Divisibility properties of power GCD matrices and power LCM matrices

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Abstract

Let a, b and n be positive integers and the set \( S = \{x_1, \ldots, x_n\} \) of n distinct positive integers be a divisor chain (i.e. there exists a permutation \( \sigma \) on \( \{1, \ldots, n\} \) such that \( x_{\sigma(1)} | \cdots | x_{\sigma(n)} \)). In this paper, we show that if \( a \mid b \), then the \( a \)th power GCD matrix \((S^a)\) having the \( a \)th power \((x_i, x_j)^a\) of the greatest common divisor of \( x_i \) and \( x_j \) as its \( i, j \)-entry divides the \( b \)th power GCD matrix \((S^b)\) in the ring \( M_n(\mathbb{Z}) \) of \( n \times n \) matrices over integers. We show also that if \( a \not\mid b \) and \( n \geq 2 \), then the \( a \)th power GCD matrix \((S^a)\) does not divide the \( b \)th power GCD matrix \((S^b)\) in the ring \( M_n(\mathbb{Z}) \). Similar results are also established for the power LCM matrices.

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1. Introduction

Smith [26] published his famous and beautiful theorem stating that for any integer \( n \geq 1 \), the determinant of the \( n \times n \) matrix \([ (i, j) ]\) having the the greatest common divisor \( (i, j) \) of \( i \) and \( j \) as its \( i, j \)-entry is the product \( \prod_{k=1}^{n} \varphi(k) \), where \( \varphi \) is Euler’s totient function. Since then many generalizations of Smith’s determinant have been published (see, for example, [1–25,27,28]).

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Let \( n \geq 1 \) be an integer and \( S = \{x_1, \ldots, x_n\} \) be a set of \( n \) distinct positive integers. Let \( a \geq 1 \) be an integer. The matrix having the \( a \)th power \((x_i, x_j)^a\) of the greatest common divisor of \( x_i \) and \( x_j \) as its \( i, j \)-entry is called the \( a \)th power greatest common divisor (GCD) matrix defined on \( S \), denoted by \((S^a)\). The eigen structure of power GCD matrices were received attentions by Wintner [27] as well as Lindqvist and Seip [24], and recently by Hong and Loewy [21, 22] and by Hong and Knoch Lee [20]. If \( a = 1 \), then the power GCD matrix defined on \( S \) is called the GCD matrix defined on \( S \), denoted by \((S)\). GCD matrices have been investigated since 1875 and especially actively in the recent decades. The matrix having the \( a \)th power \((x_i, x_j)^a\) of the least common multiple of \( x_i \) and \( x_j \) as its \( i, j \)-entry is called the \( a \)th power least common multiple (LCM) matrix defined on \( S \), denoted by \((S^a)\). If \( a = 1 \), then the power LCM matrix defined on \( S \) is called the LCM matrix defined on \( S \). Nonsingularity of power LCM matrices has been extensively studied by some authors [3, 7, 11, 16–20, 23]. In the field of power GCD matrices and power LCM matrices, questions of divisibility are central. The set \( S \) is said to be factor closed (FC) if it contains every divisor of \( x \) for any \( x \in S \). Bourque and Ligh [3] showed that if \( S \) is an FC set, then the GCD matrix \((S)\) divides the LCM matrix \((S)\) in the ring \( M_n(Z) \) of \( n \times n \) matrices over the integers. That is: There exists an \( A \in M_n(Z) \) such that \([S] = (S)A \) or \([S] = A(S)\). The set \( S \) is said to be gcd closed if \((x_i, x_j) \in S \) for all \( 1 \leq i, j \leq n \). It is clear that a factor-closed set is gcd closed but not conversely. Hong [13] showed that such factorization is no longer true in general if \( S \) is gcd closed. Bourque and Ligh [6] extended their result showing that if \( S \) is factor closed, then for any positive integer \( a \), the power GCD matrix \((S^a)\) divides the power LCM matrix \((S^a)\) in the ring \( M_n(Z) \).

