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Operators Generated by Countably Many Differential Operators

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1. INTRODUCTION

We consider a differential operator on $\mathbb{R}$ together with boundary conditions imposed at a sequence of points. Our idea is to study the relationship between extensions of a certain symmetric operator, and the associated "admissable" boundary conditions. This correspondence has been analyzed in the case of a finite number of boundary conditions (see [3, 5, 7]). The generalization of these results to our new context is not trivial because the deficiency indices may be infinite. Some progress in certain cases has already been made in [2]. In our Theorems 3.13 and 3.14, we analyze this relationship completely for operators a little more general than differential operators.

Let $\{I_j: 1 \leqslant j \leqslant \kappa\}$ ($\kappa \leqslant \infty$) be a set of *disjoint open* intervals and $I = \bigcup_{j=1}^{\kappa} I_j$. I may not be an interval. When $\kappa = \infty$, the set of end points of all I_j can have infinitely many cluster points in the extended real line. For each $j = 1, 2, \ldots, \kappa$, let τ_j denote a formally self-adjoint ordinary linear differential expression with complex coefficients in the interval I_j such that the leading coefficients of τ_j does not vanish on I_j. (For definition, see, for example, [8].) Let $\mathscr{L}$ denote the linear expression

$$\mathscr{L}y = \tau y + Ay + \iota\chi^t MB(y) \tag{1.1}$$

where $\iota = (-1)^{1/2}$. Here A is a given bounded linear operator defined everywhere in $L_2(I)$, $\chi \in (L_2(I))^m$ ($m < \infty$), M is a $m \times N$ Hilbert matrix, $B(y)$ is an $N \times 1$ "boundary" column vector in l_2^N, and τ is the (possible infinite) direct sum expression acting on the functions y for $t \in I$ defined by

$$(\tau y)(t) = (\tau_j y)(t) \qquad \text{if} \quad t \in I_j, \qquad 1 \leqslant j \leqslant \kappa.$$

We briefly summarize the contents of our paper. In Section 2, we prove or state some results on Hilbert matrices which we need later. In Section 3, we define the closed symmetric operator $T_1(\tau)$ generated by the expressions τ_j.

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We perturb $T_1(\tau)$ to get the maximal operator $T_1(\mathscr{L})$ generated by $\mathscr{L}$. The rest of Section 3 is devoted to studying the densely defined closed restrictions of $T_1(\mathscr{L})$. Using "adjoint pairs," the adjoint operators to the restrictions of $T_1(\mathscr{L})$ are found (Theorem 3.13). All possible closed symmetric restrictions T of $T_1(\mathscr{L})$ of the type $T^* \subset T_1(\mathscr{L})$ are found (Theorem 3.14). Finally all possible self-adjoint restrictions of $T_1(\mathscr{L})$ are characterized using Hilbert matrices (Theorem 3.15). In Section 3 we give examples of self-adjoint operators.

The following notation is used in this paper. The set of positive integers, real numbers, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. If D is a matrix, then the transpose and conjugate transpose are denoted by D^t and D^*. If $D = (d_{kj})$ and $Q = (q_{kj})$ are $p \times q$ and $q \times r$ $(p, q, r \leqslant \infty)$ matrix functions of $t \in I$, the set in which all of our analysis will take place, then $(D \mid Q)$ denotes the $p \times r$ matrix $\int_I D(t)\, \bar{Q}_2(t)\, dt$ whose (k, j) entry is $\sum_{i=1}^{q} \int_I d_{ki}(t)\, \bar{q}_{ij}(t)\, dt$. The algebraic direct sum of linear subspaces G_1 and G_2 are denoted by $G_1 \oplus G_2$. If D_j $(j = 1, 2,..., \kappa)$ are finite matrices, then the usual matrix direct sum (block matrix) of all D_j are denoted by $\bigoplus_{j=1}^{\kappa} D_j$. The $k \times k$ $(k \leqslant \infty)$ identity matrix is denoted by E_k. The closure of I_j is denoted by $\bar{I}_j$. For an operator T, its domain and adjoint operator are denoted by $\mathscr{D}(T)$ and T^*.

2. Hilbert Matrices

Let l_2^N $(N \leqslant \infty)$ denote the Hilbert space of all $N \times 1$ complex constant *column* vector α such that $\alpha^* \alpha < \infty$. The norm of α is denoted by $\| \alpha \|_{l_2^N}$. Thus, if $N < \infty$, then $l_2^N = \mathbb{C}^N$. If $N = \infty$, l_2^N will be denoted by l_2. An $m \times N$ $(m \leqslant \infty,\ N \leqslant \infty)$ matrix D is called a Hilbert matrix if, for every $\alpha \in l_2^m$ and $\beta \in l_2^N$, the sum (possibly infinite) $\alpha^* D \beta$ converges (see also [4]). Thus, a $m \times \infty$ $(m < \infty)$ matrix D is a Hilbert matrix iff the m rows of D are in $l_2^t = \{\alpha^t : \alpha \in l_2\}$.

The following can be found in [4].

Proposition 2.1. *Let D be an $\infty \times \infty$ Hilbert matrix. Then*

(1) *D^t, D^* are Hilbert matrices.*

(2) *Every row of D is in l_2^t, and every column of D is in l_2.*

(3) *If D_1 and D_2 are the $\infty \times \infty$ matrices such that every row of D_1 is in l_2^t, and every column of D_2 is in l_2, then $(D_1 D) D_2 = D_1 (D D_2)$ (associative).*

Proposition 2.2. *Let D be an $\infty \times \infty$ matrix. Then the following is equivalent:*

(1) *D is a Hilbert matrix.*

(2) *For each $\alpha \in l_2$, $D\alpha$ exists and belongs to l_2.*

(3) *The map* $\alpha \to D\alpha$ *defines a bounded linear operator from* l_2 *into* l_2.

(4) *There exists a constant* $K < \infty$ *such that*

$$|\alpha^* D\beta| \leqslant K \|\alpha\|_{l_2} \|\beta\|_{l_2}$$

for all $\alpha, \beta \in l_2^0$. *Here* l_2^0 *is the set of all* $\alpha = (\alpha_j) \in l_2$ *such that* α_j *vanishes for all but finitely many* j.

Proof. (1) $\Leftrightarrow$ (2) is a direct consequence of the definition of Hilbert matrix. (3) $\Leftrightarrow$ (4) and (3) $\Rightarrow$ (2) are easy to check. To see (2) $\Rightarrow$ (3), let $\alpha = (\alpha_j) \in l_2$, and let α^n be the vector in l_2 whose kth entry is α_k if $k \leqslant n$, and zero elsewhere. Then $\alpha^n \to \alpha$ in l_2, and, by (2), $D\alpha^n \to D\alpha$ as $n \to \infty$. However, for each $n \in \mathbb{N}$ fixed, the map $\alpha \to D\alpha^n$ defines a bounded operator defined everywhere in l_2. Therefore by the Banach–Steinhaus closure theorem (see [9]), the map $\alpha \to D\alpha$ defines a bounded operator from l_2 into l_2.

