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Anyon wave equations and the noncommutative plane

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Abstract

The "Jackiw–Nair" nonrelativistic limit of the relativistic anyon equations provides us with infinite-component wave equations of the Dirac–Majorana–Lévy-Leblond type for the "exotic" particle, associated with the two-fold central extension of the planar Galilei group. An infinite dimensional representation of the Galilei group is found. The velocity operator is studied, and the observable coordinates describing a noncommutative plane are identified.

1. Introduction

What kind of wave equation do anyons satisfy? The question has haunted researchers since the beginning [1–4]. A related problem is to find a wave equation for "exotic" particles [5–7], associated with the two-fold central extension of the planar Galilei group [8]. (Remember that exotic Galilean symmetry implies the noncommutativity of the position coordinates [5, 7] and, at the classical level, can be deduced from spinning anyons as a tricky nonrelativistic limit [9].)

Thirty years ago [10] Dirac proposed a new wave equation for particles with internal structure. Adapting his ideas to the plane, one of us (M.P.) has derived a $(2+1)$-dimensional version of Dirac's new equation that describes relativistic anyons and usual fields of integer and half-integer spin in a unified way [4]. This theory, briefly summarized in Section 2 below, will be the starting point for our new developments here. Using the Fock space representation, we consider two kinds of nonrelativistic limits. Both limits yield infinite sets of first-order equations. In the first type the spin is kept fixed and, at each step, the new component gets a factor of c^{-1}. The first two components yield the nonrelativistic "Dirac equation" put forward by Lévy-Leblond in the sixties [11]. The "Jackiw–Nair (JN) limit" [9] yields instead a genuinely infinite component system, (3.4), of "Lévy-Leblond-type" equations. These latter are invariant with respect to the two-fold centrally extended exotic Galilei group, whose action can be derived from the Poincaré-symmetry of the anyon. Presented in terms of the initial commuting coordinates, our system undergoes a nonrelativistic Zitterbewegung. Identifying the observable, Zitterbewegung-free quantities results

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in Galilei-covariant coordinates, which describe a noncommutative plane.

2. Wave equations for relativistic anyons

Let us briefly summarize the theory of anyons we start with. It was emphasised by Jackiw and Nair [1] (see also [2]) that anyons, just like usual fields of integer and half-integer spin, correspond to irreducible representations of the planar Poincaré group [12], labeled with two Casimir invariants, namely

$$\big(p_\mu p^\mu + m^2c^2\big)|\Psi\rangle = 0, \tag{2.1}$$

$$\big(p_\mu J^\mu - smc\big)|\Psi\rangle = 0, \tag{2.2}$$

where J_μ is the "spin part" of the total angular momentum operator

$$\mathcal{J}_\mu = -\epsilon_{\mu\nu\lambda}x^\nu p^\lambda + J_\mu. \tag{2.3}$$

The $\mathcal{J}_\mu$, together with p_μ, generate the $(2+1)$D Poincaré group, via the commutation relations

$$\begin{aligned}
&[x_\mu, p_\nu] = i\eta_{\mu\nu},\\
&[x_\mu, x_\nu] = [p_\mu, p_\nu] = [J_\mu, x_\nu] = [J_\mu, p_\nu] = 0,\\
&[J_\mu, J_\nu] = -i\epsilon_{\mu\nu\lambda}J^\lambda,
\end{aligned} \tag{2.4}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1)$, $\epsilon^{012} = 1$. The choice $J_\mu = -\frac{1}{2}\gamma_\mu$ and $s = \pm 1/2$ reduces Eq. (2.2) to the Dirac equation. Similarly, the choice $(J_\mu)^\sigma{}_\rho = -i\epsilon^\sigma{}_{\mu\rho}$ for $s = \pm 1$ leads to the topologically massive vector field equation [1]. In these two cases (only), the quadratic equation (2.1) follows from the linear equation (2.2).

To describe anyons for which the spin parameter s can take any real value, we have to resort to the infinite-dimensional half-bounded unitary representations of the planar Lorentz group of the discrete type series D^+_α (or, D^-_α) [1,2,4]. For these representations, characterized by $J_\mu J^\mu = -\alpha(\alpha-1)$ and $J_0 = \mathrm{diag}(\alpha+n)$ $(-J_0 = \mathrm{diag}(\alpha+n))$, $\alpha > 0$, $n = 0, 1, \ldots,$ Eq. (2.2) reduces to a $(2+1)$-dimensional analog of the Majorana equation [13]. The pair of Eqs. (2.1), (2.2) describes an anyon field with spin $s = \alpha$ (or, $s = -\alpha$ for the choice of D^-_α). Here, as for usual fields with $|s| = j > 1$, the Klein–Gordon equation (2.1) is independent of the Majorana equation.

The wave functions chosen by Jackiw and Nair [1] (see also [3]) carry a reducible representation of the planar Lorentz group. Their wave equation, (4.10), needs therefore be supplemented by further conditions, their Eqs. (4.12a) and (4.12b).

Our primary aim here is to derive a nonrelativistic model from the relativistic anyon. This could be attempted in the framework of [1]; subsidary conditions would lead to complications, though.

