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Linearization of two-dimensional systems of ODEs without conditions on small denominators

Vladimir V. Basov [a], Valery G. Romanovski [b,c,*]

[a] Faculty of Mathematics and Mechanics, St. Petersburg State University, Universitetsky prospekt, 28, 198504, Peterhof, St. Petersburg, Russia

[b] CAMTP - Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SI-2000 Maribor, Slovenia

[c] Faculty of Natural Science and Mathematics, University of Maribor, Koroška cesta 160, SI-2000 Maribor, Slovenia

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ABSTRACT

We study the problem of linearizability for two-dimensional systems of ODEs in a neighborhood of the saddle type singular point with rationally incommensurable eigenvalues. It is shown that if the linearizing transformation is convergent in one of the variables, then it is absolutely convergent.

1. Introduction

In recent years many studies have been devoted to the investigation of the problem of linearizability for plane systems of ordinary differential equations (see for instance, [1–3] and references therein). In this work we consider real two-dimensional systems of differential equations

$$\dot{x}_i = \lambda_i x_i + X_i(x_1, x_2) \quad (i = 1, 2), \tag{1}$$

where $X_i = \sum_{p:|p|=2}^{\infty} X_i^{(p)} x_1^{p_1} x_2^{p_2}$, $p = (p_1, p_2)$, $p_i \in \mathbb{Z}_+$, $|p| = p_1 + p_2$, $\lambda_i \in \mathbb{R}$, the ratio λ_1/λ_2 is irrational and negative, and it is supposed that the series $X_i(x_1, x_2)$ are real and convergent in a neighborhood of the origin. Then the origin is a non-degenerate saddle for (1). Define

$$\delta_{ip} = \lambda_1 p_1 + \lambda_2 p_2 - \lambda_i \quad (|p| \geq 2, \ \ i = 1, 2).$$

Obviously, $\delta_{ip} \neq 0$. As is well-known (see e.g. [4,5]), there exists a unique formal change of coordinates

$$x_i = y_i + h_i(y_1, y_2) \quad (i = 1, 2), \tag{2}$$

where $h_i = \sum_{p:|p|=2}^{\infty} h_i^{(p)} y_1^{p_1} y_2^{p_2}$, which transforms system (1) into the linear system

$$\dot{y}_i = \lambda_i y_i \quad (i = 1, 2). \tag{3}$$

Since $\lambda_1/\lambda_2 < 0$, there exists a sequence of vectors $p^{(1k)}, p^{(2k)}$ such that $|p^{(ik)}| \to \infty$, $|\delta_{ip^{(ik)}}| \to 0$ for $k \to \infty$. Therefore, for the system (1) there arises the so-called problem of small denominators, which means that fast decreasing of numbers $|\delta_{ip}|$ can lead to the divergence of (2).

* Corresponding author at: CAMTP - Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SI-2000 Maribor, Slovenia.
E-mail addresses: vlvlbasov@rambler.ru (V.V. Basov), Valery.Romanovski@uni-mb.si (V.G. Romanovski).

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By Theorem 4 of [5, Section 4] the transformation (2) is convergent if the eigenvalues λ_i ($i = 1, 2$) satisfy the condition ω which requires that

$$-\sum_{k=1}^{\infty} 2^{-k} \ln \omega_k < +\infty \quad (\omega_k = \min |\delta_{ip}| \text{ over } p : 2 \le |p| \le 2^k).$$

In the present work we do not require that the condition ω holds, that is, arbitrary "bad" small denominators are acceptable. For instance, the special transcendental numbers can be chosen as λ_i, and then a special Brjuno analytic system of the form (1) with divergent linearizing substitution can be constructed (Theorem 2 of [5, Section 6]). Since we do not impose any restriction on the small denominators in order to have analytic equivalence of systems (1) and (3) we need some additional assumptions. In this work we suppose that the transformation (2) is convergent in one of the variables, and prove that this assumption yields the absolute convergence of the series $h_i(y_1, y_2)$ ($i = 1, 2$).