An $N \times N$ ($N \leqslant \infty$) Hilbert matrix D is called nonsingular if the map $\alpha \to D\alpha$ defines an isomorphism from l_2^N onto l_2^N. By an isomorphism we mean a one-to-one, onto and continuous linear map. If D is nonsingular in l_2^N, then it admits a unique matrix inverse D^{-1} (see [1]) and D^{-1} is also a nonsingular Hilbert matrix in l_2^N.

PROPOSITION 2.3.

(1) *If* $D = (d_{k,j})$ *is an* $\infty \times \infty$ *nonsingular Hilbert matrix, then* $\sum_{k,j=1}^{\infty} |d_{kj}|^2 = \infty$.

(2) *If* D *is an* $\infty \times \infty$ *matrix with* $D^*D = E_\infty$, *then the map* $\alpha \to D\alpha$ *defines an isometry from* l_2 *into* l_2.

(3) *For an* $\infty \times \infty$ *matrix* D, *the following are equivalent*:

(3-i) D *is a unitary matrix, i.e.,* $DD^* = D^*D = E_\infty$.

(3-ii) *The map* $\alpha \to D\alpha$ *defines a unitary operator from* l_2 *onto* l_2.

(3-iii) *The set of the rows forms an orthonormal basis for* l_2^t.

Proof. (1) If $\sum_{k,j=1}^{\infty} |d_{kj}|^2 < \infty$, then the map $\alpha \to D\alpha$ defines a compact operator on l_2 which is one-to-one and onto. This is impossible because l_2 is infinite dimensional. For (2), note first that $(D\alpha)^*(D\beta) = \alpha^*\beta$ for every $\alpha, \beta \in l_2^0$. Since l_2^0 is dense in l_2, we can extend the above map to the unique isometry V, say, on l_2, and thus the matrix representation of V must coincide with D. The rest of the theorem is easy to check.

3. DIFFERENTIAL OPERATORS

For each $j = 1, 2, \ldots, \kappa$, let $T_0(\tau_j)$ denote the minimal closed symmetric differential operator in $L_2(I_j)$ associated with τ_j, and put $T_1(\tau_j) \equiv T_0^*(\tau_j)$.

Let $\mathscr{D}_1(\tau)$ denote the linear space of all $f \in L_2(I)$ such that for each $j = 1, 2,..., \kappa$, $f|_{I_j} \in \mathscr{D}_1(\tau_j) \equiv \mathscr{D}(T_1(\tau_j))$, and $\tau f \in L_2(I)$. Let $T_1(\tau)$ denote the operator in $L_2(I)$ defined by

$$T_1(\tau)y = \tau y, \qquad \mathscr{D}(T_1(\tau)) = \mathscr{D}_1(\tau).$$

Clearly $T_1(\tau)$ is a densely defined closed operator in $L_2(I)$, and its adjoint operator $T_0(\tau) \equiv T_1^*(\tau)$ is a closed symmetric operator. Let $(f \mid g)_*$ denote the inner product in the Hilbert space $\mathscr{D}_1(\tau)$ (see, for example, [5]). The topology in $\mathscr{D}_1(\tau)$ generated by the above inner product will be called the τ-topology or $T_1(\tau)$-topology. Let $\mathscr{D}_0(\tau)$ denote the domain of $T_0(\tau)$. Then we have

$$\mathscr{D}_1(\tau) = \mathscr{D}_0(\tau) \oplus \mathscr{N}(-\iota) \oplus \mathscr{N}(\iota),$$

where the direct sum is orthogonal in the Hilbert space $\mathscr{D}_1(\tau)$. Here $\mathscr{N}(\pm\iota) = \{f \in \mathscr{D}_1(\tau): \tau f = \pm\iota f\}$. It is easy to see that the τ-topology and the norm topology in $\mathscr{N}(-\iota) \oplus \mathscr{N}(-\iota)$ are equivalent.

Let

$$N_j^{\pm} = \dim\{f \in \mathscr{D}_1(\tau_j): \tau_j f = \pm\iota f\} \qquad (j = 1, 2,..., \kappa),$$
$$N^{\pm} = \dim \mathscr{N}(\pm\iota), \qquad N = N^- + N^+.$$

Then

$$0 \leqslant N^- = \sum_{j=1}^{\kappa} N_j^- \leqslant \infty, \qquad 0 \leqslant N^+ = \sum_{j=1}^{\kappa} N_j^+ \leqslant \infty.$$

Clearly if each $T_0(\tau_j)$ is self-adjoint, then $N^- = N^+ = 0$, and if each τ_j is regular and if $\kappa = \infty$, then $N_j^{\pm} = n_j$, $N^{\pm} = \infty$, where n_j is the order of τ_j.

Throughout this paper we identify any $f \in L_2(I_j)$ as an element of $L_2(I)$ by letting $f(t) = 0$ for $t \notin I_j$. *Let $\{\phi_j: 1 \leqslant j \leqslant N\}$ be an arbitrary but fixed basis for $\mathscr{N}(-\iota) \oplus \mathscr{N}(\iota)$ such that in the case when $N = \infty$, the basis is orthonormal with respect to the τ-topology.*

Let Φ denote the $N \times 1$ column vector whose jth entry is ϕ_j. Let B denote the operator

$$B(y) = (y \mid \Phi)_*, \qquad y \in \mathscr{D}_1(\tau). \tag{2.1}$$

THEOREM 3.4.

(1) *B annihilates $\mathscr{D}_0(\tau)$, and B restricted to $\mathscr{N}(-\iota) \oplus \mathscr{N}(\iota)$ defines an isomorphism onto l_2^N. If $N = \infty$ (thus by assumption $\{\phi_j: j \in \mathbb{N}\}$ is τ-orthonormal), then B restricted to $\mathscr{N}(-\iota) \oplus \mathscr{N}(\iota)$ provided with the τ-topology defines a unitary operator onto l_2.*

(2) *There exists a unique $N \times N$ nonsingular Hermitian Hilbert matrix C (depending on B) such that*

$$(\tau y \mid g) - (y \mid \tau g) = \iota(B(g))^* CB(y)$$

for $y, g \in \mathscr{D}_1(\tau)$. In particular, if $N = \infty$, then $C = C^* = C^{-1}$.