Another line of attack is provided by the "new wave equation" proposed by Dirac in 1971 [10], which describes a particle in $(3+1)$D, endowed with an internal structure. Dirac's idea has been applied to anyons in $2+1$ dimensions [4]: one starts with the one-dimensional deformed Heisenberg algebra

$$\big[a^-, a^+\big] = 1 + \nu R, \qquad R^2 = 1, \qquad \big\{a^\pm, R\big\} = 0, \tag{2.5}$$

where ν is a real (deformation) parameter. $N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}(\nu+1)$ plays the role of a number operator, $[N, a^\pm] = \pm a^\pm$, allowing us to present a reflection operator R in terms of $a^\pm$: $R = (-1)^N = \cos \pi N$. For any $\nu > -1$, this algebra admits an infinite-dimensional unitary representation realized on Fock space.[3] The vacuum state is distinguished by the relation $a^-|0\rangle = 0$, and $|n\rangle$, $N|n\rangle = n|n\rangle$, is given by $|n\rangle = C_n(a^+)^n|0\rangle$, where C_n is a normalization coefficient. Fock space is decomposed into even and odd subspaces defined by $R|\psi\rangle_\pm = \pm|\psi\rangle_\pm$, which correspond to n even or odd. The quadratic operators

$$J_0 = \frac{1}{4}\big\{a^+, a^-\big\}, \qquad J_\pm \equiv J_1 \pm iJ_2 = \frac{1}{2}\big(a^\pm\big)^2 \tag{2.6}$$

generate the $so(1,2)$ Lorentz algebra, see the last relation in (2.4). The $so(1,2)$ Casimir is $J_\mu J^\mu = -s(s-1)$ with $s = \frac{1}{4}(1 \pm \nu)$ on the even/odd subspaces. The quadratic operators (2.6), together with the linear operators

$$L_1 = \frac{1}{\sqrt{2}}\big(a^+ + a^-\big), \qquad L_2 = \frac{i}{\sqrt{2}}\big(a^+ - a^-\big),$$

extend the Lorentz algebra into an $osp(1|2)$ superalgebra: $\{L_\alpha, L_\beta\} = 4i(J\gamma)_{\alpha\beta}$ $[J_\mu, L_\alpha] = \frac{1}{2}(\gamma_\mu)_\alpha{}^\beta L_\beta$, where the two-by-two γ-matrices are in the Majorana representation, $(\gamma^0)_\alpha{}^\beta = -(\sigma^2)_\alpha{}^\beta$, $(\gamma^1)_\alpha{}^\beta =$

[3] For negative odd integer values $\nu = -(2k+1)$, $k = 1, 2, \ldots,$ one gets finite, namely $(2k+1)$-dimensional representations. Then Eqs. (2.7) describe a spin-j field with $j = k/2$, which has states with both signs of the energy [4]. From now on, we only consider the infinite-dimensional case.

$i(\sigma^1)_\alpha{}^\beta$, $(\gamma^2)_\alpha{}^\beta = i(\sigma^3)_\alpha{}^\beta$. The antisymmetric tensor $\epsilon_{\alpha\beta}$, $\epsilon_{12} = 1$, provides us with a metric for the spinor indices, $(\gamma^\mu)_{\alpha\beta} = (\gamma^\mu)_\alpha{}^\rho \epsilon_{\rho\beta}$. The space–time coordinates x^μ and conjugate momenta p_μ are independent from the internal variables, $[x^\mu, a^\pm] = [p_\mu, a^\pm] = 0$; hence, due to (2.3), the L_α form a $(2+1)$D spinor. With all these ingredients at hand, now we posit our linear-in-p_μ anyon equations [4]

$$Q_\alpha|\psi\rangle = 0, \quad \text{where}$$
$$Q_\alpha = \big(R\big(p_\mu\gamma^\mu\big)_\alpha{}^\beta + mc\epsilon_\alpha{}^\beta\big)L_\beta. \tag{2.7}$$

Since the Q_α and the total angular momentum operator given by Eqs. (2.3) and (2.6) satisfy the relation $[\mathcal{J}_\mu, Q_\alpha] = \frac{1}{2}(\gamma_\mu)_\alpha{}^\beta Q_\beta$, (2.7) is a covariant *spinor* set of equations. (In contrast, the corresponding operators in (2.1) and (2.2) are $\mathrm{so}(1,2)$ scalars.) In our Dirac-type approach the consistency conditions

$$\{Q_\alpha, Q_\beta\}|\psi\rangle = 0 \quad \text{and} \quad [Q_\alpha, Q_\beta]|\psi\rangle = 0, \tag{2.8}$$

restricted to the even subspace, imply Eqs. (2.1) and (2.2), whereas they force the odd part to vanish identically, $|\psi\rangle_- = 0$ [4]. We shall restrict our considerations to even states henceforth. Then our wave functions carry an irreducible representation of the planar Lorentz group and do not necessitate therefore any subsidary condition. Compared to the wave functions of [1] (or, of [3]) which carry additional vector (spinor) indices, ours are more economical, as they boil down to a scalar-like field.

