Nonexistence of Ovoids in $\Omega^+(10, 3)$.

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1. INTRODUCTION

An ovoid in a nondegenerate polar space is a collection $\mathcal{O}$ of points which meets each maximal singular subspace at a single point. (For a survey of the existence and non-existence theorems for polar ovoids see Kantor [1, 2] and Thas [5].) An ovoid of the quadric $Q$ of type $\Omega^+(2n, q)$ must contain $1 + q^{n-1}$ points, and conversely any collection of pairwise non-collinear points of $Q$ of this cardinality constitutes an avoid. Such ovoids are known to exist for $n \leq 4$, but the existence or non-existence of these ovoids in dimension 10 or higher is unknown for any $q$ except 2. In [2] W. M. Kantor proves that $\Omega^+(2n, 2)$, $n \geq 5$, has no ovoids. If $\mathcal{O}$ is an ovoid of the quadric $Q$ of type $\Omega^+(2n, q)$, and $s$ is a point of the quadric not on $\mathcal{O}$, then the set $((s^{-1} \cap \mathcal{O}) + s)/s$ is an ovoid of the quadric induced on $s^{-1}/s$ of type $\Omega^+(2n-2, q)$. Thus the nonexistence of an ovoid for $\Omega^+(10, q)$ would, for any fixed $q$, imply the nonexistence of an ovoid in any quadric of type $\Omega^+(2n, q)$ for all $n \geq 5$. In this note we show the nonexistence of an ovoid in $\Omega^+(10, 3)$, thereby eliminating ovoids for all $\Omega^+(2n, 3)$, $n \geq 5$.

This proof depends strongly on the very short supply of ovoids of the quadric of type $\Omega^+(8, 3)$. There is essentially only one.

PROPOSITION (Patterson [3]). If $\mathcal{O}$ is an ovoid of a quadric $Q$ of type $\Omega^+(8, 3)$ then $\mathcal{O}$ is unique up to semi-isometry.

Remark. Patterson’s result is phrased in terms of 4-by-4 Kerdock sets over $GF(3)$, which are in 1-1 correspondence with spreads of the $\Omega^+(8, 3)$ polar space, which are in turn in 1-1 correspondence with ovoids of the same polar space via triality.

We give here presentations of representative ovoids of each of the two isometry classes—referred to here as classes I and II. For this purpose, let
$e_1, \ldots, e_8$ be an orthonormal basis for the vector space $V$ over $GF(3)$. Thus $V$ has an inner product $(, )$, where $(e_i, e_j) = \delta_{ij}$. We take the liberty of also representing these vectors $\sum \alpha_i e_i$ by 8-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_8)$. This inner product $(, )$ makes $V$ into a nondegenerate space of type $\Omega^+(8, 3)$, and is derived from the quadratic form $Q: (\alpha_1, \ldots, \alpha_8) \rightarrow \sum \alpha_i^2$.

**Presentation of an Ovoid of Class I.** Impose on $I = \{1, 2, \ldots, 7\}$ the structure of a Fano plane $(I, \mathcal{L})$ with line set $\mathcal{L} = \{ [1, 2, 4], [2, 3, 5], [3, 4, 6], \ldots, [7, 1, 3] \}$. The points of $O$ are the 28 singular 1-spaces generated by vectors $v$ of the form $\pm e_i \pm e_j \pm e_k$, where $[i, j, k]$ is in $\mathcal{L}$. All 1-spaces of this ovoid are perpendicular to the vector $e_8$ of norm $(e_8, e_8) = 1$.

**Presentation of an Ovoid of Class II.** $O$ consists of the 28 singular 1-spaces $\langle v \rangle$, where $v$ is any vector of shape $(1^6, 0^2)$—i.e., it has six of its 8-tuple entries equal to $+1$ and two of them equal to 0. All of these vectors are perpendicular to the vector $j = (1, \ldots, 1)$ of norm $-1$.