Proof. We shall prove the theorem for the case when $N = \infty$ because the case $N < \infty$ follows easily. Clearly B annihilates $\mathscr{D}_0(\tau)$. Let $y, g \in \mathscr{D}_1(\tau)$ and write $y = y_0 + y_1$, $g = g_0 + g_1$ where $y_0, g_0 \in \mathscr{D}_0(\tau)$ and $y_1, g_1 \in \mathscr{N}(-\iota) \oplus \mathscr{N}(\iota)$. Since $\{\phi_j : j \in \mathbb{N}\}$ is an orthonormal basis for $\mathscr{N}(-\iota) \oplus \mathscr{N}(\iota)$, the entries of $B(y_1)$ are the Fourier coefficients of the expansion of y_1 with respect to $\{\phi_j : j \in \mathbb{N}\}$. Therefore the map $y \to B(y)$ defines a unitary operator from $\mathscr{N}(-\iota) \oplus \mathscr{N}(\iota)$ provided with the τ-topology onto l_2^N. Let J denote the unitary operator from $L_2(I) \times L_2(I)$ onto $L_2(I) \times L_2(I)$ defined by $J(\{a, b\}) = \{b, -a\}$. Let C denote the $N \times N$ hermitian matrix whose (j, k)-entry is $((J(\{\phi_j, \tau\phi_j\}) \mid \{\phi_k, \tau\phi_k\}))/\iota$ where $((\ \mid\ ))$ denotes the inner product of $L_2(I) \times L_2(I)$. Then since J is a unitary operator on $L_2(I) \times L_2(I)$ and $\{\hat{\phi}_j : j \in \mathbb{N}\}$ is an orthonormal basis for $\mathscr{G}(T_1(\tau)) \ominus \mathscr{G}(T_0(\tau))$, C is a unitary Hilbert matrix. Here $\mathscr{G}$ denotes "the graph of" and $\hat{\phi}_j = \{\phi_j, \tau\phi_j\}$. Let $y \in \mathscr{N}(-\iota) \oplus \mathscr{N}(\iota)$. Then

$$\{y, \tau y\} = (B(y))^t \hat{\Phi}$$

where the sum converges in $L_2(I) \times L_2(I)$ and $\hat{\Phi}$ is the $\infty \times 1$ column vector whose jth entry is $\{\phi_j, \tau\phi_j\}$. Thus for $y, g \in \mathscr{N}(-\iota) \oplus \mathscr{N}(-\iota)$,

$$\begin{aligned}(\tau y \mid g) - (y \mid \tau g) &= ((J(\{y, \tau y\}) \mid \{g, \tau g\})) \\ &= ((J((B(y))^t \hat{\Phi}) \mid (B(g))^t \hat{\Phi})) = \iota (B(g))^* C B(y).\end{aligned}$$

This completes the proof.

Remark. If $\kappa < \infty$ and each τ_j is regular in $\bar{I}_j$, then C can be obtained from the Green's formula for τ_j.

PROPOSITION 3.5. *Suppose that $F(t, y)$ is a map from $I \times \mathscr{D}_1(\tau)$ into $\mathbb{C}$ such that* (i) *for a.a. $t \in I$, the map $y \to F(t, y)$ is a τ-continuous linear functional on $\mathscr{D}_1(\tau)$, annihilating $\mathscr{D}_0(\tau)$,* (ii) *for each $y \in \mathscr{D}_1(\tau)$ fixed, $h(t) = F(t, y) \in L_2(I)$,* (iii) *$\{F(t, y) : y \in \mathscr{D}_1(\tau)\}$ is an m-dimensional subspace of $L_2(I)$ ($m < \infty$) with a basis $\chi_1, \ldots, \chi_m$. Then there exists a $m \times N$ Hilbert matrix D such that*

$$F(t, y) = \chi^t(t)\, D B(y) \qquad (*)$$

for a.a. $t \in I$ and $y \in \mathscr{D}_1(\tau)$ where $\chi = (\chi_1, \chi_2, \ldots, \chi_m)$.

Conversely, if D is a $m \times N$ ($m < \infty$) Hilbert matrix and if $\chi \in (L_2(I))^m$, the function $F(t, y)$ defined by $(*)$ satisfies (i) *and* (ii), *and is of finite-dimensional range with its dimension $\leqslant m$.*

Proof. Assume (i)–(iii). In particular, (i) implies that for a.a. $t \in I$ fixed, there exists a $N \times 1$ column vector $\psi(t)$ such that $F(t, y) = \psi^t(t)\, B(y)$ for all $y \in \mathscr{D}_1(\tau)$ where the sum converges and $\psi(t) \in l_2^N$. Since $B(y)$ can be an arbitrary member in l_2^N and $h(t) = F(t, y) \in L_2(I)$, each row of $\psi(t)$ is in $L_2(I)$. There exists a $m \times N$ matrix D such that $\psi^t(t) = \chi^t(t) D$ for a.a. $t \in I$. Thus $F(t, y) =$

$\chi^t(t)\, DB(y)$ for a.a. $t \in I$ and $y \in \mathscr{D}_1(\tau)$. Since $\{\chi_j: 1 \leqslant j \leqslant m < \infty\}$ is linearly independent, each row of D must be in $(l_2^N)^t$, so that D is a $m \times N$ Hilbert matrix in l_2^N. The converse is clear.

THEOREM 3.6. *Let M be a $m \times N$ ($m \leqslant N$, $m < \infty$) Hilbert matrix and $\chi \in (L_2(I))^m$ and A be a bounded linear operator defined everywhere in $L_2(I)$. If $T_1(\mathscr{L})$ is the operator*

$$T_1(\mathscr{L})\, y = \mathscr{L}y, \qquad \mathscr{D}(T_1(\mathscr{L})) = \mathscr{D}_1(\tau),$$

then

$$T_1^*(\mathscr{L})z = \tau z + A^*z,$$

$$\mathscr{D}(T_1^*(\mathscr{L})) = \{z : z \in \mathscr{D}_1(\tau),\ (B(z))^*C + (\chi^t \mid z)M = 0_{1\times N}\}.$$

Proof. Let $T = T_1(\mathscr{L})$. Take $z \in \mathscr{D}(T^*)$. Then

$$0 = (\tau y \mid z) - (y \mid T^*z - A^*z) + \iota(\chi^t \mid z)\, MB(y)$$

for all $y \in \mathscr{D}_1(\tau)$. This is in particular true for all $y \in \mathscr{D}_0(\tau)$. Thus $z \in \mathscr{D}_1(\tau)$ and $T^*z = \tau z + A^*z$. Now for $y \in \mathscr{D}_1(\tau)$

$$(\tau y + Ay + \iota\chi^t MB(y) \mid z) = (y \mid \tau z + A^*z).$$

Thus

$$((B(z))^*C + (\chi^t \mid z)M)\, B(y) = 0_{1\times N}$$

for all $y \in \mathscr{D}_1(\tau)$. It follows from Theorem 3.4, $(B(z))^*C + (\chi^t \mid z)M = 0_{1\times N}$.