An insight is gained when we write (2.7) on Fock space. Expanding as $|\psi\rangle = \sum_{n\geqslant 0}\psi_n|2n\rangle$ yields, for each $n \geqslant 0$, a pair of coupled equations which only involve neighbouring components, namely

$$\begin{cases}\sqrt{n+2s}(mc+p_0)\psi_n - \sqrt{n+1}\,p_+\psi_{n+1} = 0,\\ \sqrt{n+2s}\,p_-\psi_n + \sqrt{n+1}(mc-p_0)\psi_{n+1} = 0,\end{cases} \tag{2.9}$$

where $p_\pm \equiv p_1 \pm ip_2$. The second equation in (2.9) yields the recurrence relation

$$\psi_{n+1} = -\sqrt{\frac{n+2s}{n+1}}\frac{p_-}{mc-p_0}\psi_n. \tag{2.10}$$

Substitution into the first equation reproduces, for each ψ_n, the Klein–Gordon equation (2.1): $(p^2 + m^2c^2)\psi_n = 0$. To see that our anyon equations imply also Eq. (2.2) which fixes the second (spin) Casimir invariant, let us first observe that the spin generators in (2.6) act on Fock space according to

$$\begin{aligned}&\langle 2n|J_0|\psi\rangle = (s+n)\psi_n,\\ &\langle 2n|J_+|\psi\rangle = \sqrt{n(n-1+2s)}\psi_{n-1},\\ &\langle 2n|J_-|\psi\rangle = \sqrt{(n+1)(n+2s)}\psi_{n+1}.\end{aligned} \tag{2.11}$$

Then taking into account Eqs. (2.11) yields the matrix form of (2.2), namely

$$\langle 2n|p_\mu J^\mu - smc|\psi\rangle = 0. \tag{2.12}$$

Eqs. (2.9) describe therefore an infinite-component field of mass m, whose (anyonic) spin, $s = \frac{1}{4}(1+\nu) > 0$, is fixed by the value of the deformation parameter $\nu > -1$. In the rest frame system $\vec{p} = 0$ we find also that, just like for the usual Majorana equation [13], the sign of the energy of those states described by (2.9) is necessarily positive [4], $p^0 > 0$. Note also that the positive energy anyonic states with negative spin values, $s = -\frac{1}{4}(1+\nu) < 0$, can be obtained by changing the sign before the mass term mc in (2.7) and positing the $\mathrm{so}(1,2)$ generators $J_0 = -\frac{1}{4}\{a^+, a^-\}$, $J_\pm = -\frac{1}{2}(a^\mp)^2$. Then repeated application of (2.10), together with Eq. (2.2) projected onto the vacuum state, provides us with the momentum-representation solution to the anyon Eqs. (2.9),

$$\begin{aligned}\psi_n(p) &= (-1)^n\sqrt{\frac{2s\cdot(2s+1)\cdots(2s+n-1)}{n!}}\\ &\quad\times\left(\frac{p_1-ip_2}{mc-p_0}\right)^n\psi_0(p),\\ \psi_0(p) &= \delta\big(p^0 - \sqrt{m^2c^2+\vec{p}^2}\big)\psi(\vec{p}).\end{aligned} \tag{2.13}$$

3. "Lévy-Leblond" equations for exotic particles

Now we consider the nonrelativistic limit of our anyon equations. For the purpose we note that, consistently with Eqs. (2.13), all components vanish in the rest frame, with the exception of ψ_0, which has energy $p^0 = mc$. Then, putting $p_0 = -ic^{-1}\partial_t$ and

$$|\psi\rangle = e^{-imc^2t}|\phi\rangle = e^{-imc^2t}\sum_{n\geqslant 0}\phi_n|2n\rangle,$$

the Fock-space equations (2.9) yield the first-order system

$$\begin{aligned}&\sqrt{n+2s}\,ic^{-1}\partial_t\phi_n + \sqrt{n+1}\,p_+\phi_{n+1} = 0,\\ &\sqrt{n+2s}\,p_-\phi_n + \sqrt{n+1}\big(2mc - ic^{-1}\partial_t\big)\phi_{n+1} = 0.\end{aligned}$$

We can now consider *two types* of nonrelativistic limits. Firstly, let us keep the spin-s fixed and let $c \to \infty$. In this limit the subsequent components get always multiplied by c^{-1}, $\phi_{n+1} \sim c^{-1}\phi_n$. Only keeping terms up to order c^{-2}, we are left with just the first two equations. Calling $\phi_0 = \Phi$ and $c\phi_1 = \chi$, they read

$$\begin{cases} i\partial_t \Phi + \frac{1}{\sqrt{2s}} p_+ \chi = 0, \\ \frac{1}{2m} p_- \Phi + \frac{1}{\sqrt{2s}} \chi = 0. \end{cases} \tag{3.1}$$