Both of these ovoids possess ovals—i.e., sets of all (four) singular 1-spaces of a non-degenerate 3-space. The ovoids do not belong to the same isometry class because of the differing norms of the 1-space $O^\perp$ in each case.

### 2. Grammians and Switching Classes of Graphs

Let $\mathcal{G}$ be the set of all graphs which have the vertex set $V = \{1, 2, \ldots, n\}$ and let $\Gamma$ be a member of $\mathcal{G}$. We can obtain from $\Gamma$ a new graph $\Gamma'$ by erasing all edges of $\Gamma$ on vertex $i$, and joining $i$ to each vertex $k$ not previously joined to $i$ in $\Gamma$. This operation is called **switching at vertex $i$**, and performing it induces a permutation $\delta_i$ of the graphs of $\mathcal{G}$. Clearly $E = \langle \delta_1, \ldots, \delta_n \rangle$ induces an elementary 2-group of order $2^{n-1}$ acting on $\mathcal{G}$. The $E$-orbits on $\mathcal{G}$ are called the **switching classes** of graphs, and are in 1–1 correspondence with 2-graphs of $n$ vertices (see Seidel [4]).

For any ovoid $O$ of a quadric in $\Omega^+(2n, 3)$, and any subset $\mathcal{P}$ of $O$, one may choose representative vectors $v_1, \ldots, v_m$, of the 1-spaces $\langle v_i \rangle$ of $\mathcal{P}$, and form the $m$-by-$m$ Gram matrix $G$ with $(i, j)$th entry, $(v_i, v_j) = +1$ or $-1$. This matrix in turn may be represented by a graph, also denoted $\mathcal{G}$, with vertex set $\{1, 2, \ldots, m\}$ and adjacency defined by declaring $i$ and $j$ to be adjacent if and only if $(v_i, v_j) = -1$. Replacing the $i$th vector $v_i$ by $-v_i$ yields a new Gram matrix whose graph is obtained from $G$ by “switching” at the vertex $i$. 
In the ovoid $\mathcal{O}$ of class I presented above, the four vectors
\[ v_1 = e_1 + e_2 + e_4, \quad v_2 = e_1 - e_2 - e_4, \quad v_3 = -e_1 + e_2 - e_4, \quad v_4 = -e_1 - e_2 + e_4 \]
generate 1-spaces of $\mathcal{O}$ and have Gram matrix $I - J$ whose associated graph belongs to the switching class of eight members with representative graphs

\[ K_4 = \begin{array}{ccc}
(1), & (2), & (3), & (4)
\end{array} \]

Since $\sum v_i = 0$, $\langle v_1 \rangle, \ldots, \langle v_4 \rangle$ form an oval in $\mathcal{O}$.

We denote the above switching class of 4-vertex graphs by the symbol $\mathcal{H}$. In the language of 2-graphs, the set $T$ of triples of vertices which make up the 2-graph are those which bear 1 or 3 edges. A 4-set is called \textit{homogeneous} if all of its 3-subsets belong to $T$. The class of graphs $\mathcal{H}$ are the homogeneous 4-sets.

Now let $\mathcal{O}$ be any ovoid of a polar space of type $\Omega^+(2n, 3)$. We call a 4-subset of $\mathcal{O}$ an $\mathcal{H}$-set if and only if the $(-1)$-graph representing its Gram matrix belongs to the switching class $\mathcal{H}$. Similarly a 4-subset of $\mathcal{O}$ is called an $\mathcal{H}'$-set if its $(-1)$-graph belongs to the switching class $\mathcal{H}'$ consisting of the eight graphs of these types:

\[ \mathcal{H}': 4\text{-}coclique, \quad \text{rectangle}, \quad \text{3-claw} \]

Let $\mathcal{O}$ be any ovoid (of any $\Omega^+(2n, 3)$-quadric for the moment) and let $\mathcal{G}(\mathcal{O})$ be its associated switching class of graphs. The associated 3-subsets are of two types: (T) those whose 3-by-3 Gram matrix has its $(-1)$-graph in the switching class of graphs having an odd number of edges (these are the triplets of the associated 2-graph); and $(T')$ those whose associated switching class consists of a 3-coclique and the 2-claw. The class of all 3-subsets of $\mathcal{O}$ of type $(T)$ (resp. $(T')$) is denoted $\mathcal{I}$ (resp. $\mathcal{I}'$).