PROPOSITION 3.7. *Let $\{\chi_j: 1 \leqslant j \leqslant m\}$ ($m < \infty$) be a set in $L_2(I)$ and let D be a $N \times m$ Hilbert matrix in l_2^N. Let χ denote the $m \times 1$ column vector whose* jth *entry is χ_j. Then*

(1) *The map $y \to B(y) + D(y \mid \chi)$ defines a τ-bounded operator from $\mathscr{D}_1(\tau)$ onto l_2^N.*

(2) *Assume that the operator B in* (2.1) *has the form*

$$B^t(y) = (B_1^t(y \mid I_1),\ldots, B_\kappa^t(y \mid I_\kappa))\ (1 \times N).$$

Here each B_j is the τ_j-continuous operator from $\mathscr{D}(T_1(\tau_j))$ onto $\mathbb{C}^{N_j^- + N_j^+}$, annihilating $\mathscr{D}(T_0(\tau_j))$. Let us write

$$D = \begin{bmatrix} D_1 \\ \vdots \\ D_\kappa \end{bmatrix}$$

where each D_j is $(N_j^- + N_j^+) \times m$. Then $\{y \in \mathscr{D}_1(\tau) \mid B(y) + D(y \mid \chi) = 0_{N\times 1}\}$ is dense in $L_2(I)$ if for each $j = 1, 2,\ldots, \kappa$, $\bar{D}_j\chi(t) = 0_{(N_j^- + N_j^+)\times 1}$ for a.a. $t \notin I_j$.

Proof. (1) Let $\alpha \in l_2^N$. We may assume that the entries of χ_k of χ are linearly independent. Thus the map $y \to (y \mid \chi)$ from $\mathscr{D}_0(\tau)$ into $\mathbb{C}^m$ is surjective. Let Z be the $m \times 1$ column vector with entries in $\mathscr{D}_0(\tau)$ such that $(Z \mid \chi^t) = E_m$. Let $B(a) = \alpha$, $a \in \mathscr{D}_1(\tau)$ and put $y = a - Z^t(a \mid \chi)$. Then $y \in \mathscr{D}_1(\tau)$, and $B(y) + D(y \mid \chi) = \alpha$.

(2) For each $j = 1, 2, \ldots, \kappa, \{y \in \mathscr{D}(T_1(\tau_j)) \mid B_j(y) + D_j(y \mid \chi)_j = 0_{(N_j^- + N_j^+) \times 1}\}$ is dense in $L_2(I_j)$ by Lemma 2.2 in [7]. Here $(y \mid \chi)_j = \int_{I_j} y(t)\, \bar{\chi}(t)\, dt$. Thus the set in the theorem is dense in $L_2(I)$.

Remark. If $\kappa < \infty$, then the set in (2) in the above proposition is always dense.

THEOREM 3.8. *Let $T_1(\mathscr{L})$ be as in Theorem 3.6. Then*

$$(\mathscr{G}(T_1(\mathscr{L})))^c = \{\{y, \tau y + Ay + \chi^t M\beta\} : y \in \mathscr{D}_1(\tau) \text{ and } \beta \text{ is any element in } l_2^N \text{ such that } (\chi^t \mid z)(M\beta - \iota MB(y)) = 0 \text{ for all } z \in \mathscr{D}(T_1^*(\mathscr{L}))\}.$$

Here c denotes "the closure of."

If $\mathscr{D}(T_1^*(\mathscr{L}))$ is dense in $L_2(I)$, then $T_1(\mathscr{L}) = T_1^{**}(\mathscr{L})$ and $T_1(\mathscr{L})$ is closed

Proof. Let us write $T = T_1(\mathscr{L})$. We will compute $(\mathscr{G}(T))^{**}$ where $(\mathscr{G}(T))^* = J(\mathscr{G}(T))^\perp$. Let $b = \{b_1, b_2\} \in (\mathscr{G}(T))^{**}$. Then for $z \in \mathscr{D}(T^*)$,

$$0 = ((J(\{z, \tau z + A^* z\}) \mid \{b_1, b_2\})). \qquad (*)$$

This is true in particular for all $z \in \mathscr{D}_0(\tau)$ with $(\chi^t \mid z)M = 0_{1 \times N}$. Let $Q = \{\{\chi^t M\beta, 0\} : \beta \in l_2^N\}$. This is finite dimensional and hence closed in $L_2(I) \times L_2(I)$. It follows that

$$\{Ab_1 - b_2, b_1\} \in (\mathscr{G}(T_0(\tau)) \cap Q^\perp)^\perp = (\mathscr{G}(T_0(\tau)))^\perp \oplus Q$$

so that

$$\{b_1, b_2 - Ab_1\} \in \{\{y, \tau y + \chi^t M\beta\} : y \in \mathscr{D}_1(\tau), \beta \in l_2^N\}.$$

Thus $b_1 \in \mathscr{D}_1(\tau)$ and $b_2 = \tau b_1 + Ab_1 + \chi^t M\beta$ for some $\beta \in l_2^N$. Returning to $(*)$

$$\iota(B(z))^* CB(b_1) + (\chi^t \mid z)\, M\beta = 0$$

for all $z \in \mathscr{D}(T^*)$. Thus $(\chi^t \mid z)(M\beta - \iota MB(b_1)) = 0$ for all $z \in \mathscr{D}(T^*)$. Since $(\mathscr{G}(T_1(\mathscr{L})))^{**} = (\mathscr{G}(T_1(\mathscr{L})))^c$ we have the first part. Assume $\mathscr{D}(T_1^*(\mathscr{L}))$ is dense. We may assume that the entries of χ_k are linearly independent. Thus $(\chi^t \mid z)(M\beta - \iota MB(y)) = 0$ for all $z \in \mathscr{D}(T_1^*(\mathscr{L}))$ implies that $M\beta = \iota MB(y)$. Thus $\mathscr{G}(T_1(\mathscr{L})) = (\mathscr{G}(T_1(\mathscr{L})))^c$, so that $T_1(\mathscr{L}) = T_1^{**}(\mathscr{L})$.

Remark. If the entries of χ_k are linearly independent, then the map $y \to (y \mid \chi)$ from $\mathscr{D}(T_1^*(\mathscr{L}))$ into $\mathbb{C}^m$ is surjective iff $\mathscr{D}(T_1^*(\mathscr{L}))$ is dense.