These equations form already a closed system, which generalizes from $s = 1/2$ to any $s > 0$ the two-component "nonrelativistic" Dirac equation introduced by Lévy-Leblond [11]. Let us emphasise that in this "ordinary" nonrelativistic limit, relativistic spin simply becomes nonrelativistic spin, still denoted by s. It should be remembered, however, that (3.1) represents only the two first leading equations in the expansion in the small parameter $|\vec{p}|/mc$, and that the NR limit of the anyon system of (any) spin-s contains, unlike for the nonrelativistic limit of the Dirac equation, an infinite number of components $\phi_n \sim (|\vec{p}|/mc)^n \phi_0$. Redefining the higher-order components as $\Phi_n = c^n \phi_n$ yields indeed

$$\begin{cases} i\partial_t \Phi_n + \sqrt{\frac{n+1}{n+2s}}\, p_+ \Phi_{n+1} = 0, \\ \frac{1}{2m} p_- \Phi_n + \sqrt{\frac{n+1}{n+2s}}\, \Phi_{n+1} = 0. \end{cases} \tag{3.2}$$

Rather then pursuing these investigations, we focus our attention to another, more subtle limit, considered by Jackiw and Nair [9]. As spin in $(2+1)$D is a continuous parameter, we can indeed let it diverge so that s/c^2 tends to a finite limit:

$$c \to \infty, \qquad s \to \infty, \qquad \frac{s}{c^2} = \kappa. \tag{3.3}$$

Then all components remain of the same order, and we get an infinite number of equations

$$\begin{cases} i\partial_t \phi_n + \sqrt{\frac{n+1}{2\kappa}}\, p_+ \phi_{n+1} = 0, \\ \frac{1}{2m} p_- \phi_n + \sqrt{\frac{n+1}{2\kappa}}\, \phi_{n+1} = 0. \end{cases} \tag{3.4}$$

These are the first-order, infinite-component Dirac–Majorana–Lévy-Leblond type equations we propose to describe our free "exotic" system.

Eliminating one component shows, furthermore, that each component satisfies the free Schrödinger equation $i\partial_t \phi_n = (\vec{p}^{\,2}/2m)\phi_n$ (cf. (2.1)). For further discussion we observe that, grouping all "upper" and all "lower equations" collectively, (3.4) can also be presented as

$$D|\phi\rangle = 0, \qquad \Lambda|\phi\rangle = 0. \tag{3.5}$$

The second equation here can be viewed as a quantum constraint which specifies the physical subspace. It allows us to express all components in terms of the first one:

$$\phi_n = (-1)^n \left(\frac{\kappa}{2}\right)^{n/2} \frac{1}{\sqrt{n!}} \left(\frac{p_1 - ip_2}{m}\right)^n \phi_0, \tag{3.6}$$

cf. (2.13).

4. Exotic Galilei symmetry

Both types of our nonrelativistic systems considered above are indeed invariant under Galilean transformations. The generators of Galilei boosts can be derived from the relativistic Lorentz generators as the $c \to \infty$ limits of

$$\mathcal{K}_i = -\frac{1}{c}\epsilon_{ij}\mathcal{J}_j. \tag{4.1}$$

Put $\delta\phi_n = i\delta b_j \langle 2n|\mathcal{K}_j|\phi\rangle$, where δb_j are the transformation parameters. Dropping terms of $o(c^{-2})$ for the (spin-s) Lévy-Leblond system (3.1) we get, using (2.3) and (2.11),

$$\delta\Phi = \delta b_i (m x_i - t p_i)\Phi,$$
$$\delta\chi = \delta b_i (m x_i - t p_i)\chi - \sqrt{\frac{s}{2}}(\delta b_1 + i\delta b_2)\Phi. \tag{4.2}$$

For spin $s = 1/2$, we recover the formula of Lévy-Leblond [11]. This representation is conventional in that the boosts commute, $[\mathcal{K}_i, \mathcal{K}_j] = 0$.

Let us stress, however, that (3.1) is just an $o(c^{-2})$ truncation of an infinite-component system (3.2), whose Galilean symmetry could be established by recursion. Let us indeed posit that Galilei boosts act on the first component Φ_0 as on a scalar, i.e., as on Φ in (4.2). Then the action on the second component Φ_1 can be deduced seeking it in the form $\delta\Phi_1 = \alpha_0\Phi_0 + \alpha_1\Phi_1$ and requiring the field equations to be satisfied. The procedure could be continued for the next component, etc. Explicit formulae that generalize

(4.2) are not illuminating as they soon become rather complicated.

Let us now identify the Galilean symmetry of the anyon field system (3.4), obtained by the JN limit (3.3). The Galilean boost generators (4.1), acting on infinite-dimensional Fock space, can be found from Eqs. (2.3) and (2.11) as

$$\langle 2n|\mathcal{K}_1|\phi\rangle = (mx_1 - tp_1)\phi_n - i\sqrt{\frac{\kappa}{2}}\big(\sqrt{n+1}\phi_{n+1} - \sqrt{n}\phi_{n-1}\big),$$
$$\langle 2n|\mathcal{K}_2|\phi\rangle = (mx_2 - tp_2)\phi_n + \sqrt{\frac{\kappa}{2}}\big(\sqrt{n+1}\phi_{n+1} + \sqrt{n}\phi_{n-1}\big). \tag{4.3}$$

The first, diagonal-in-ϕ_n terms here represent the ordinary Galilean symmetry (as for a scalar particle). The terms which mix the components $\phi_{n\pm1}$ are associated with the "internal" structure of the system. The remarkable feature of the boosts (4.3) is that, owing precisely to the internal structure, they satisfy

$$[\mathcal{K}_1, \mathcal{K}_2] = -i\kappa \tag{4.4}$$

rather then commute. Relation (4.4) is the hallmark of "exotic" Galilean symmetry [5,7,8], further discussed in the next section.