2. Preliminary transformations

We show that by a change of the coordinates analytic at the origin we can eliminate in system (1) the terms $X_i^{(p)} x_1^{p_1} x_2^{p_2}$ such that $p \in \pi_1 \cup \pi_2$, where $\pi_1 = \{p \mid 0 \le p_1 < p_1^{(0)}, 0 \le p_2 < +\infty\}$, $\pi_2 = \{p \mid 0 \le p_2 < p_2^{(0)}, 0 \le p_1 < +\infty\}$ (that is, π_1 and π_2 are infinite strips from the integer grid of the first quadrant).

Proposition 1. *For any $p_1^{(0)}, p_2^{(0)} \in \mathbb{Z}_+$ such that $p_1^{(0)} + p_2^{(0)} \ge 2$, there exists a convergent substitution (2) with $h_i = \sum_{p \in \pi_1 \cup \pi_2} h_i^{(p)} y_1^{p_1} y_2^{p_2}$, which transforms (1) into the system*

$$\dot{y}_i = \lambda_i y_i + y_1^{p_1^{(0)}} y_2^{p_2^{(0)}} Y_i(y_1, y_2), \tag{4}$$

where $Y_i = \sum_{p_1+p_2=0}^{\infty} Y_i^{(p)} y_1^{p_1} y_2^{p_2}$ and Y_i are convergent in a neighborhood of the origin ($i = 1, 2$).

Proof. Differentiation of (2) yields

$$\lambda_i(y_i + h_i) + X_i(y_1 + h_1, y_2 + h_2) = \lambda_i y_i + y_1^{p_1^{(0)}} y_2^{p_2^{(0)}} Y_i + \sum_{j=1}^{2} \frac{\partial h_i}{\partial y_j} (\lambda_j y_j + y_1^{p_1^{(0)}} y_2^{p_2^{(0)}} Y_j).$$

Equating in this equality the coefficients of $y_1^{p_1} y_2^{p_2}$ for $p \in \pi_1 \cup \pi_2$, we obtain

$$(\lambda_1 p_1 + \lambda_2 p_2 - \lambda_i) h_i^{(p_1, p_2)} = \{X_i(y_1 + h_1(y_1, y_2), y_2 + h_2(y_1, y_2))\}^{(p_1, p_2)}, \tag{5}$$

where $|\delta_{ip}| = |\lambda_1 p_1 + \lambda_2 p_2 - \lambda_i| \ge \varepsilon > 0$ for all admissible p, because for such p either p_1 or p_2 in δ_{ip} is bounded. Therefore,

$$|h_i^{(p_1, p_2)}| \le \varepsilon^{-1} \{\widehat{X}_i(y_1 + \widehat{h}_1, y_2 + \widehat{h}_2)\}^{(p_1, p_2)}.$$

Here and below we use the notation $\widehat{Z}_i(z_1, z_2) = \sum_{p_1+p_2=2}^{\infty} |Z_i^{(p)}| z_1^{p_1} z_2^{p_2}$. The convergence of $h_i(y)$ is now easily proved by the Cauchy majorant method. □

Denote by R the radius of convergence of the series $X_i(x_1, x_2)$ in the system (1). If (a, a) is a point of absolute convergence of the series $X_i(x_1, x_2)$, we can perform the linear change $x_i = a\widetilde{x}_i$ to obtain a series with the radius of convergence greater than 1. Therefore, without loss of generality we can assume that in (1)

$$X_i = \sum_{p_1=2}^{\infty} \sum_{p_2=0}^{\infty} X_i^{(p)} x_1^{p_1} x_2^{p_2}, \quad R > 1 \tag{6}$$

(the first condition of (6) follows from Proposition 1 if we choose $p_1^{(0)} = 2$, $p_2^{(0)} = 0$). It follows from (6) that, in particular,

$$\widehat{X}_i(1, 1) = \sum_{p_1=2}^{\infty} \sum_{p_2=0}^{\infty} |X_i^{(p_1, p_2)}| < \infty.$$

Any formal series $Z = \sum_{p_1=2}^{\infty} \sum_{p_2=0}^{\infty} Z^{(p)} z_1^{p_1} z_2^{p_2}$ can be considered as a series in one variable with the coefficients depending on the other variable; in such a case we write $Z = \sum_{p_1=2}^{\infty} Z^{[p_1]}(z_2) z_1^{p_1}$, where $Z^{[p_1]} = \sum_{p_2=0}^{\infty} Z^{(p_1, p_2)} z_2^{p_2}$.