It turns out in the next section that for an ovoid $\mathcal{O}$ of class I above, every $\mathcal{H}$-set is an oval in $\mathcal{O}$, and that any $\mathcal{H}'$-set generates a 4-subspace with a nontrivial radical and that this property can be used to distinguish class I from class II.
3. Important Properties of the Classical Ovoids of $\Omega^+(8,3)$

The Weyl group $W(E_7)$ acts on the lattice $A^*$ generated by the system of 28 equiangular lines $\{\langle u+v \rangle | u, v \in A, the E_7$-lattice, $(u, v) = -1 \}$. The ovoid $O$ of class I is just the system of singular 1-spaces generated by the images of these 56 norm 3 vectors $u+v$ under the morphism $A^* \to A^*/3A^*$.

In its action on the 2-graph, $W(E_7) \cong Sp(6,2)$ is doubly transitive, and the homogeneous 4-sets form the 315 blocks of a block design with parameters $(28, 4, 5)$. Similarly, the images of these homogeneous 4-sets, form a system of 315 ovals in $O$, all conjugate to the oval $\{ \langle v_1 \rangle, ..., \langle v_4 \rangle \}$ above.

**Lemma 1.1.** (i) If $O$ is an ovoid of class I, then any 3-subset belongs to $\mathcal{J}$ and only if the oval of the 3-space it generates lies within $O$. (In particular, this means that if $u_1, u_2$, and $u_3$ are any three vectors of the ambient space which generate 1-spaces of $O$ and which pair-wise have inner product $-1$, then the oval of the 3-space $\langle u_1, u_2, u_3 \rangle$ lies in $O$—i.e., $\langle -\sum u_i = u_4 \rangle$ lies in $O$. Similarly if $(u_1, u_2) = (u_1, u_3) = -1$ and $(u_2, u_3) = 1$, then the fourth member of the oval of $\langle u_1, u_2, u_3 \rangle$ (namely $\langle u_4 \rangle$ where $u_4 = -u_1 + u_2 + u_3$), does not lie in $O$).

(ii) If $O$ is an ovoid belonging to class II then any 3-subset belongs to $T'$ if and only the oval of the 3-space it generates lies within $O$. (This means that given the two sets of hypotheses on the inner products among $u_1, u_2$, and $u_3$ in (i) above, the conclusions whether $\langle u_4 \rangle$ lies in $O$ are to be transposed.)

**Proof.** (i) We may assume $O$ is the ovoid of class I presented in the previous section. A typical 3-subset of $\mathcal{J}$ may be represented by vectors $v_1, v_2$, and $v_3$ with pair-wise inner products $-1$. By the 2-transitivity of $W(E_7)$ on $O$, one may assume $v_1 = e_1 + e_2 + e_4$ and $v_2 = e_1 - e_2 - e_4$. Clearly, if $v_3$ is in $\langle e_1, e_2, e_4 \rangle$ we obtain the oval given above just before the definition of switching class $\mathcal{K}$. Thus we may assume $v_3$ lies in $\langle e_i, e_j, e_k \rangle$, where $[i, j, k] \cap [1, 2, 4] = 1$. Applying the appropriate isometries one may assume $v_3 = e_1 + e_5 + e_6$. Then $-\sum v_i = v_4 = e_1 - e_5 - e_6$ generates an element of $O$.