To identify the generator of rotations of the system, we note that the JN limit of the relativistic angular momentum $\mathcal{J}_0$ (just like of the energy) diverges [15]. Omitting the divergent part and taking into account Eq. (2.3) and the first equation from (2.11) yields

$$\langle 2n|\mathcal{J}|\phi\rangle \equiv \langle 2n|\mathcal{J}_0|\phi\rangle_{\text{renorm}} = (\epsilon_{ij}x_i p_j + n)\phi_n. \tag{4.5}$$

The internal contribution $n\phi_n$ is essential for establishing the correct commutation relations $[\mathcal{J}, \mathcal{K}_j] = i\epsilon_{jk}\mathcal{K}_k$.

5. The velocity operator

The Hamiltonian of the system is the generator of time translations. It is convenient to introduce the nonrelativistic counterparts of the relativistic bosonic operators we denote (for reasons which will become clear later) by $v_\pm$. They act on "nonrelativistic" Fock space spanned by $|n\rangle_v = |2n\rangle$, $n = 0, 1, \ldots,$

$$v_+|n\rangle_v = -(2/\kappa)^{1/2}\sqrt{n+1}|n+1\rangle_v,$$
$$v_-|n\rangle_v = -(2/\kappa)^{1/2}\sqrt{n}|n-1\rangle_v. \tag{5.1}$$

The factor $-(2/\kappa)^{1/2}$ is included for later convenience.

The operators $v_\pm$ here are in fact the JN limits of the translation invariant part of relativistic Lorentz generators, $-(c/s)J_\pm \to v_\pm$, cf. (4.1). The JN limit of $\langle 2n|J_+|\psi\rangle$ yields, e.g., by (2.11), ${}_v\langle n|(-c/s)J_+|n-1\rangle_v \to -(2/\kappa)^{1/2}\sqrt{n}$, i.e., (5.1). These "internal" operators span an (undeformed) Heisenberg algebra $[v_-, v_+] = 2\kappa^{-1}$, or, putting $v_\pm = v_1 \pm iv_2$,

$$[v_j, v_k] = -i\kappa^{-1}\epsilon_{jk}. \tag{5.2}$$

As it follows from the relativistic relations (2.4), the v_i commute with the "external" canonical coordinates x_i and momenta p_i, $[x_i, p_j] = i\delta_{ij}$, $[x_i, x_j] = [p_i, p_j] = [v_i, x_j] = [v_i, p_j] = 0$.

Any state of our system can now be decomposed over Fock space, $|\phi\rangle = \sum_{n\geqslant 0}\phi_n|n\rangle_v$, where the field components ϕ_n are taken either in coordinate (x_i), or in momentum (p_i) representation. Writing the field equations (3.4) in the form (3.5) with $D = i\partial_t - \frac{1}{2}p_+v_-$, $\Lambda = v_- - m^{-1}p_-$, one would be tempted to view $\frac{1}{2}p_+v_-$ as a Hamiltonian. This is, however, not correct, as $\frac{1}{2}p_+v_-$ is not Hermitian. Our clue is that (3.5) is equivalent to a set of equations of the same form, but with D changed into $D - \frac{1}{2}mv_+\Lambda$. The system of field equations (3.4) can finally be represented in the equivalent form

$$\begin{cases} D|\phi\rangle = 0, \quad D = i\partial_t - \mathcal{H}, \quad \mathcal{H} = \vec{p}\cdot\vec{v} - \frac{1}{2}mv_+v_-, \\ \Lambda|\phi\rangle = 0, \quad \Lambda = v_- - \frac{1}{m}p_-. \end{cases} \tag{5.3}$$

The Hermitian operator $\mathcal{H}$ here can be identified with the Hamiltonian of the system. On the physical subspace defined by $\Lambda|\phi\rangle_{\text{phys}} = 0$, our $\mathcal{H}$ reduces to the free expression,

$$\mathcal{H}|\phi\rangle_{\text{phys}} = H|\phi\rangle_{\text{phys}}, \quad H = \frac{\vec{p}^{\,2}}{2m}. \tag{5.4}$$

Thus, we recover the framework proposed in [5] on grounds of canonical quantization. The *quadratic* expression in (5.4) comes from our eliminating ϕ_{n+1} using (5.1). Note that, consistently with (5.3), the physical states are just the coherent states of the Heisenberg algebra corresponding to the operators $v_\pm$,

$$v_-|\phi\rangle_{\text{phys}} = m^{-1}p_-|\phi\rangle_{\text{phys}},$$
$$|\phi\rangle_{\text{phys}} \propto e^{-\frac{1}{2}\sqrt{\theta}\,p_-v_+}|0\rangle_v,$$

where we have introduced the notation [7]

$$\theta = \frac{\kappa}{m^2}. \tag{5.5}$$

The generators (4.3), (4.5) can be presented in the form

$$\mathcal{K}_i = m x_i - t p_i + \kappa \epsilon_{ij} v_j,$$
$$\mathcal{J} = \epsilon_{ij} x_i p_j + \frac{1}{2} \kappa v_+ v_-. \tag{5.6}$$