Let $\tilde{\chi} = (\tilde{\chi}_i)$ be a $\tilde{m} \times 1$ ($\tilde{m} < \infty$) vector function for $t \in I$ such that each $\tilde{\chi}_j \in L_2(I)$. Let $\tilde{M}$ be a $\tilde{m} \times N$ Hilbert matrix, and let $\tilde{A}$ be a bounded linear operator in $L_2(I)$ defined on $L_2(I)$. Let $\tilde{\mathscr{L}}$ be the expression

$$\tilde{\mathscr{L}}z = \tau z + \tilde{A}z + \iota\tilde{\chi}^t\tilde{M}B(z), \tag{3.2}$$

for $z \in \mathscr{D}_1(\tau)$. Let $T_1(\tilde{\mathscr{L}})$ be the operator

$$T_1(\tilde{\mathscr{L}})z = \tilde{\mathscr{L}}z, \qquad \mathscr{D}(T_1(\tilde{\mathscr{L}})) = \mathscr{D}_1(\tau).$$

Then by Theorem 3.8, if $T_1^*(\tilde{\mathscr{L}})$ is densely defined, then $T_1(\tilde{\mathscr{L}}) = T_1^{**}(\tilde{\mathscr{L}})$ and $T_1(\tilde{\mathscr{L}})$ is closed.

Definition. The expression $\tilde{\mathscr{L}}$ in (3.2) is adjoint to the expression $\mathscr{L}$ defined by (1.1) if

$$\tilde{A}z = A^*z + \iota\tilde{\chi}^t\tilde{M}C^{-1}M^*(z \mid \chi), \qquad z \in L_2(I).$$

Notice that, since $C = C^*$, $\tilde{\mathscr{L}}$ is adjoint to $\mathscr{L}$ iff $\mathscr{L}$ is adjoint to $\tilde{\mathscr{L}}$.

Let us define the operators $V\colon \mathscr{D}_1(\tau) \to l_2^N$, $\tilde{V}\colon \mathscr{D}_1(\tau) \to l_2^N$ by

$$\begin{aligned} \tilde{V}(y) &= B(y) + C^{-1}M^*(y \mid \chi) \\ V(z) &= B(z) + C^{-1}\tilde{M}^*(z \mid \tilde{\chi}) \end{aligned} \tag{3.3}$$

Then the following is easy to check.

Proposition 3.9. *Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be an adjoint pair. Then*

(1) $(\mathscr{L}y \mid z) - (y \mid \tilde{\mathscr{L}}z) = \iota\tilde{V}^*(z)\, CV(y), \qquad y, z \in \mathscr{D}_1(\tau).$

(2) $T_1^*(\mathscr{L}) \subset T_1(\tilde{\mathscr{L}}), \qquad T_1^*(\tilde{\mathscr{L}}) \subset T_1(\mathscr{L}),$

$\tilde{V}^{-1}(0) = \mathscr{D}(T_1^*(\mathscr{L})), \qquad V^{-1}(0) = \mathscr{D}(T_1^*(\tilde{\mathscr{L}})).$

Proposition 3.10. *Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be an adjoint pair and assume that $T_1^*(\mathscr{L})$ and $T_1^*(\tilde{\mathscr{L}})$ are densely defined. Then the $T_1(\tau)$-topology, $T_1(\mathscr{L})$-topology, and $T_1(\tilde{\mathscr{L}})$-topology in $\mathscr{D}_1(\tau)$ are all equivalent.*

Proof. We shall only show that the $T_1(\tau)$-topology and $T_1(\mathscr{L})$-topology are equivalent because the rest follows easily. Note first that the map $y \to MB(y)$ from the Hilbert space $\mathscr{D}_1(\tau)$ with the *-inner product into the Euclidean space $\mathbb{C}^m$ is a bounded linear operator, and the map $\alpha \to \chi^t\alpha$ from the Euclidean space $\mathbb{C}^m$ into $L_2(I)$ is also a bounded operator. Therefore the map $y \to \chi^tMB(y)$ from the Hilbert space $\mathscr{D}_1(\tau)$ with the $T_1(\tau)$-topology into $L_2(I)$ is a bounded linear operator. Hence there exists a constant $K_1 < \infty$ such that $\| \chi^t MB(y)\| \leqslant K_1 \| y \|_{T_1(\tau)}$ for every $y \in \mathscr{D}_1(\tau)$. Thus

$$\begin{aligned} \| \mathscr{L}y \| &\leqslant \| \tau y \| + \| Ay \| + \| \chi^t MB(y)\| \\ &\leqslant K_2 \| y \|_{T_1(\tau)} \end{aligned}$$

for some constant $K_2 < \infty$. Thus

$$\| y \|_{T_1(\mathscr{L})} = (\| y \|^2 + \| \mathscr{L}y \|^2)^{1/2} \leqslant (1 + K_2)^{1/2} \| y \|_{T_1(\tau)} .$$

Hence the identity operator from the Hilbert space $\mathscr{D}_1(\tau)$ with the *-inner product to the Hilbert space $\mathscr{D}_1(\tau)$ with the $T_1(\mathscr{L})$-norm is continuous. Therefore by the closed graph theorem, the two topologies must be equivalent. This completes the proof.

Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be the same as in (1.1) and (3.2) (need not be an adjoint pair). Then we define "minimal" operators $T_0(\mathscr{L})$ and $T_0(\tilde{\mathscr{L}})$ by

$$T_0(\mathscr{L}) = T_1^*(\tilde{\mathscr{L}}), \qquad T_0(\tilde{\mathscr{L}}) = T_1^*(\mathscr{L}).$$

Then, if $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are an adjoint pair and if $T_1^*(\mathscr{L})$ and $T_1^*(\tilde{\mathscr{L}})$ are densely defined, we have the following four closed operators satisfying

$$\begin{array}{ccc} T_0(\mathscr{L}) & \subset & T_1(\mathscr{L}) \\ & \nwarrow \!\!\!\! \nearrow \\ & \swarrow \!\!\!\! \searrow \\ T_0(\tilde{\mathscr{L}}) & \subset & T_1(\tilde{\mathscr{L}}) \end{array}$$

where $\leftrightarrow$ means one is adjoint to the other.

PROPOSITION 3.11. *Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be an adjoint pair and assume that $T_0(\mathscr{L})$ and $T_0(\tilde{\mathscr{L}})$ are densely defined. Then*

(1) *$F \in (\mathscr{D}(T_1(\mathscr{L}))/\mathscr{D}(T_0(\mathscr{L})))^*$ iff there exists a $\alpha \in l_2^N$ such that $F(y) = \alpha^t V(y)$ for all $y \in \mathscr{D}_1(\tau)$.*

(2) *$F \in (\mathscr{D}(T_1(\tilde{\mathscr{L}}))/\mathscr{D}(T_0(\tilde{\mathscr{L}})))^*$ iff there exists a $\alpha \in l_2^N$ such that $F(y) = \alpha^t \tilde{V}(y)$ for all $y \in \mathscr{D}_1(\tau)$.*

(3) *V defines an isomorphism from the Hilbert space $\mathscr{D}_1(\tau) \ominus \mathscr{D}(T_0(\mathscr{L}))$ onto l_2^N.*

(4) *$\tilde{V}$ defines an isomorphism from the Hilbert space $\mathscr{D}_1(\tau) \ominus \mathscr{D}(T_0(\tilde{\mathscr{L}}))$ onto l_2^N.*

Proof. (1) Assume $F(y) = \alpha^t V(y)$ for all $y \in \mathscr{D}_1(\tau)$. Then F is the pointwise limit of τ-continuous functionals on the Hilbert space $\mathscr{D}_1(\tau)$. Thus by the

Banach–Steinhaus closure theorem F is a τ-continuous functional. Clearly F annihilates $\mathscr{D}(T_0(\mathscr{L}))$. Thus $F \in (\mathscr{D}(T_1(\mathscr{L}))/\mathscr{D}(T_0(\mathscr{L})))^*$. The converse is clear. (2) can be proved in a similar way. (3) and (4) follow easily from Proposition 3.7.