Lemma 1. *Assume that for the system (1) the condition (6) is fulfilled and the substitution (2) transforms (1) into the linear form (3). Then*

$$\forall\, p_1 \ge 2 : \quad \widehat{h}_i^{[p_1]}(1) = \sum_{p_2=0}^{\infty} |h_i^{(p_1, p_2)}| < +\infty.$$

Proof. The series X and h satisfy the relation

$$(\partial h_i / \partial y_1) \lambda_1 y_1 + (\partial h_i / \partial y_2) \lambda_2 y_2 - \lambda_i h_i = X_i(y_1 + h_1, y_2 + h_2).$$

Equating the coefficients of $y_1^{p_1}y_2^{p_2}$ for $p_1 \geq 2,\ p_2 \geq 0$ we obtain Eq. (5). From (5) it follows that an upper bound for $h_i^{(p)}$ is given by

$$|h_i^{(p)}| \leq |\delta_{ip}|^{-1}\Psi_i^{(p)}, \tag{7}$$

where $\Psi_i^{(p_1,p_2)} = \{\widehat{X}_i(y_1+\widehat{h}_1, y_2+\widehat{h}_2)\}^{(p_1,p_2)}$. Note that the series with non-negative coefficients $\Psi_i(y_1,y_2) = \widehat{X}_i(y_1 + \widehat{h}_1(y_1,y_2), y_2+\widehat{h}_2(y_1,y_2))$ converge if $|y_i+\widehat{h}_i(y_1,y_2)| < R$,. From the Taylor expansion we have

$$\Psi_i(y_1,y_2) = \widehat{X}_i(y_1,y_2) + \sum_{q_1+q_2=1}^{\infty} \frac{\partial^{\,|q|}\widehat{X}_i(y_1,y_2)}{\partial x_1^{q_1}\partial x_2^{q_2}} \frac{\widehat{h}_1^{q_1}\widehat{h}_2^{q_2}}{q_1!\,q_2!}.$$

We re-expand the obtained series as a series in the variable y_1. Taking into account that the expansions of the function $X_i,\ h_i$ in y_1 start from terms of at least the second order, we conclude that $\Psi_i^{[p_1]}$ is a polynomial with positive coefficients of the functions $\widehat{X}_i^{[k]}(y_2)(2 \leq k \leq p_1)$, the derivatives of these functions up to the order $p_1 - 1$ and the series $\widehat{h}^{[2]}(y_2), \ldots, \widehat{h}^{[p_1-1]}(y_2)$. Using (6) we conclude that the series $\widehat{X}_i^{[p_1]}(1)$ are convergent; therefore, for $y_2 = 1$ the derivatives of $\widehat{X}_i^{[p_1]}(y_2)$ of any order converge as well.

Going back to inequalities (7) we note that for any fixed $p \geq 2$ there exists an $\varepsilon_{p_1} > 0$ such that $|\delta_{ip}| \geq \varepsilon_{p_1}$ for all $p_2 \geq 0$. Therefore, summing up (7) over p_2 we obtain

$$\sum_{p_2=0}^{\infty} |h_i^{(p_1,p_2)}| \leq \varepsilon_{p_1}^{-1} \sum_{p_2=0}^{\infty} \Psi_i^{(p_1,p_2)} \quad \text{or} \quad \widehat{h}_i^{[p_1]}(1) \leq \varepsilon_{p_1}^{-1}\Psi_i^{[p_1]}(1).$$

It remains to show that $\widehat{h}_i^{[p_1]}(1) < \infty$. We show this by induction on p_1. For $p_1 = 2$, obviously, $\widehat{h}_i^{[2]}(1) \leq \varepsilon_2^{-1}\Psi_i^{[2]}(1) = \widehat{X}_i^{[2]}(1) < \infty$. Assuming now that the series $\widehat{h}_i^{[k]}(1)$ are convergent for $k < p_1$ we obtain that $\Psi_i^{[p_1]}(1)$ is a finite sum of finite products of finite quantities. Therefore, the series $\widehat{h}_i^{[p_1]}(y_2)$ are convergent for $y_2 = 1$. □

We write the formal series h_i of (2) as the series in y_1 with coefficients depending on $y_2 : h_i = \sum_{p_1=2}^{\infty} h_i^{[p_1]}(y_2)y_1^{p_1}$. Then, by Lemma 1 the series $h_i^{[p_1]}(y_2)$ are convergent for $|y_2| \leq 1$ and we always can compute their values at $y_2 = 1$.