Let us note for further reference that

$$\mathcal{H} = \kappa^{-1}(\epsilon_{ij} \mathcal{K}_i p_j - m\mathcal{J}). \tag{5.7}$$

In the representation (5.3), (5.6), we obtain the nontrivial commutation relations of the two-fold centrally extended Galilei group [8], namely

$$[\mathcal{K}_i, p_j] = im\delta_{ij}, \qquad [\mathcal{K}_i, \mathcal{K}_j] = -i\kappa\epsilon_{ij},$$
$$[\mathcal{K}_i, \mathcal{H}] = ip_i, \qquad [\mathcal{J}, p_i] = i\epsilon_{ij} p_j,$$
$$[\mathcal{J}, \mathcal{K}_i] = i\epsilon_{ij} \mathcal{K}_j. \tag{5.8}$$

Conversely, the exotic relations (5.8) fix the additional terms in (5.6). Seeking indeed the generators of Galilei boosts and rotations in the from $\mathcal{K}_i = mx_i - tp_i + \Gamma_i$ and $\mathcal{J} = \epsilon_{ij} x_i p_j + \Sigma$, respectively, where Γ_i and Σ commute with the external operators x_i and p_i, the Galilei relations in (5.8) imply that necessarily $[\Gamma_i, \Gamma_j] = -i\kappa\epsilon_{ij}$, i.e., the Γ_i span a Heisenberg algebra. Similarly, these relations also fix Σ as $\Sigma = \frac{1}{2\kappa}\Gamma_i\Gamma_i + s_0$ where s_0 is a constant. Calling $\Gamma_i = \kappa\epsilon_{ij} v_j$ and putting $s_0 = 0$ results in (5.6). Our previous (infinite-dimensional) formulae can be recovered by representing the $v_\pm$ operators on Fock space according to (5.1).

Using the nonrelativistic commutation relations given above, the Hamiltonian $\mathcal{H}$ is seen to generate the Heisenberg equations of motion

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dp_i}{dt} = 0,$$
$$\frac{dv_i}{dt} = \frac{m}{\kappa}\epsilon_{ij}\big(v_j - m^{-1}p_j\big), \tag{5.9}$$

which is the quantum counterpart of the classical equations studied in [5,6], and can be integrated at once to give $p_i(t) = \text{const}$ and,

$$x_i(t) = m^{-1}\big(p_i t - \kappa\epsilon_{ij} V_j(t)\big) + \text{const},$$
$$V_\pm(t) = \exp\big(\mp im\kappa^{-1}t\big) V_\pm(0), \tag{5.10}$$

where $V_\pm = V_1 \pm iV_2$, and

$$V_i = v_i - m^{-1} p_i. \tag{5.11}$$

The noncommuting operators v_i describe therefore the velocity of the system and, as for a Dirac particle, the coordinates perform a "nonrelativistic Zitterbewegung". The internal variable V_i measures the extent the momentum, p_i, differs from [m-times] the velocity. In its terms, the operator Λ in (5.3) which defines the physical-state constraint is $\Lambda = V_-$.

6. Observable coordinates in the noncommutative plane

According to Eq. (5.10), the time evolution of the initial (commuting) coordinates x_i is different from that of a usual free nonrelativistic particle. This happens because the x_i do not commute with the operator Λ that singles out the physical subspace, $[x_1, \Lambda] = -im^{-1}$, $[x_2, \Lambda] = -m^{-1}$. As a result, the position operators do not leave the physical subspace invariant, $\Lambda x_i|\phi\rangle_{\text{phys}} \neq 0$. Hence, following Dirac, they cannot be viewed as observable operators. Like for a Dirac particle [16], one can identify the observable coordinates as those that do commute with Λ. To find them, let us consider a unitary transformation generated by the operator $U = \exp(i\kappa m^{-1}\epsilon_{jk} p_j v_k)$,

$$Uv_iU^{-1} = V_i, \qquad Ux_iU^{-1} = X_i,$$
$$Up_iU^{-1} = p_i, \tag{6.1}$$

where

$$X_i = \mathcal{X}_i + \frac{\theta}{2}\epsilon_{ij} p_j, \quad \mathcal{X}_i = x_i + \kappa m^{-1}\epsilon_{ij} V_j. \tag{6.2}$$

Here the constant θ is given by (5.5). Since $[X_i, V_j] = [x_i, v_j] = 0$ by construction, the X_i can be viewed as observable coordinate operators. In terms of the new operators, the nontransformed Hamiltonian (5.3) reads

$$\mathcal{H} = \frac{\vec{p}^{\,2}}{2m} - \frac{m}{2} V_+ V_-, \tag{6.3}$$

and we conclude that, consistently with (5.10), the X_i evolve as coordinate operators of a free nonrelativistic particle, namely as $X_i(t) = X_i(0) + tm^{-1}p_i$. The dynamics of $V_\pm$ is in turn that of a harmonic oscillator. Note, however, that V_+ is not observable ($[V_+, \Lambda] = -2\kappa^{-1} \neq 0$), whereas $V_-|\phi\rangle_{\text{phys}} = 0$, cf. [5]. Thus, the physical states are the vacuum states of the harmonic oscillator-like internal operators $V_\pm$.