The main purpose of this paper is to find all the closed operators including symmetric ones between $T_0(\mathscr{L})$ and $T_1(\mathscr{L})$.

Proposition 3.12. *Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be an adjoint pair and let $T_0(\mathscr{L})$ and $T_0(\tilde{\mathscr{L}})$ be densely defined. Then*

(1) *A linear operator T between $T_0(\mathscr{L})$ and $T_1(\mathscr{L})$ is closed iff there exists a $m \times N$ $(m \leqslant N)$ Hilbert matrix P such that*

$$\mathscr{D}(T) = \{y \in \mathscr{D}_1(\tau): PCV(y) = 0_{m\times 1}\}.$$

(2) *A linear operator $\tilde{T}$ between $T_0(\tilde{\mathscr{L}})$ and $T_1(\tilde{\mathscr{L}})$ is closed iff there exists a $\tilde{m} \times N$ $(\tilde{m} \leqslant N)$ Hilbert matrix $\tilde{P}$ such that*

$$\mathscr{D}(\tilde{T}) = \{z \in \mathscr{D}_1(\tau): \tilde{P}C\tilde{V}(z) = 0\}_{\tilde{m}\times 1}.$$

Proof. (1) Assume T is closed. Then $T = T^{**}$. Thus for every $z \in \mathscr{D}(T^*)$

$$0 = (\mathscr{L}y \mid z) - (y \mid \tilde{\mathscr{L}}z) = \iota(\tilde{V}(z))^* \, CV(y).$$

Let $G = \{(\tilde{V}(z))^*: z \in \mathscr{D}(T^*)\}$. This is a closed subspace of $(l_2^N)^t$. Let m denote its dimension. Let $\{p_j: 1 \leqslant j \leqslant m\}$ be an orthonormal basis for G. Here each p_j is $1 \times N$. Let P denote the $m \times N$ matrix whose jth row is p_j. Clearly, P is a Hilbert matrix and $PCV(y) = 0_{m\times 1}$. Conversely, if $y \in \mathscr{D}_1(\tau)$ and $PCV(y) = 0_{m\times 1}$, then $0 = (\mathscr{L}y \mid z) - (y \mid \tilde{\mathscr{L}}z)$ for all $z \in \mathscr{D}(T^*)$. This proves that $\mathscr{D}(T) = \{y \in \mathscr{D}_1(\tau): PCV(y) = 0_{m\times 1}\}$. The converse of (1) is clear. This proves (1). (2) can be proved in a similar way.

Definition. If D is a $N \times d$ $(d \leqslant N \leqslant \infty)$ Hilbert matrix, then $\langle D \rangle$ will denote the closure of the linear span of $\{d_j: 1 \leqslant j \leqslant d\}$ where each d_j is the jth *column* of D (thus $\langle D \rangle \subset l_2^N$). $(\langle D \rangle)^\perp$ will denote the orthogonal compliment of $\langle D \rangle$ in l_2^N. If $d < \infty$, then the rank of D, denoted by $\rho(D)$, is the maximum number of linearly independent columns of D.

Theorem 3.13. *Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be the same as in* (1.1) *and* (3.2) (*need not be an adjoint pair*), *and assume $T_1(\mathscr{L})$ and $T_1(\tilde{\mathscr{L}})$ are densely defined. Let T and $\tilde{T}$ denote the operators defined by*

$$Ty = \mathscr{L}y, \qquad \mathscr{D}(T) = \{y \in \mathscr{D}_1(\tau): PCV(y) = 0_{m\times 1}\},$$
$$\tilde{T}z = \tilde{\mathscr{L}}z, \qquad \mathscr{D}(\tilde{T}) = \{z \in \mathscr{D}_1(\tau): \tilde{P}C\tilde{V}(z) = 0_{\tilde{m}\times 1}\}.$$

Here P is a $m \times N$ ($m \leqslant N$) Hilbert matrix such that $PP^* = I_\infty$ if $m = \infty$, and $\rho(P^t) = m$ if $m < \infty$. $\tilde{P}$ is a $\tilde{m} \times N$ ($\tilde{m} \leqslant N$) Hilbert matrix such that $\tilde{P}\tilde{P}^* = I_\infty$ if $\tilde{m} = \infty$, and $\rho(P^t) = \tilde{m}$ if $\tilde{m} < \infty$. Then the following statements are equivalent:

(1) $\tilde{T} = T^*$.

(2) $\tilde{T}^* = T$.

(3) $l_2^N = \langle CP^* \rangle \oplus \langle \tilde{P}^* \rangle$ *(orthogonal), $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are an adjoint pair.*

(4) $l_2^N = \langle P^* \rangle \oplus \langle C\tilde{P}^* \rangle$ *(orthogonal), $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are an adjoint pair.*

Proof. We prove (1) $\Leftrightarrow$ (3) only because the rest can be proved easily. Assume (1). Since $T \subset T_1(\mathscr{L})$, $T_1^*(\mathscr{L}) \subset T_1(\tilde{\mathscr{L}})$. Thus $\tau y + A^* y = \tau y + \tilde{A} y + \iota \bar{\chi}^t MB(y)$ for all y with $\tilde{V}(y) = 0$. This implies that $\tilde{\mathscr{L}}$ is adjoint to $\mathscr{L}$ because $\tilde{V}^{-1}(0)$ is dense in $L_2(I)$. Let us write $l_2^N = \langle CP^* \rangle \oplus \langle \tilde{P}_1^* \rangle$ where the direct sum is orthogonal. Here $\tilde{P}_1$ is a $\tilde{m} \times N$ ($\tilde{m} \leqslant N$) Hilbert matrix such that $\tilde{P}_1 \tilde{P}_1^* = I_\infty$ if $\tilde{m} = \infty$, and $\rho(\tilde{P}_1)^t = \tilde{m}$ if $\tilde{m} < \infty$. Then $\mathscr{D}(T) = V^{-1}((\langle CP^* \rangle)^\perp) = V^{-1}(\langle \tilde{P}_1^* \rangle)$, so that $\mathscr{D}(T^*) = \tilde{V}^{-1}((\langle C\tilde{P}_1^* \rangle)^\perp)$. Hence $\tilde{T} = T^*$ implies that $\langle \tilde{P}_1^* \rangle = \langle \tilde{P}^* \rangle$. Thus $l_2^N = \langle CP^* \rangle \oplus \langle \tilde{P}^* \rangle$, (3) $\Rightarrow$ (1) can be shown easily.