Let us assume that for $y_2 = 1$ the series $h_i(y_1, 1) = \sum_{p_1=2}^{\infty} h_i^{[p_1]}(1)y_1^{p_1}$ are absolutely convergent in y_1. This means that

$$\exists\, c,\ C > 0: \quad |h_i^{[p_1]}(1)| \leq Cc^{p_1}. \tag{8}$$

After minor generalization of the theorem on convergence of Dulac's integrals [6, Section 14] we obtain the following statement.

Lemma 2. *For any absolutely convergent real series $\sum_{k=2}^{\infty} c_i^{(k)}y_1^k$ there exists a linearizing substitution*

$$x_i = y_i + \sum_{k=2}^{\infty} g_i^{(k)}(y_2)y_1^k \tag{9}$$

with (perhaps non-analytic) coefficients $g_i^{(k)}(y_2)$, satisfying the initial conditions $g_i^{(k)}(1) = c_i^{(k)}$, which is absolutely convergent in y_1 uniformly over $y_2 \in [-1, 1]$.

Therefore, if (8) is not fulfilled, then among substitutions (9) there is no substitution of the form (2) having (by Lemma 2) coefficients which are analytic in y_2. Using (8) we apply to the substitution (2) Lemma 2 and obtain that it is analytic in y_1 uniformly over y_2.

However, for arbitrary real series in $y_1,\ y_2$ the condition (8) of the pointwise convergence and even the stronger assumption of convergence of the series in y_1 uniformly over y_2 do not guarantee the absolute convergence of the series in both variables, as is shown in the following statement.

Proposition 2. *For any divergent real series $\sum_{k=1}^{\infty} a_k z^k$ it is possible to construct a series*

$$f(u,v) = \sum_{k=1}^{\infty}\sum_{m=k}^{m_k} f^{(k,m)}u^k v^m, \qquad f^{(k,k)} = a_k,$$

which is, obviously, divergent (because the series $\sum_{k=1}^{\infty} f^{(k,k)}(uv)^k$ is divergent); however if we write down f as a series in u with the polynomial coefficients in v, that is, $f = \sum_{k=1}^{\infty} f^{[k]}(v)u^k$, where $f^{[k]} = v^k\big(a_k + f^{(k,k+1)}v + \cdots + f^{(k,m_k)}v^{m_k}\big)$, then $|f^{[k]}(v)| \leq 1$ for $|v| \leq 1$.

Proof. We make a suitable choice of degrees m_k and coefficients $f^{(k,k+j)}$ $(1 \leq j < m_k)$ as follows.

For $k \geq 1$ such that $|a_k| \leq 1$, we let $m_k = 0$. Then $|f^{[k]}(v)| = |a_k v^k| \leq 1$ for $|v| \leq 1$. Now, for $k \geq 1$ such that $|a_k| > 1$, we define $I_k = [-|3a_k|^{-1}, |3a_k|^{-1}]$ and consider on $[-1, 1]$ the continuous function

$$g(v) = \{\, 3a_k^2|v| \quad \text{for } v \in I_k, \qquad |a_k| \quad \text{for } v \in [-1,1] \setminus I_k \,\}.$$

By the Weierstrass theorem there is a polynomial $G_k(v)$ of degree $m_k \geq 1$ such that $|g(v) - G_k(v)| \leq 1/2$ for $|v| \leq 1$. Since $g(0) = 0$, we see that $|G_k(0)| < 1/2$. Let

$$F_k(v) = G_k(v) - G_k(0), \qquad f^{[k]}(v) = v^k(a_k - \operatorname{sign} a_k\, F_k(v)),$$

that is, $\sum_{j=1}^{m_k} f^{(k,j)} v^j = -\operatorname{sign} a_k\, F_k(v)$.