At last, the $\mathcal{X}_i = X_i - \frac{\theta}{2}\epsilon_{ij}p_j$ commute with the V_j [and hence with Λ], and are therefore also observable. Due to the conservation of p_i, they have the same evolution law as the X_i. However, the coordinates $\mathcal{X}_i$, $i = 1,2$, unlike X_i, do not commute between themselves,

$$[\mathcal{X}_j, \mathcal{X}_k] = i\theta\epsilon_{jk}. \tag{6.4}$$

Their noncommutativity stems from the noncommutativity of the velocity operators and the related Zitterbewegung. In terms of the operators (6.2), (5.11),

$$\mathcal{K}_i = m\mathcal{X}_i - tp_i + m\theta\epsilon_{ij}p_j,$$
$$\mathcal{J} = \epsilon_{ij}\mathcal{X}_i p_j + \frac{1}{2}\theta p_i^2 + \frac{1}{2}\kappa V_+ V_- \tag{6.5}$$

cf. [5,6]. The operators X_i, $\mathcal{X}_i$ and V_i are 2D vectors by construction. Unlike the initial v_i, the V_i are invariant under Galilei boosts, $[\mathcal{K}_i, V_j] = 0$, whereas for X_i and $\mathcal{X}_i$ we get

$$[\mathcal{K}_j, \mathcal{X}_k] = it\delta_{jk}, \qquad [\mathcal{K}_j, X_j] = it\delta_{jk} - i\frac{1}{2}m\theta\epsilon_{ij}.$$

This means that the (observable) $\mathcal{X}_i$ transform under Galilei boosts as planar coordinate operators and, consistently with (6.4), they describe a noncommutative plane. (In contrast, the operators X_i *are not Galilei-covariant*.) Note that the $\mathcal{X}_i$ and X_i are analogous to the Foldy–Wouthuysen and Newton–Wigner coordinates for the Dirac particle, respectively, [16,17].

We conclude that our system described by Eqs. (3.4) represents a free, massive, nonrelativistic field on the noncommutative plane. Its reduction to the physical subspace $V_-|\phi\rangle_{\rm phys} = 0$ yields the free exotic particle introduced in [7]. The latter is described by the noncommutative coordinates $\mathcal{X}_i$ and momenta p_i (see Eq. (6.4) and $[\mathcal{X}_i, p_j] = i\delta_{ij}$, $[p_i, p_j] = 0$), whose dynamics is given by the usual quadratic Hamiltonian $H = \vec{p}^{\,2}/2m$; the generators of the Galilei boosts are given by Eq. (6.5), while the angular momentum operator is $\mathcal{J} = \epsilon_{ij}\mathcal{X}_i p_j + \frac{1}{2}\theta\vec{p}^{\,2}$, cf. [7].

7. The JN limit of the Majorana equation

In the anyon context, the Majorana equation appeared as the equation describing the quantum theory of the higher derivative $(2+1)$D model of a relativistic particle with torsion [2,14], whose Euclidean version emerged originally in relation to Fermi–Bose transmutation mechanism [18]. Like the original $(3+1)$D Majorana equation [13], its $(2+1)$D analog admits three types of solutions, namely massive ($p^2 < 0$ with spectrum $M_n = ms/S_n$, where $S_n = s + n$ is spin of the corresponding state, $n = 0, 1, \ldots$), massless ($p^2 = 0$) and tachyonic ($p^2 > 0$) ones. Then requiring also the Klein–Gordon equation eliminates the massless and tachyonic sectors and singles out the only massive state with spin $S_0 = s$ and mass m. We focus therefore our attention to the massive sector, and inquire about the JN limit of the Majorana equation.

In Fock space associated with the deformed Heisenberg algebra, the matrix form of the Majorana equation, $\langle 2n|p_\mu J^\mu - smc|\psi\rangle = 0$, reads

$$-\big[p_0(s+n) + smc\big]\psi_n + \frac{1}{2}\Big[\sqrt{n(n-1+2s)}\,p_-\psi_{n-1} + \sqrt{(n+1)(n+2s)}\,p_+\psi_{n+1}\Big] = 0. \tag{7.1}$$

Separating, as before, the divergent part of energy by putting $\psi_n = e^{-imc^2 t}\phi_n$, the JN limit yields

$$i\partial_t\phi_n = -\frac{m}{\kappa}n\phi_n - \frac{1}{\sqrt{2\kappa}}\big(\sqrt{n}\,p_-\phi_{n-1} + \sqrt{n+1}\,p_+\phi_{n+1}\big). \tag{7.2}$$