If the 3-set is in $T'$, it can be represented by vectors $u_1, u_2$, and $u_3$, where we may assume $u_1 = v_1$ and $u_2 = v_2$ as in the previous paragraph and as $(u_3, u_1) = -1$, $(u_3, u_2) = 1$, and vectors $\{e_1 \pm e_2 \pm e_4\}$ form a homogenous 4-set, $u_3 \in \langle e_i, e_j, e_k \rangle$, where $[i, j, k] \cap [1, 2, 4] \neq \{1\}$. Applying appropriate isometries, without loss, $u_3 = -2 + e_3 + e_5$. The fourth singular 1-space of $\langle u_1, u_2, u_3 \rangle$ is generated by $u_4 = -u_1 + u_2 + u_3 = e_3 + e_4 + e_5$ and so is not in $O$.

(ii) The ovoid of Presentation 2 is isometric to that which would appear for Presentation 1 if the quadratic form $Q$ were replaced by $-Q$. In
terms of the graph-theoretic descriptions it is as if we had made "adjacency" = "inner product + 1," and accordingly replaced each relevant graph by its complement. The results of (i) then give the conclusions of (ii).

Remark. Note that in case (ii) the 4-sets comprising the ovals in \( O \) are those whose Gram matrices have their \((-1)\)-graphs in the class \( \mathcal{H}' \).

We may rephrase these results:

**Lemma 1.2.** Let \( O \) be an ovoid of a polar space of type \( \Omega^+(8, 3) \). Then the following are equivalent:

(i) \( O \) belongs to class I.

(ii) Every \( \mathcal{H}' \)-set of \( O \) is an oval.

(iii) At least one \( \mathcal{H} \)-set is an oval.

(iv) Every \( \mathcal{H}' \)-set generates a 4-space with 1-dimensional radical.

(v) At least one \( \mathcal{H}' \)-set generates a 4-space with 1-dimensional radical.

**Proof.** We have seen from Lemma 1.1(i) that if \( O \) is in class I then a 3-set belongs to \( J \) or \( J' \) according as the oval of the 3-space it generates belongs to \( O \) or not. Similarly, from part (ii), if \( O \) is in class II, a 3-set belongs to \( J' \) if and only if the oval of the 3-space it generates belongs to \( O \). Together these imply these equivalence of (i) \( (O \) belongs to class I) and

(a) For some 3-set in \( J \), the oval of the 3-space it generates belongs to \( O \).  

(b) For some 3-set in \( J' \), the 3-space it generates has a point of its oval not in \( O \).*

In case (b) we can assume the 3-set is \( \langle v_1, v_2, v_3 \rangle \), where \((v_1, v_2) = -1 = (v_1, v_3), (v_2, v_3) = 1 \). If \( v_4 = -v_1 + v_2 + v_3 \), then \( \langle v_4 \rangle \) is the fourth point of the oval and does not lie in \( O \). Now suppose \( X \) is an \( \mathcal{H}' \)-set of \( O \) containing \( \langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle \) with fourth member \( \langle x \rangle \). Then \( x \) can be chosen so that \( \{v_1, v_2, v_3, x\} \) has the same Gram matrix as \( \{v_1, \ldots, v_4\} \). Then \( r = \langle v_4 - x \rangle = \text{Rad}(v_1, \ldots, v_3, x) \). There are, in fact, exactly 10 choices of \( x \), given \( v_1, v_2, \) and \( v_3 \).

We have only to show (v) or (iii) imply (i). Suppose \( Y \) is an \( \mathcal{H}' \)-set in \( O \) and is an oval. Then any 3-subset \( X \) of \( Y \) satisfies the hypothesis of (a), whence (i). Suppose instead \( Y \) is an \( \mathcal{H} \)-set in \( O \) and that \( \langle Y \rangle \) has a 1-dimensional radical \( r \). Choose any 3-subset \( X \) of \( Y \); then \( X \) belongs to \( J' \). Suppose the oval on \( \langle X \rangle \) lies in \( O \). Then the point \( \langle x \rangle \) of \( Y - X \) is perpendicular to some point of \( X \) since the singular points of \( \langle Y \rangle \) form a cone. This contradicts the fact that \( O \) is an ovoid. Thus \( X \) satisfies the hypothesis of (b) and so (i) holds.
Lemma 1.3. \( \mathcal{O} \) is an ovoid of a polar space of type \( \Omega^+(8, 3) \). The following are equivalent:

(i) \( \mathcal{O} \) is in class II.
(ii) Every \( \mathcal{H}' \)-set forms an oval in \( \mathcal{O} \).
(iii) At least one \( \mathcal{H}' \)-set forms an oval in \( \mathcal{O} \).
(iv) Every \( \mathcal{H} \)-set spans a 4-space with a 1-space radical.
(v) At least one \( \mathcal{H} \)-set spans a 4-space with a 1-dimensional radical.