The set \( S \) is called a divisor chain if there exists a permutation \( \sigma \) on \( \{1, \ldots, n\} \) such that \( x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)} \). Obviously a divisor chain is gcd closed but the converse is not true. The set \( S \) is called multiple closed \( (S) \) such that the least common multiple of all elements in \( S \). Hong [14] showed that for any divisor chain \( S \) with \(|S| = n \) and for any multiple-closed set \( S \) with \(|S| = n \), if \( a \) is a positive integer, then the power GCD matrix \((S^a)\) divides the power LCM matrix \((S^a)\) in the ring \( M_n(Z) \). It should be noted that Zhao et al. [28] showed that for any given \( n \geq 4 \), there exists an odd-lcm-closed set \( S = \{x_1, \ldots, x_n\} \) (namely, each element in \( S \) is an odd number and \( \{x_i, x_j\} \in S \) for all \( 1 \leq i, j \leq n \)) such that the power GCD matrix \((x_i, x_j)^a\) on \( S \) does not divide the power LCM matrix \((x_i, x_j)^a\) in the ring \( M_n(Z) \). We remark that Hong [15], He [8] and He–Zhao [9] obtained some results about the divisibilities of determinants of power LCM matrices.

In this paper we will concentrate on the questions of divisibility. We provide a new and interesting idea by considering the divisibility among power GCD matrices and among power LCM matrices. Let \( a, b \geq 1 \) be integers. We show that if \( a \mid b \), then for any divisor chain \( S \), the power GCD matrix \((S^a)\) divides the power GCD matrix \((S^b)\) in the ring \( M_n(Z) \). But such factorization should not hold if \( a \nmid b \). We also show that if \( a \nmid b \), then for any divisor chain \( S \), the power LCM matrix \((S^a)\) divides the power LCM matrix \((S^b)\) in the ring \( M_n(Z) \). But such result fails to be true if \( a \nmid b \).

For any permutation \( \sigma \) on \( \{1, \ldots, n\} \), define \( S_\sigma := \{x_{\sigma(1)}, \ldots, x_{\sigma(n)}\} \). Then one can easily check that \((S^a)^{-1}(S^b) = P^t(S_\sigma^a)^{-1}(S_\sigma^b)P\), where \( P \) is the \( n \times n \) permutation matrix whose \( i \)th row equals \((0, \ldots, 0, 1, 0, \ldots, 0)(1 \leq i \leq n) \). It follows that \((S^a)^{-1}(S^b) \in M_n(Z) \Leftrightarrow (S_\sigma^a)^{-1}(S_\sigma^b) \in M_n(Z) \). Similarly, we have \([S^a]^{-1}[S^b] \in M_n(Z) \Leftrightarrow [S_\sigma^a]^{-1}[S_\sigma^b] \in M_n(Z) \). So for our purpose of divisibility, without loss of generality, we assume throughout this paper that \( x_i \mid x_{i+1} \) for \( 1 \leq i \leq n - 1 \) and \( x_1 = 1 \).
2. Divisibility among power GCD matrices

In this section we discuss the divisibility among power GCD matrices. First we give a formula for the inverse of the GCD matrix on a divisor chain.

**Lemma 2.1.** Let $S$ be a divisor chain such that $1 = x_1 | x_2 | \ldots | x_n$. Then the inverse of the GCD matrix $(S)$ is tridiagonal. Furthermore, we have

$$(S)^{-1} = \begin{pmatrix}
    x_2 r_2 & -r_2 & 0 & \ldots & 0 & 0 \\
    -r_2 & r_2 + r_3 & -r_3 & \ldots & 0 & 0 \\
    0 & -r_3 & r_3 + r_4 & \ldots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \ldots & r_{n-1} + r_n & -r_n \\
    0 & 0 & 0 & \ldots & -r_n & r_n
  \end{pmatrix},$$

where $r_i = \frac{1}{x_i - x_{i-1}}$ for $2 \leq i \leq n$.

**Proof.** By direct computation, the result follows immediately. $\square$

We are now in a position to give the first main result of this paper.

**Theorem 2.2.** Let $a, b \geq 1$ be integers and $S$ be a divisor chain.

(i) If $a | b$, then the power GCD matrix $(S^a)$ divides the power GCD matrix $(S^b)$ in the ring $M_n(\mathbb{Z})$.