We show that the polynomial $f^{[k]}(v)$ has the required property. Since

$$|g(v) - F_k(v)| \leq |g(v) - G_k(v)| + |G_k(0)| \leq 1,$$

we obtain $|F_k(v)| \leq g(v) + 1$. By the definition, $|f^{[k]}(v)| = |v^k|\,||a_k| - F_k(v)|$.

If $v \in [-1,1] \setminus I_k$, then $|a_k| = g(v)$ and $|f^{[k]}(v)| \leq |g(v) - F_k(v)| \leq 1$. If $v \in I_k$, $g(v) \leq |a_k|$,

$$|f^{[k]}(v)| \leq |3a_k|^{-k}(|a_k| + (g(v)+1)) \leq |3a_k|^{1-k}(1/3 + 1/3 + |3a_k|^{-1}) \leq 1,$$

because $k \geq 1$ and $a_k > 1$.

Thus, $|f^{[k]}(v)| \leq 1$ for all $v \in [-1,1]$. Therefore, we have constructed the series f with polynomial coefficients in v, which is absolutely convergent in u for $|u| < 1$ uniformly over $v \in [-1,1]$, but it is divergent as a series in two variables. □

3. Convergence of the linearization

We have seen that for arbitrary series the condition (8) does not guarantee the analyticity of the series. However, if (8) holds not for an arbitrary series, but for the coefficients $h_i(y)$ of the linearizing transformation (2), then the transformation (2) is analytic in a neighborhood of the origin.

Theorem 1. *Assume that the normalizing substitution (2) transforms the system (1) satisfying (6) into the linear system (3) and for (2) the condition (8) is fulfilled. Then the transformation (2) converges for any rationally incommensurable λ_1 and λ_2.*

Proof. Consider the inequality (7) from Lemma 1. The quantity δ_{ip} is different from zero and for a fixed $p_1 \delta_{ip}$ is a linear function in p_2 with $\lambda_2 \neq 0$. Hence, there exists an index $p_1^{(i)} \geq 0$ such that $\delta_{i(p_1,p_1^{(i)})}\delta_{i(p_1,p_1^{(i)}+1)} < 0$. Then, $|\delta_{i(p_1,p_1^{(i)})}| + |\delta_{i(p_1,p_1^{(i)}+1)}| = |\lambda_2|$ and $|\delta_{i(p_1,p_1^{(i)})}| \neq |\delta_{i(p_1,p_1^{(i)}+1)}|$ (otherwise there exists an index p such that $\delta_{ip} = 0$).

Assume, for instance, that $|\delta_{i(p_1,p_1^{(i)})}| < |\delta_{i(p_1,p_1^{(i)}+1)}|$. Then,

$$|\delta_{i(p_1,p_1^{(i)})}| < |\lambda_2|/2 = \varepsilon, \quad \forall\, p_2 \neq p_1^{(i)} : \ |\delta_{i(p_1,p_2)}| > \varepsilon.$$

From the condition (8), $\forall\, p_1 \geq 2$: $\left| h_i^{(p_1,p_1^{(i)})} + \sum_{p_2 \neq p_1^{(i)}} h_i^{(p_1,p_2)} \right| \leq C c^{p_1}$; therefore,

$$|h_i^{(p_1,p_2)}| \leq C c^{p_1} + \left| \sum_{p_2 \neq p_1^{(i)}} h_i^{(p_1,p_2)} \right| \leq C c^{p_1} + \sum_{p_2 \neq p_1^{(i)}} |h_i^{(p_1,p_2)}|.$$

Hence, from (7) we obtain

$$|h_i^{(p_1,p_2)}| < \varepsilon^{-1} \Psi_i^{(p_1,p_2)}\ (p_2 \neq p_1^{(i)}), \qquad |h_i^{(p_1,p_1^{(i)})}| < C c^{p_1} + \varepsilon^{-1} \sum_{p_2 \neq p_1^{(i)}} \Psi_i^{(p_1,p_2)}.$$