Remark. Since V and $\tilde{V}$ restricted to $\mathscr{D}(T_1(\mathscr{L})) \ominus \mathscr{D}(T_0(\mathscr{L}))$ and $\mathscr{D}(T_1(\tilde{\mathscr{L}})) \ominus \mathscr{D}(T_0(\tilde{\mathscr{L}}))$ are isomorphisms onto l_2^N, (3) implies that $N = m + \tilde{m}$ and

$$\dim(T_1(\mathscr{L})/T) = m, \qquad \dim(T_1(\tilde{\mathscr{L}})/T^*) = \tilde{m}$$

where we have identified the operators as their graphs.

Suppose that the maximal operator $T_1(\mathscr{L})$ is closed and has a closed symmetric restriction T of the type $T^* \subset T_1(\mathscr{L})$. This implies that $T_1^*(\mathscr{L})$ is densely defined and $T_1^*(\mathscr{L}) \subset T \subset T^* \subset T_1(\mathscr{L})$. Thus $\mathscr{L}$ is adjoint itself. This means that $\tilde{V}$ is sufficient in describing the symmetric restrictions of $T_1(\mathscr{L})$. We have the following

THEOREM 3.14. *Let the $\mathscr{L}$ in* (1.1) *be adjoint to itself and $T_1^*(\mathscr{L})$ be densely defined. Let T denote the operator (closed) defined by*

$$Ty = \mathscr{L} y, \qquad \mathscr{D}(T) = \{y \in \mathscr{D}_1(\tau) \colon PC\tilde{V}(y) = 0_{m \times 1}\}.$$

Here P is a $m \times N$ ($m \leqslant N$) Hilbert matrix such that $PP^* = I_\infty$ if $m = \infty$, and $\rho(P^t) = m$ if $m < \infty$. Then the following statements are equivalent:

(1) *T is a symmetric restriction of $T_1(\mathscr{L})$.*

(2) *There exists a $\tilde{m} \times m$ ($\tilde{m} \leqslant m \leqslant N$) Hilbert matrix D such that*

$$l_2^N = \langle CP^* \rangle \oplus \langle (DP)^* \rangle \ (orthogonal).$$

In particular, $\mathscr{D}(T^*) = \{y \in \mathscr{D}_1(\tau) \colon DPC\tilde{V}(y) = 0_{\tilde{m} \times 1}\}$.

Proof. Let $\tilde{P}$ be the $\tilde{m} \times N$ ($\tilde{m} \leqslant N$) Hilbert matrix such that $l_2^N = \langle CP^* \rangle \oplus \langle \tilde{P}^* \rangle$ where the direct sum is orthogonal. Furthermore, $\tilde{P}\tilde{P}^* = I_\infty$ if $\tilde{m} = \infty$, and $\rho(\tilde{P}^t) = \tilde{m}$ if $\tilde{m} < \infty$. Then by Theorem 3.13, $\mathscr{D}(T^*) = V^{-1}((\langle C\tilde{P}^* \rangle)^\perp)$ where we take $\tilde{V} = V$. Assume (1). Then $T \subset T^*$ so that

$$V^{-1}((\langle CP^* \rangle)^\perp) \subset V^{-1}((\langle C\tilde{P}^* \rangle)^\perp).$$

This implies that $\langle \tilde{P}^* \rangle \subset \langle P^* \rangle$ and so $\tilde{m} \leqslant m$ and $\tilde{P} = DP$ for some $\tilde{m} \times m$ Hilbert matrix D satisfying the condition in the theorem. This proves that (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (1) is easy. The last part of the theorem is clear. This completes the proof.

If we allow D in the above theorem to be a nonsingular square Hilbert matrix, then we get all possible self-adjoint restrictions of $T_1(\mathscr{L})$. More precisely

THEOREM 3.15. *Let $\mathscr{L}$ and T be the same as in Theorem 3.14. Then T is self-adjoint iff $l_2^N = \langle CP^* \rangle \oplus \langle P^* \rangle$ where the direct sum is orthogonal.*

Remark. In the above theorem we did not assume that the deficiency indices of $T^*(\mathscr{L})$ are equal. However the existence of such a Hilbert matrix satisfying the above is equivalent to the fact that the deficiency indices are equal. This theorem also implies that if $\mathscr{L}$ in (1.1) generates a self-adjoint restriction, then τ also generates a self-adjoint operator.

Remark. If we wish to construct a nonsymmetric operator, then we simply delete some row vectors of P which defines the symmetric operator T in Theorem 3.14.

4. EXAMPLES

We shall give examples of self-adjoint operators in the case when the number κ of intervals is infinite, and $N^- = N^+ = N = \infty$. Let T be the same as in Theorem 3.15. We shall show in the following examples how to construct the corresponding operators B, $\tilde{V}$ and $\infty \times \infty$ unitary matrix C and $\infty \times \infty$ matrix P. These will be dependent on the direct sum expression τ.

EXAMPLE i. For each $j \in \mathbb{N}$, $\tau_j y = -\iota y'$, $t \in I_j \equiv (a_j, b_j)$,

$$(\tau y)(t) = -\iota y'(t) \qquad \text{if} \quad t \in I_j .$$

For each $j \in \mathbb{N}$, let

$$\begin{aligned} B_j(y) &= (e^{b_j} y(b_j-) - e^{a_j} y(a_j+))/(e^{2b_j} - e^{2a_j})^{1/2} \\ &= (e^{-b_j} y(b_j-) - e^{-a_j} y(a_j+))/(e^{-2a_j} - e^{-2b_j})^{1/2} \end{aligned}$$

$$C = \bigoplus_{j=1}^{\infty} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B^t(y) = (B_1^t(y), B_2^t(y), \ldots).$$

As a special case, take $M \equiv 0$, $A = A^*$, and

$$P = \frac{1}{2^{1/2}} \bigoplus_{1}^{\infty} (\theta, 1) \qquad (|\,\theta\,| = 1,\ \theta \notin \mathbb{R}).$$