(ii) If $a \nmid b$ and $n \geq 2$, then the power GCD matrix $(S^a)$ does not divide the power GCD matrix $(S^b)$ in the ring $M_n(\mathbb{Z})$.

**Proof.** (i) First we consider the case $a = 1$. By Lemma 2.1 we get

$$(S)^{-1}(S^a) = \begin{pmatrix}
    1 & t_1 & t_1 & \ldots & t_1 & t_1 \\
    0 & t_2 & t_2 - t_3 & t_2 - t_3 & \ldots & t_2 - t_3 & t_2 - t_3 \\
    0 & 0 & t_3 & t_3 - t_4 & \ldots & t_3 - t_4 & t_3 - t_4 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & \ldots & t_{n-1} & t_{n-1} - t_n \\
    0 & 0 & 0 & 0 & \ldots & 0 & t_n
  \end{pmatrix},$$

where $t_1 = \frac{x_2 - x_2^b}{x_2 - 1}$ and $t_i = \frac{x_i^b - x_{i-1}^b}{x_i - x_{i-1}}$ for $2 \leq i \leq n$.

Clearly $t_i \in \mathbb{Z}$ for $1 \leq i \leq n$. So we have $(S)^{-1}(S^b) \in M_n(\mathbb{Z})$. This concludes part (i) for the case $a = 1$.

Now consider the general case: $a > 1$. Let $T = \{y_1, \ldots, y_n\}$ with $y_i = x_i^a$ for $1 \leq i \leq n$. Since $S$ is a divisor chain, $T$ is also a divisor chain. Note that for any $1 \leq i, j \leq n$

$$(y_i, y_j) = (x_i^a, x_j^a) = (x_i, x_j)^a.$$  

Hence the GCD matrix $(T)$ on $T$ is equal to the $a$th power GCD matrix $(S^a)$ on $S$, namely $(T) = (S^a)$. Let $c = \frac{b}{a}$. Then $c \in \mathbb{Z}$ since $a | b$. Since $(y_i, y_j)^c = (x_i, x_j)^b$ for all $1 \leq i, j \leq n$, we have $(T^c) = (S^b)$.
On the other hand, the result for the case \( a = 1 \) tells us that in the ring \( M_n(\mathbb{Z}) \), we have \((T)(T^r) = 0\). So the desired result \((S^a)(S^b) = 0\) follows immediately. Part (i) is proved.

(ii) Let \( n \geq 2 \) be an integer and \( a \nmid b \). Then \( a \neq b \). Since the set \( \{x_1^a, x_2^d, \ldots, x_n^a\} \) is a divisor chain, we get by Lemma 2.1

\[
(S^a)^{-1} = \begin{pmatrix}
 x_2^a - r_2 & 0 & \ldots & 0 & 0 \\
 -r_2 & x_2^a + r_3 & -r_3 & \ldots & 0 \\
 0 & -r_3 & x_2^a + r_4 & \ldots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & \ldots & 0 \\
 0 & 0 & 0 & \ldots & -r_n \end{pmatrix},
\]

(1)

where \( r_i = \frac{1}{x_i - x_{i-1}} \) for \( 2 \leq i \leq n \). Using (1) we can compute the \((2, 2)\) entry of the product \((S^a)^{-1}(S^b)\) and get that

\[
((S^a)^{-1}(S^b))_{22} = \frac{x_2^b - 1}{x_2^a - 1}.
\]

We claim that \(((S^a)^{-1}(S^b))_{22} \notin \mathbb{Z}\). By the claim we have immediately that \((S^a)^{-1}(S^b) \notin \mathbb{Z}\) which concludes part (ii). In what follows we show the claim. If \( a > b \), then \( 0 < \frac{x_2^a - 1}{x_2^b} < 1 \) since \( x_2 > 1 \). It follows that \( 0 < \frac{x_2^b - 1}{x_2^a - 1} < 1 \) which means that \(((S^a)^{-1}(S^b))_{22} \notin \mathbb{Z}\) as claimed. If \( a < b \) and \( a \nmid b \), then \( b > a \geq 2 \). So there are unique integers \( q \geq 1 \) and \( 1 \leq r \leq a - 1 \) such that \( b = qa + r \). From this we then deduce that

\[
\frac{x_2^b - 1}{x_2^a - 1} = x_2^r(1 + x_2^a + \cdots + x_2^{a(q - 1)}) + \frac{x_2^r - 1}{x_2^a - 1} \notin \mathbb{Z}
\]

since \( 0 < r < a \) together with \( x_2 > 1 \) implying that \( 0 < \frac{x_2^b - 1}{x_2^a - 1} < 1 \). Therefore the claim is proved and the proof of part (ii) of Theorem 2.2 is complete. \( \square \)