For any $p_1 \geq 2$ we sum up over all $p_2 \geq 0$ and obtain

$$\sum_{p_2=0}^{\infty} |h_i^{(p_1,p_2)}| < C c^{p_1} + 2\varepsilon^{-1} \sum_{p_2 \neq p_1^{(i)}} \Psi_i^{(p_1,p_2)} \leq C c^{p_1} + 2\varepsilon^{-1} \sum_{p_2=0}^{\infty} \Psi_i^{(p_1,p_2)},$$

or, equivalently, $\widehat{h}_i^{[p_1]}(1) < C c^{p_1} + 2\varepsilon^{-1} \Psi_i^{[p_1]}(1)$. It is proved in Lemma 1 that $\Psi_i^{[p_1]}(1)$ are finite numbers. Multiplying the latter inequalities by $y_1^{p_1}$ and summing up over all $p_1 \geq 2$ and $i = 1, 2$ we obtain the majorant inequality

$$\widehat{h}_1(y_1, 1) + \widehat{h}_2(y_1, 1) \prec \Theta(y_1) + 2\varepsilon^{-1}(\Psi_1(y_1, 1) + \Psi_2(y_1, 1)), \tag{10}$$

where $\Theta(y_1) = 2C \sum_{p_1=2}^{\infty} c^{p_1} y_1^{p_1}$ is the convergent series.

By (6) the series $X_i(x_1, x_2)$ are convergent for $|x_1|, |x_2| < R$, where $R > 1$. Therefore, for small $y_1, \eta\,\Xi_i(y_1, \eta) = \widehat{X}_i(y_1 + \eta, 1 + \eta)$ are analytic functions of y_1, η and their expansions start from the terms of at least the second order.

Let $\eta = \widehat{h}_1(y_1, 1) + \widehat{h}_2(y_1, 1)$; then $\Psi_i(y_1, 1) \prec \widehat{X}_i(y_1 + \eta, 1 + \eta)$. Thus, (10) yields the majorant inequality

$$\eta \prec \Theta(y_1) + 2\varepsilon^{-1}(\Xi_1(y_1, \eta) + \Xi_2(y_1, \eta)).$$

Replacing now the symbol of majorant by the symbol of equality and η by $\widetilde{\eta}$, we obtain the equation $\Phi(y_1, \widetilde{\eta}) = 0$, where $\Phi(y_1, \widetilde{\eta}) = 0$ is an analytic function. It is obvious that the equation $\Phi(y_1, \widetilde{\eta}) = 0$ satisfies the Implicit Function Theorem. Hence it has an analytic solution $\widetilde{\eta}(y_1)$ which majorizes the series $\eta = \widehat{h}_1(y_1, 1) + \widehat{h}_2(y_1, 1)$.

Thus, the series $\widehat{h}_i(y_1, 1) = \sum_{p_1=2}^{\infty} y_1^{p_1} \sum_{p_1=2}^{\infty} |h_i^{(p_1,p_2)}|$ are convergent yielding that the series $h_i(y_1, y_2)$ are absolutely convergent for $|y_1| \leq y_1^*$ and $|y_2| \leq 1$. □

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References

[1] X. Chen, V.G. Romanovski, W. Zhang, Linearizability conditions of time-reversible quartic systems having homogeneous nonlinearities, Nonlinear Anal. TMA 69 (2008) 1525–1539.
[2] C. Christopher, P. Mardešić, C. Rousseau, Normalizable, integrable, and linearizable saddle points for complex quadratic systems in $\mathbb{C}^2$, J. Dyn. Control Syst. 9 (2003) 311–363.
[3] J. Gine, V.G. Romanovski, Linearizability conditions for Lotka–Volterra planar complex cubic systems, J. Phys. A: Math. Theor. 42 (22) (2009) 225206–225221.
[4] Y.N. Bibikov, Local Theory of Nonlinear Analytic Ordinary Differential Equations, in: Lecture Notes in Mathematics, vol. 702, Springer-Verlag, New York, 1979.
[5] A.D. Brjuno, Analytic form of differential equations. I, Tr. Mosk. Mat. Obs. 25 (1971) 119–262. (in Russian).
[6] H. Dulac, Sur les cycles limites, Bull. Soc. Math. France 51 (1923) 45–188.