising from different geophysical research fields could have been considered. Among these we only mention kriging. Ordinary kriging [16] is a statistical interpolation method which relies on the spatial correlation structure of the data to determine the weighting values. It is basically a form of generalized linear regression since the weights are chosen to minimize the error variance. This leads to a system of n linear equations in the unknown weights for every interpolation point, where n is the number of data points. Due to its high computational effort, kriging has proved to be effective mainly in the case of irregularly spaced or strongly regionalized data.

3.3. Cubic interpolation

(h) Hermite-type interpolation methods require the matching of derivative information: once we locate the point P in the triangulation, the function values and gradient estimates are required for the triangle vertices P_i. Early papers [13,18] adopt this approach and construct the approximation as a convex combination of the three cubic Hermite interpolators along the geodesic arcs from P_i through P to the opposite edge; the interpolation weights are related to the barycentric coordinates of P: $w_i = (b_j b_k)/(b_i b_j + b_j b_k + b_i b_k)$.

This procedure is quite expensive since it requires the preliminary estimates of the gradient vectors; moreover, any inaccuracy in these estimates will strongly affect the interpolation results.

(i) A more interesting approach is based on Lagrange-type interpolation on triangular domains. Once identified in the triangulation 10 neighbors of point P which form a triangle T, the approximation at P is written in Bernstein–Bézier form [1] as

$$p(P) = \sum_{i+j+k=3} c_{ijk} B^3_{ijk}(P),$$

where the $B^3_{ijk}(P)$ are the homogeneous Bernstein basis polynomials of degree 3 on T:

$$B^3_{ijk}(P) = \frac{3!}{i!j!k!} b_1^i b_2^j b_3^k$$

and (b_1, b_2, b_3)'s are the barycentric coordinates of P in T. The coefficients c_{ijk} have to be computed imposing interpolation at the 10 fixed neighbors, which requires the solution of a linear system.

Many other cubic interpolation schemes could have been considered, such as the Clough–Tocher interpolant described in [1], or the hybrid (rational) cubic patches introduced in [14]; however, they are all very similar to the last one considered here.

4. Numerical experiments

We compare the performance of the considered interpolation techniques in the case of solid-body rotation of a passive scalar on the surface of the sphere, which is a typical test problem for NWP, since the scalar F can be viewed as the surface pressure divided by gravity in a shallow water model, or as the pressure difference divided by gravity between the top and the bottom of a model layer in a global circulation model or also as the mass per unit area of any single atmospheric component, for example, water vapor or a chemical constituent.

Table 1
Error indicators for the considered schemes after a single time step on the test problem

Level	L_1			L_∞		
	L	RBF	C	L	RBF	C
4	2.34e − 6	4.05e − 8	2.159e − 8	4.46e − 6	5.23e − 7	9.533e − 8
5	1.18e − 6	1.25e − 8	2.263e − 9	2.23e − 6	2.66e − 7	1.177e − 8
6	5.88e − 7	8.22e − 9	2.536e − 10	1.11e − 6	1.39e − 7	1.441e − 9
7	2.92e − 7	5.99e − 9	2.976e − 11	5.54e − 7	7.36e − 8	1.761e − 10

The problem of solid-body rotation in (λ, θ) coordinates is formulated in [25], where an initial height profile (for simplicity, a cosine bell) rotates, with constant angular velocity Ω, around the Earth axis. To avoid symmetry effects, we consider this rotation in a spherical coordinate system (λ, θ) having its NP at the point P (not coinciding with the physical NP). Then, if we indicate by θ_0 the angle between the physical and the numerical NP, so that $(0, \pi/2 - \theta_0)$ are NP coordinates in the new system, the advected field is given by

$$F = h_0 - \frac{1}{g}\left[\Omega R u_0 + \frac{u_0^2}{2}\right](\cos\lambda\cos\theta\sin\theta_0 + \sin\theta\cos\theta_0)^2 \tag{1}$$

with R being Earth radius and g gravity, whereas wind components are

$$\begin{cases} u = u_0[\cos\theta\cos\theta_0 + \cos\lambda\sin\theta\sin\theta_0], \\ v = -u_0\sin\lambda\sin\theta_0. \end{cases} \tag{2}$$

For the presented tests, the following values of the parameters have been fixed: $u_0 = 140\,\mathrm{km/h}$, $h_0 = 1\,\mathrm{km}$, $\theta_0 = 0.05$.

We choose three schemes as representatives of the considered classes: linear scheme (b) (denoted by L in the following tables), multiquadrics scheme (f) (denoted by RBF) and cubic Lagrange scheme (denoted by C). For every scheme, executions were made for four grid resolutions, corresponding to refinement levels 4 (2562 grid points), 5 (10 242 grid points), 6 (40 962 grid points) and 7 (163 842 grid points).