Proof. Replacing the quadratic form \( Q \) by \(-Q\) converts a class I ovoid to a class II ovoid and vice versa. This replaces each graph representing a Gram matrix by its complementary graph, and the result is a restatement of Lemma 1.2.

Remark. In an ovoid of class I, there are many more \( \mathcal{H}' \)-sets than \( \mathcal{H} \)-sets. Every 3-set of \( \mathcal{J} \) lies in exactly one \( \mathcal{H} \)-set (namely, the oval of the 3-space it generates); but each 3-set of \( \mathcal{J}' \) lies in 10 \( \mathcal{H}' \)-sets. The situation is reversed for an ovoid of class II: Each subset of \( \mathcal{J}' \) lies in a unique \( \mathcal{H}' \)-set forming an oval, while every 3-subset of \( \mathcal{J} \) lies in 10 \( \mathcal{H} \)-sets.

Let us assume \( \mathcal{O} \) is an ovoid of \( \Omega^+(8, 3) \) of class I. \( W(E_7) \) acts on \( \mathcal{O} \) and permutes the points of the quadric in two orbits: one lying in the 7-space \( \langle \mathcal{O} \rangle \), the other consisting of all singular 1-spaces not in \( \langle \mathcal{O} \rangle \). We refer to the former as Desarguesian points relative to \( \mathcal{O} \), since these induce 6-dimensional ovoids corresponding to Desarguesian translation planes (see Kantor [2]). One notices that the 1-spaces \( r = \text{Rad} \langle p_1, \ldots, p_2 \rangle \), where \( \{p_1, \ldots, p_2\} \) is an \( \mathcal{H}' \)-set, belong to \( \langle \mathcal{O} \rangle \) and so are Desarguesian points. But since \( W(E_7) \) is transitive on this set we have

Lemma 1.4. Suppose \( \mathcal{O} \) is an ovoid of \( \Omega^+(8, 3) \) of class I. Then every Desarguesian point appears as the radical of some 4-space generated by an \( \mathcal{H}' \)-set.

Similarly if \( \mathcal{O} \) is class II, every Desarguesian point appears as the radical of a 4-space generated by an \( \mathcal{H} \)-set.

Finally we need

Lemma 1.5. Suppose \( \mathcal{O} \) is an ovoid of \( \Omega^+(8, 3) \) of class I. If \( p \) is a Desarguesian point of the ambient polar space, then \( p \cap \mathcal{O} \) contains an oval.

Proof. Without loss we may assume \( p = \langle e_1 + e_2 + e_3 \rangle \), since \([1, 2, 3]\) is not a line of the Fano plane. Then the 1-spaces generated by the vectors \( e_4 \pm e_5 \pm e_7 \) form the points of an oval in \( p^\perp \cap \mathcal{O} \).
4. Proof of the Main Result

We begin our proof that no ovoid can exist in a polar space of type \( \Omega^+(10,3) \). Assume, by way of contradiction that \( \mathcal{O} \) is such an ovoid. For each point \( s \) of the quadric not on \( \mathcal{O} \), the set \( \mathcal{O}_s = (s^\perp \cap \mathcal{O}) + s/s \) is an ovoid of the space \( s^\perp/s \) of type \( \Omega^+(8,3) \). Thus \( \mathcal{O}_s \) is either of class I or class II. We let \( S^- \) be all singular 1-spaces \( s \) of the 10-dimensional quadric \( Q \) not on \( \mathcal{O} \) for which \( \mathcal{O}_s \) is class I. Similarly \( S^+ \) denotes the set of all points \( s \) of \( Q - \mathcal{O} \) for which \( \mathcal{O}_s \) is in class II. Replacing the quadratic form \( Q \) by \(-Q\) if necessary, we may assume that \( S^- \) is non-empty.