Then

$$PC = \frac{1}{2^{1/2}} \bigoplus_{1}^{\infty} (-\theta, 1) \qquad \text{and} \qquad y \in \mathscr{D}(T)$$

iff $y \in \mathscr{D}_1(\tau)$ and, for each $j \in \mathbb{N}$,

$$\theta(e^{-2a_j} - e^{-2b_j})^{1/2}(e^{b_j}y(b_j-) - e^{a_j}y(a_j+))$$
$$= (e^{2b_j} - e^{2a_j})^{1/2}(e^{-b_j}y(b_j-) - e^{-a_j}y(a_j+)).$$

EXAMPLE ii. For each $j \in \mathbb{N}$, let

$$\tau_j y = \iota p_j(t)(p_j y)'(t) + q_j(t)y, \qquad t \in I_j \equiv (a_j, b_j),$$
$$(\tau y)(t) = (\tau_j y)(t) \qquad \text{if} \quad t \in I_j.$$

Here for each $j \in \mathbb{N}$, p_j is a continuously differentiable complex-valued function of $t \in I_j$ such that p_j^2 is real-valued, and $p_j(t) \neq 0$ for every $t \in \overline{I_j}$ except at a_j, and $p_j(a_j) = 0$. q_j is a continuous complex-valued function of $t \in I_j$. We assume further that sign $p_j^2(b_j) < 0$ if j is odd, sign $p_j^2(b_j) > 0$ if j is even, and that

$$(\tau_j f \mid g) - (f \mid \tau_j g) \equiv \langle f \mid g \rangle_g = \iota \bar{g}(b_j-)\, p_j^2(b_j)\, f(b_j-)$$

for every $f, g \in \mathscr{D}_1(\tau_j)$. Thus $N_j^- + N_j^+ = 1$ for every $j \in \mathbb{N}$ (see [8]). Because of the condition on the sign of $p_j^2(b_j)$,

$$N_j^- = N_j^+ = 0 \qquad \text{if} \quad j \text{ is even},$$
$$N_j^+ = N_j^- = 1 \qquad \text{if} \quad j \text{ is odd}$$

(see [6]). Thus $\sum_{j=1}^{\infty} N_j^- = \sum_{j=1}^{+\infty} N_j^+ = +\infty$, and

$$B_j(y) = (-p_j^2(b_j))^{1/2}\, y(b_j-) \qquad \text{if} \quad j \text{ is odd},$$
$$= (p_j^2(b_j))^{1/2}\, y(b_j-) \qquad \text{if} \quad j \text{ is even}.$$

$$C \equiv \bigoplus_{1}^{\infty} (-1)^j, \qquad B^t(y) = (B_1(y), B_2(y), \ldots).$$

As a special case, take $M \equiv 0$, $A = A^*$ and let $\mathscr{D}(T)$ be the set of all $y \in \mathscr{D}_1(\tau)$ such that

$$\theta(-p_{2j-1}^2(b_{2j-1}))^{1/2}\, y(b_{2j-1}-) = (p_{2j}^2(b_{2j}))^{1/2}\, y(b_{2j}-)$$

for every $j \in \mathbb{N}$ where θ is a given nonreal complex number with $|\,\theta\,| = 1$.

EXAMPLE iii. In this example, the order of the differential expression can "blow up."

For each $j \in \mathbb{N}$, let τ_{2j} denote the formally self-adjoint differential expression:

$$\tau_{2j} y = -(p_{2j} y')' + q_{2j} y, \qquad t \in I_{2j} \equiv (a_{2j}, b_{2j}).$$

Here p_{2j} and q_{2j} are real-valued continuous functions of $t \in I_{2j}$, p_{2j} is continuously differentiable on $\bar{I}_{2j}$ such that $p_{2j}(t) \neq 0$ for every $t \in \overline{I_{2j}}$ except at $t = a_{2j}$, and $p_{2j}(a_{2j}) = 0$. We further assume that

$$[y \mid z](a_{2j}+) \equiv \lim_{t \to a_{2j}^+} (p_{2j}(t)(y(t)\,\bar{z}'(t) - y'(t)\,\bar{z}(t))) = 0$$

for every $y, z \in \mathscr{D}_1(\tau_{2j})$. Thus the deficiency indices $\{N_{2j}^-, N_{2j}^+\}$ of the minimal closed symmetric differential operator $T_0(\tau_{2j})$ associated with τ_{2j} are $\{1, 1\}$ (see, Lee [8]). Let ϕ_{2j} and ψ_{2j} be functions in $\mathscr{D}_1(\tau_{2j})$ such that $\tau_{2j}\phi_{2j} = -\iota\phi_{2j}$, $\tau_{2j}\psi_{2j} = \iota\psi_{2j}$, $\|\phi_{2j}\| = \|\psi_{2j}\| = 1/(2^{1/2})$. Put

$$B_{2j}(y) = \begin{pmatrix} (y \mid \phi_{2j})_* \\ (y \mid \psi_{2j})_* \end{pmatrix}$$

where $(y \mid z)_* = (y \mid z) + (\tau y \mid \tau z)$. Thus for every $y, z \in \mathscr{D}_1(\tau_{2j})$

$$(\tau_{2j} y \mid z) - (y \mid \tau_{2j} z) \equiv \langle y \mid z \rangle_j = \iota B_{2j}^*(z)\, C_{2j} B_{2j}(y)$$

where

$$C_{2j} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose that τ_{2j-1} ($j \in \mathbb{N}$) is a formally self-adjoint differential expression in I_{2j-1} such that the deficiency indices of the minimal symmetric operator $T_0(\tau_{2j-1})$ are $\{0, 0\}$. Thus $N_{2j-1}^- = N_{2j-1}^+ = 0$, and the leading coefficient of τ_{2j-1} must vanish at the end points of I_{2j-1}. We note that there is no restriction on the size of the order of τ_{2j-1}. Finally let

$$\begin{aligned} (\tau y)(t) &= (\tau_{2j-1} y)(t) && \text{if } t \in I_{2j-1} \\ &= (p_{2j} y')'(t) + q_{2j}(t) y && \text{if } t \in I_{2j}. \end{aligned}$$

In this case

$$B^t(y) = (B_2{}^t(y), B_4{}^t(y), \ldots), \qquad C = \bigoplus_{j=1}^{\infty} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As a specific example, take $P = \bigoplus_1^\infty (\theta, 1)$ ($\theta \notin \mathbb{R}$, $|\theta| = 1$), $M \equiv 0$. Then $y \in \mathscr{D}(T)$ iff $y \in \mathscr{D}_1(\tau)$ and

$$\begin{aligned} &\theta\{y(b_{2j}-)\,\bar{\phi}_{2j}'(b_{2j}) - y'(b_{2j}-)\,\bar{\phi}_{2j}(b_{2j})\} \\ &\quad = -\{y(b_{2j}-)\,\bar{\psi}_{2j}'(b_{2j}) - y'(b_{2j} - 1)\,\bar{\psi}_{2j}(b_{2j})\} \end{aligned}$$

for every $j \in \mathbb{N}$.

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