As a first test, we consider the semi-Lagrangian advection of the initial height profile for a single time step. Then we evaluate F in all the n_i locations and compare the results with the analytical field. This test allows us to evaluate the performance of the considered schemes for a single interpolation on every node of the grid. The global relative errors on the advected field in L_1 and L_∞ norms have been calculated and reported in Table 1.

As expected, in the case of a smooth field, such as the considered test problem, all the schemes give good approximation results; however, the cheap linear scheme is the less accurate; less expected, also the RBF scheme, even if definitely more accurate, shows an almost linear rate of convergence (although, as remarked by one of the referees, an improvement could be achieved by adding a polynomial term to the approximation). The cubic scheme has a very good performance, especially for high resolution. Although computational cost and conservation properties of the three schemes will be discussed in the second test, where difference in performance is enhanced by the repeated interpolation, we want just to outline that the linear scheme only requires the computation of the barycentric coordinates of P. The RBF scheme, as we implemented it on this grid structure, uses for the interpolation the grid point closest to point P along with its 6 neighbors (5 in the case of a pentagonal point); then, for every node the solution of a linear system of size 7 (6) is required. Finally, cubic interpolation requires the solution of a linear system of size 10 and the computation of the barycentric coordinates for the 10 nodes plus point P.

As a second, and more sophisticated test, we consider the effect of interpolation after a full rotation for the considered advection problem. Then, we run the simulation for a final time of 120 h with a fixed time step of 60 min and consider the accumulation of the interpolation error for the three methods. Results are shown in Table 2 along with CPU time of the simulations in seconds (on a Pentium IV 3.0 GHz workstation). In the same table we also report, as an indicator for the conservation properties of the schemes, the relative difference between the theoretical and computed total mass: so we define

$$\Delta F = \frac{\sum_{n_i} F_{(\mathrm{th})} - \sum_{n_i} F_{(\mathrm{comp})}}{\sum_{n_i} F_{(\mathrm{th})}}.$$

Table 2
Error indicators for the considered schemes on the test problem after a full solid body rotation (120 h)

Level	Courant	L_1	L_∞	$\Delta F/F$	CPU time
Linear					
4	0.31	2.81e − 5	5.36e − 5	−2.39e − 5	1.81
5	0.63	1.42e − 5	2.69e − 5	−1.21e − 5	6.47
6	1.26	7.10e − 6	1.34e − 5	−6.10e − 6	55.45
7	2.52	3.98e − 6	3.83e − 3	−3.49e − 6	346.66
RBF					
4	0.31	8.05e − 7	1.17e − 5	−9.26e − 9	5.78
5	0.63	1.83e − 7	5.87e − 6	−1.01e − 8	22.34
6	1.26	4.14e − 8	2.95e − 6	−1.06e − 8	124.00
7	2.52	4.75e − 7	3.56e − 3	−3.83e − 7	599.91
Cubic					
4	0.31	2.56e − 7	1.13e − 6	8.76e − 8	11.7
5	0.63	2.68e − 8	1.36e − 7	5.75e − 9	46.8
6	1.26	3.03e − 9	1.09e − 7	3.62e − 10	221.25
7	2.52	4.09e − 10	3.77e − 7	2.16e − 11	999.26

Results of this test generally confirm the rate of convergence estimated in the previous comments, even if we can see that, looking at the L_∞ indicator, at higher resolution the accumulation of the error in linear and RBF scheme becomes more remarkable. The cubic scheme seems to be more robust. Simulations with smaller time step (30 min) have been done, leading to L_∞ errors smaller and more coherent with the already shown linear trend for the first two schemes, but at the price of a CPU time comparable with the one of cubic interpolation. It must be stressed that the highest L_∞ errors in general are located on the pentagonal points, i.e., on the corners of the 10 diamonds in which the grid is computationally divided. This suggests the need for a more accurate treatment of these points.

As a final remark, we see that the linear scheme exhibits a little lack of conservation (decreasing with resolution): looking at the relative mass indicator ΔF we see that in this scheme we have a slight mass increase; the RBF scheme has a minor mass increase, but it does not reduce when resolution increases; on the contrary, the cubic scheme has just a little mass loss, which strongly reduces at higher resolution.

5. Conclusions

Several interpolation techniques arising in different research fields and suitable for use in NWP models have been analyzed. Our aim was not to establish an absolute hierarchy of their performance, but to show advantages and drawbacks of these schemes in this concrete application field. Even if RBF methods seem to be very promising and some related extensions (varying stencil size, stabilizing polynomial terms) need to be analyzed further, theoretical considerations and numerical experiments led us to definitely suggest the use of Lagrange-type cubic patches for the interpolation on geodesic grids. Indeed, they have shown a very good performance in the test case involving a smooth field typical of global circulation models. However, performance of these schemes in the case of rapidly varying fields with steep gradients, as in the case of local or limited area models, should be investigated further and probably more sophisticated procedures for choosing the interpolation nodes should be developed.

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