**Remark.** By Theorem 2.4 (i), we know immediately that for any integer \( a \geq 1 \) and any divisor chain \( S \), the GCD matrix \( (S) \) divides the \( a \)th power GCD matrix \( (S^a) \).

### 3. Divisibility among power LCM matrices

In the present section, we consider the divisibility among power LCM matrices. We need to compute the inverse of the LCM matrix on a divisor chain.

**Lemma 3.1.** Let \( S \) be a divisor chain such that \( 1 = x_1|x_2| \ldots|x_n \). Then the inverse of the LCM matrix \( [S] \) is tridiagonal. Furthermore, we have

\[
[S]^{-1} = \begin{pmatrix}
 u_1 & -u_1 & 0 & \ldots & 0 & 0 \\
 -u_1 & u_1 + u_2 & -u_2 & \ldots & 0 & 0 \\
 0 & -u_2 & u_2 + u_3 & \ldots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \ldots & u_{n-2} + u_{n-1} & -u_{n-1} \\
 0 & 0 & 0 & \ldots & -u_{n-1} & u_{n-1} + u_n
\end{pmatrix},
\]

where \( u_i = \frac{-1}{x_i - x_{i+1}} \) for \( 1 \leq i \leq n \) and \( x_{n+1} := 0 \).
Proof. By direct computation, the result follows immediately. □

We can now show the second main result of this paper.

Theorem 3.2. Let \( a, b \geq 1 \) be integers and \( S \) be a divisor chain.

(i) If \( a \mid b \), then the power LCM matrix \([S^a]\) divides the power LCM matrix \([S^b]\) in the ring \( M_n(\mathbb{Z}) \);

(ii) If \( a \not\mid b \) and \( n \geq 2 \), then the power LCM matrix \([S^a]\) does not divide the power LCM matrix \([S^b]\) in the ring \( M_n(\mathbb{Z}) \).

Proof. (i) First we consider the case \( a = 1 \). By Lemma 3.1, we obtain

\[
[S]^{-1}[S^b] = \begin{pmatrix}
v_1 & 0 & 0 & \cdots & 0 & 0 \\
v_2 - v_1 & v_2 & 0 & \cdots & 0 & 0 \\
v_3 - v_2 & v_3 - v_2 & v_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
v_{n-1} - v_{n-2} & v_{n-1} - v_{n-2} & v_{n-1} - v_{n-2} & \cdots & v_{n-1} & 0 \\
v_n - v_{n-1} & v_n - v_{n-1} & v_n - v_{n-1} & \cdots & v_n - v_{n-1} & v_n
\end{pmatrix},
\]

where

\[
v_i = \frac{x_i^b - x_i^{b+1}}{x_{i+1} - x_i} \text{ for } 1 \leq i \leq n
\]

and

\[
x_{n+1} := 0.
\]

Clearly \( v_i \in \mathbb{Z} \) for \( 1 \leq i \leq n \). So we have

\([S]^{-1}[S^b] \in M_n(\mathbb{Z})\).

This concludes part (i) for the case \( a = 1 \).

Now consider the general case: \( a > 1 \). Let \( T = \{y_1, \ldots, y_n\} \) with \( y_i = x_i^a \) for \( 1 \leq i \leq n \). Then \( T \) is a divisor chain since \( S \) is a divisor chain. Note that for any \( 1 \leq i, j \leq n \), we have

\([y_i, y_j] = [x_i^a, x_j^a] = [x_i, x_j]^a\).