Eq. (7.2) is nothing else as the component form of the first of Eqs. (5.3): its equivalent presentation is

$${}_v\langle n|i\partial_t - \mathcal{H}|\phi\rangle = 0, \tag{7.3}$$

with $\mathcal{H}$ the Hamiltonian operator in Eq. (5.3). The spectrum can be extracted at once from the equivalent form (6.3) of the Hamiltonian. In the representation where both the momentum operator $\vec{p}$ and V_+V_- are diagonal, we find that the energy of the state $|\vec{p}, n\rangle$, $n = 0, 1, \ldots$, is

$$E_n(\vec{p}) = \frac{1}{2m}\vec{p}^{\,2} - \frac{m}{\kappa}n. \tag{7.4}$$

Taking into account the form of the angular momentum operator (6.5), the quantum number n can be interpreted as internal angular momentum. In other words, the JN limit of the Majorana equation is a kind of nonrelativistic rotator, whose internal angular momentum can take only nonnegative integer values, and whose spectrum, (7.4), is unbounded from below.

Therefore, the only but crucial difference between our "exotic" system (3.4) and the JN limit of the $(2+1)$D Majorana equation (7.2) is that the first system also requires the additional equation $V_-|\phi\rangle = 0$ (whose component form is the second equation in (3.4)). Like Gupta–Bleuler quantization of the electromagnetic field, this additional condition "freezes" the internal "rotator" degree of freedom, which is responsible for the negative contribution to the energy.

To conclude this section, we note that our results here are consistent with those obtained in our previous paper [6], where, on the one hand, we demonstrated that the JN limit of the relativistic particle with torsion yields, at the classical level, the acceleration-dependent model of Lukierski–Stichel–Zakrzewski [5], and, on the other hand, showed that the quantum version of the latter can be described in terms of our noncommutative coordinates $\mathcal{X}_i$, momenta p_i, and noncommuting internal "rotator" variables V_i [denoted in [6] by Q_i]. The dynamics and transformation properties with respect to exotic Galilean boosts and rotations are given by (6.3), (6.5).

8. Discussion

The origin of the noncommutative plane can be traced back to the noncommutativity of velocities obtained as the JN limit of anyons (2.7). The structure of the associated Fock space as well as that of the deformed Heisenberg algebra of the initial relativistic anyon system are rooted in the hidden Fock space structure of the Majorana equation which underlies anyon theory. The $(2+1)$D Majorana equation is based on the half-bounded infinite-dimensional unitary representations of the $(2+1)$-dimensional Lorentz group.

In close analogy with the relativistic case, (2.3), the structure of the two-fold extended Galilei group fixes the additional "exotic" terms in the conserved quantities so that they generate the Heisenberg algebra and provides us with a noncommuting velocity operator.

The "lower" set of equations in (3.4) singles out the physical subspace of the system on which, according to the second equation from (5.3), the nonrelativistic Zitterbewegung disappears, $\langle|v_i - m^{-1}p_i|\rangle = 0$, $|\rangle = |\phi\rangle_{\text{phys}}$. On the physical subspace the system is hence described by the Galilei-covariant coordinates $\mathcal{X}_i$ of the noncommutative plane and by the momenta p_i.

Our free system (3.4) is equivalent to the exotic particle model in [7]. The two systems *are not* equivalent, however, if we switch on an interaction: the nonphysical polarizations $|\vec{p}, n\rangle$, $(\kappa/2)V_+V_-|\vec{p}, n\rangle = n|\vec{p}, n\rangle$ $(n = 1, 2 \ldots)$, could reveal themselves as virtual states.

From the viewpoint of representation theory, the two-fold centrally extended planar Galilei group has two Casimir operators, namely

$$\mathcal{C}_1 = \epsilon_{ij}\mathcal{K}_i p_j - m\mathcal{J} - \kappa\mathcal{H}, \qquad \mathcal{C}_2 = p_i^2 - 2m\mathcal{H}. \tag{8.1}$$

By Eq. (5.7), the first Casimir is fixed as a strong operator relation, $\mathcal{C}_1 = 0$. On the other hand, due to Eq. (6.3), we have $\mathcal{C}_2 = m^2 V_+V_-$. Hence, the physical subspace constraint $V_-|\phi\rangle = 0$ fixes the second Casimir to vanish, $\mathcal{C}_2|\phi\rangle = 0$ (whereas in the model of Ref. [7] $\mathcal{C}_2 = 0$ is a strong operator relation). An (up to a constant factor) equivalent form of the second Casimir is

$$\tilde{\mathcal{C}}_2 = \mathcal{J} - m^{-1}\epsilon_{ij}\mathcal{K}_i p_j - \frac{1}{2}\theta p_i^2.$$

Since in the rest frame system $p_i = 0$ it reduces to $\mathcal{J}$ (that, in turn, becomes proportional to V_+V_-), our second equation requires the internal spin to vanish. These two conditions play a role analogous to that of (2.1), (2.2) for relativistic anyons.

In contrast, for the JN limit of the Majorana equation, only the first Casimir operator is fixed, $\mathcal{C}_1 = 0$, and the second is left free. This results in the appearance of negative-energy states in the model of Ref. [5].

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