Let \( s \in S^- \). Then for a Desarguesian point \( \langle r, s \rangle/s \) of \( s^\perp/s \) with respect to the class I ovoid \( \mathcal{O}_s \), Lemma 1.4 says that \( \langle r, s \rangle/s \) is the radical of a 4-space generated by four ovoid points, \( \langle p_1, s \rangle/s, ..., \langle p_4, s \rangle/s \) forming an \( \mathcal{H}' \)-set of \( \mathcal{O}_s \). We may assume that \( \{p_1, ..., p_4\} \) is an \( \mathcal{H}' \)-set of \( \mathcal{O} \cap s^\perp \). We may also take \( r = \text{Rad}(p_1, ..., p_4) \).

Then \( \langle p_1, r \rangle/r, ..., \langle p_4, r \rangle/r \) is an \( \mathcal{H}' \)-set of \( \mathcal{O}_r \) forming an oval. It thus follows from Lemma 1.3(iii) that \( \mathcal{O}_r \) is a type II ovoid of \( \Omega^+(8,3) \) or \( \mathcal{O} \cap s^\perp \) and so \( r \) lies in \( S^+ \).

Now choose any 1-space \( s' \) of the isotropic 2-subspace \( \langle r, s \rangle \) of the \( \Omega^+(10,3) \)-space \( V \) so that \( s' \neq r \). Then \( \{\langle p_1, s' \rangle/s', ..., \langle p_4, s' \rangle/s'\} \) is an \( \mathcal{H}' \)-set of \( \mathcal{O}_{s'} \) which generates a 4-space with non-trivial radical \( \langle r, s' \rangle/s' \). Hence, by Lemma 1.3 (equivalence of (i) and (v)), \( s' \in S^- \).

Now by Lemma 1.5, since \( \langle r, s \rangle/s \) is a Desarguesian point of \( s^\perp/s \) with respect to \( \mathcal{O}_s \), there exists an oval \( \langle q_1, s \rangle/s, ..., \langle q_4, s \rangle/s \) in \( \mathcal{O}_s \cap \langle r, s \rangle/s \). We may choose the 1-spaces \( q_1, ..., q_4 \) so that they form an \( \mathcal{H} \)-set of \( \mathcal{O} \) lying in \( \mathcal{O} \cap \langle r, s \rangle^\perp \). Being an oval of \( s^\perp/s \) we know

\[
\text{Rad}(q_1, q_2, q_3, q_4) \subseteq s.
\]

But from the second paragraph above, if \( s' \) is a 1-subspace of \( \langle r, s \rangle \) distinct from \( r \), then \( s' \in S^- \). Then as \( \{q_1, ..., q_4\} \) is an \( \mathcal{H} \)-set, also \( \langle q_1, s' \rangle/s', ..., \langle q_4, s' \rangle/s' \) is an \( \mathcal{H} \)-set of \( \mathcal{O}_{s'} \). Since \( \mathcal{O}_{s'} \) is type I, this \( \mathcal{H} \)-set must form an oval there (Lemma 1.2). Hence

\[
\text{Rad}(q_1, ..., q_4) \subseteq s'.
\]

Taking \( s' \neq s \), we see

\[
\text{Rad}(q_1, ..., q_4) = 0,
\]

so \( \{q_1, ..., q_4\} \) is a bonafide oval of \( \mathcal{O} \). But then \( \langle q_1, r \rangle/r, ..., \langle q_4, r \rangle/r \) is an oval of \( \mathcal{O}_r \) whose elements form an \( \mathcal{H} \)-set. By Lemma 1.2 (equivalence of (i) and (iii)), we have \( r \in S^- \).

We now have a contradiction since \( r \) cannot belong to both \( S^- \) and \( S^+ \).
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