So the LCM matrix \([T]\) on \( T \) is equal to the power LCM matrix \([S^a]\) on \( S \), namely \([T] = [S^a]\). Let \( c = \frac{b}{a} \). Then \( c \in \mathbb{Z} \) since \( a \mid b \). Since for all \( 1 \leq i, j \leq n \), \([y_i, y_j]^c = [x_i, x_j]^b\). From this we derive that \([T^c] = [S^b]\).

On the other hand, it follows form the result for the case \( a = 1 \) that in the ring \( M_n(\mathbb{Z}) \), we have

\([T][T^c]\). Thus the desired result \([S^a][S^b]\) follows immediately. Part (i) is proved.

(ii) Let \( n \geq 2 \) be an integer and \( a \not\mid b \). Then \( a \neq b \). Since the set \( \{1, x_2^a, \ldots, x_n^a\} \) is a divisor chain, we get by Lemma 3.1

\[
[S^a]^{-1} = \begin{pmatrix}
\tilde{u}_1 & -\tilde{u}_1 & 0 & \cdots & 0 & 0 \\
-\tilde{u}_1 & \tilde{u}_1 + \tilde{u}_2 & -\tilde{u}_2 & \cdots & 0 & 0 \\
0 & -\tilde{u}_2 & \tilde{u}_2 + \tilde{u}_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \tilde{u}_{n-2} + \tilde{u}_{n-1} & -\tilde{u}_{n-1} \\
0 & 0 & 0 & \cdots & -\tilde{u}_{n-1} & \tilde{u}_{n-1} + \tilde{u}_n
\end{pmatrix},
\]
Proof. (i) First it follows from [14] that 
\( (S^a)^{-1}S^b \) and obtain that 
\[
([S^a]^{-1}[S^b])_{11} = \frac{x_2^b - 1}{x_2^a - 1},
\]
which is not an integer, by the proof of part (ii) of Theorem 2.2, since \(a \nmid b\) and \(x_2 \geq 2\). This implies that \([S^a]^{-1}[S^b] \notin M_n(\mathbb{Z})\). Part (ii) is proved. This completes the proof of Theorem 3.2.  

Remark. Evidently for \(n = 1\), we have \((S^a)|(S^b)\) and \([S^a][S^b]\) if \(a < b\) and \(a \nmid b\). By Theorem 3.2 (i) one knows immediately that for any integer \(a \geq 1\) and any divisor chain \(S\), the LCM matrix \([S]\) divides the \(a\)th power LCM matrix \([S^a]\).

4. Divisibility of \([S^b]\) by \((S^a)\)

From the results presented in [14] and Sections 2 and 3 of this paper, we can derive the third main result of this paper as follows.

**Theorem 4.1.** Let \(a, b \geq 1\) be integers and \(S\) be a divisor chain. 

(i) If \(a | b\), then the power GCD matrix \((S^a)\) divides the power LCM matrix \([S^b]\) in the ring \(M_n(\mathbb{Z})\); 

(ii) If \(a \nmid b\) and \(n \geq 2\), then the power GCD matrix \((S^a)\) does not divide the power LCM matrix \([S^b]\) in the ring \(M_n(\mathbb{Z})\).

**Proof.** (i) First it follows from [14] that \((S^a)|(S^b)\). On the other hand, since \(a | b\), by Theorem 2.2(i) we have \((S^a)|(S^b)\). So we have \((S^a)][S^b]\) as desired. Part (i) also follows from [14] and Theorem 3.2(i).

(ii) By the inverse formula (1), we can calculate the (1, 1) entry of the product \((S^a)^{-1}[S^b]\) and get that 
\[
((S^a)^{-1}[S^b])_{11} = \frac{x_2^a - x_2^b}{x_2^a - 1} = 1 - \frac{x_2^b - 1}{x_2^a - 1},
\]
which is not an integer since \(a \nmid b\) together with \(x_2 > 1\) implying that \(\frac{x_2^b - 1}{x_2^a - 1} \notin \mathbb{Z}\) by the proof of part (ii) of Theorem 2.2. It implies that \((S^a)^{-1}[S^b] \notin M_n(\mathbb{Z})\). So part (ii) is proved. This completes the proof of Theorem 4.1.  

Remark. (1) Clearly for \(n = 1\), we have \((S^a)][S^b]\) if \(a < b\) and \(a \nmid b\). By Theorem 4.1(i) one knows immediately that for any integer \(a \geq 1\) and any divisor chain \(S\), the GCD matrix \((S)\) divides the \(a\)th power LCM matrix \([S^a]\).

(2) Let \(a, b \geq 1\) be integers and \(S\) a divisor chain with \(|S| \geq 2\). It follows from Theorems 2.2, 3.2 and 4.1 that \(\text{det}(S^a)\text{det}(S^b)\), \(\text{det}[S^a]\text{det}[S^b]\) and \(\text{det}(S^a)\text{det}[S^b]\) if \(a | b\), and \((S^a) \nmid (S^b)\), \([S^a] \nmid [S^b]\) and \((S^a) \nmid [S^b]\) if \(a \nmid b\). However, the non-divisibility of matrices does not imply the non-divisibility of determinants. It is still unclear whether we have \(\text{det}(S^a) \nmid \text{det}(S^b)\), \(\text{det}[S^a] \nmid \text{det}[S^b]\) and \(\text{det}(S^a) \nmid \text{det}[S^b]\) if \(a \nmid b\) for any divisor chain \(S\) with \(|S| \geq 2\). We guess that the answer to this question should be affirmative. In a forthcoming paper, we will discuss this topic.
(3) Bhowmik and Hong [2] established the similar results as in the present paper when $S$ is factor closed or multiple closed. One can show that there is a gcd-closed set $S$ such that $(S^a) \not| (S^b)$ (resp. $[S^a] \not| [S^b]$ and $(S^a) \not| ([S^b])$ in the ring $M_n(S)(\mathbb{Z})$ if $a∤b$. Furthermore, we propose several conjectures about the gcd-closed case. For this purpose, we recall the concept of greatest-type divisor introduced in [11]. For any $d, x \in S$ with $d < x$, we say that $d$ is a greatest-type divisor of $x$ in $S$ if $d|x$ and there is no other $y \in S$ such that $d|y$ and $y|x$. For $x \in S$, denote by $G_S(x)$ the set of all greatest-type divisors of $x$ in $S$.

**Conjecture 4.2.** Let $a, b \geq 1$ be integers such that $a|b$ and $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set with $\max_{x \in S}(|G_S(x)|) = 1$. Then the $a$-th power GCD matrix $(S^a)$ on $S$ divides the $b$-th power GCD matrix $(S^b)$ on $S$ in the ring $M_n(\mathbb{Z})$.

**Conjecture 4.3.** Let $a, b \geq 1$ be integers such that $a|b$ and $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set with $\max_{x \in S}(|G_S(x)|) = 1$. Then the $a$-th power LCM matrix $[S^a]$ on $S$ divides the $b$-th power LCM matrix $[S^b]$ on $S$ in the ring $M_n(\mathbb{Z})$.

**Conjecture 4.4.** Let $a, b \geq 1$ be integers such that $a|b$ and $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set with $\max_{x \in S}(|G_S(x)|) = 1$. Then the $a$-th power GCD matrix $(S^a)$ on $S$ divides the $b$-th power LCM matrix $[S^b]$ on $S$ in the ring $M_n(\mathbb{Z})$.

Obviously, Theorems 2.2(i), 3.2(i) and 4.1(i) give evidences to Conjectures 4.2, 4.3 and 4.4, respectively. We can also prove that $(S^a)(S^b)$ (resp. $[S^a][S^b]$ and $(S^a)[[S^b])$ in the ring $M_n(S)(\mathbb{Z})$ for any gcd-closed set $S$ with $|S| \leq 3$. Let now $n \geq 4$ and $a, b \geq 1$ be integers such that $a|b$. The problem of determining the necessary and sufficient conditions on the gcd-closed set $S$ with $|S| = n$ such that $(S^a)(S^b)$ (resp. $[S^a][S^b]$ and $(S^a)[[S^b])$ in the ring $M_n(\mathbb{Z})$ keeps widely open.

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**References**

[22] S. Hong, R. Loewy, Asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions (mod r), preprint.