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Rook placements in Young diagrams and permutation enumeration

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ABSTRACT

Given two operators $\hat{D}$ and $\hat{E}$ subject to the relation $\hat{D}\hat{E} - q\hat{E}\hat{D} = p$, and a word w in $\hat{D}$ and $\hat{E}$, the rewriting of w in normal form is combinatorially described by rook placements in Young diagrams. We give enumerative results about these rook placements, particularly in the case where $p = (1-q)/q^2$. This case naturally arises in the context of the PASEP, a random process whose partition function and stationary distribution can be derived using two operators D and E subject to the relation $DE - qED = D + E$ (matrix Ansatz). Using the link obtained by Corteel and Williams between the PASEP, permutation tableaux and permutations, we prove a conjecture of Corteel and Rubey about permutation enumeration. This result gives the generating function for permutations of given size with respect to the number of ascents and occurrences of the pattern 13-2, this is also the moments of some q-Laguerre orthogonal polynomials.

1. Introduction

In recent work of Postnikov [14], permutations were linked with new objects which are pattern-avoiding fillings of Young diagrams. More precisely, he made a correspondence between positive Grassmann cells, these pattern-avoiding fillings called Ⅎ-diagrams, and decorated permutations (which are permutations with a weight 2 on each fixed point). In particular, the usual permutations are in bijection with permutation tableaux, a subset of Ⅎ-diagrams. Permutation tableaux have then been studied by Steingrímsson, Williams, Burstein, Corteel, Nadeau [2,6,7,16], and are quite useful in the combinatorics of permutations.

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Corteel and Williams observed, and explained, a rather surprising link between these permutation tableaux and the stationary distribution of a classical process of statistical physics, the Partially Asymmetric Self-Exclusion Process (PASEP). This model is described in [7,8]. More precisely, the stationary probability of a given state in the process is proportional to the sum of weights of permutation tableaux of a given shape. The factor behind this proportionality is the partition function, which is the sum of weights of permutation tableaux of a given half-perimeter.

Another way of computing the stationary distribution of the PASEP is the "matrix Ansatz" of Derrida et al. [8]. Suppose that we have operators D and E, a row vector $\langle W|$ and a column vector $|V\rangle$ such that

$$DE - qED = D + E, \qquad \langle W|E = \langle W|, \qquad D|V\rangle = |V\rangle, \quad \text{and} \quad \langle W||V\rangle = 1.$$

Then, coding any state of the process by a word w of length n in D and E, the stationary probability of the state w is given by $\langle W|w|V\rangle(\langle W|(D+E)^n|V\rangle)^{-1}$. This denominator $\langle W|(D+E)^n|V\rangle$ is the partition function.

We briefly describe how the matrix Ansatz is related to permutation tableaux [7]. First, notice that there are unique polynomials $n_{i,j} \in \mathbb{Z}[q]$ such that

$$(D+E)^n = \sum_{i,j\geqslant 0} n_{i,j} E^i D^j.$$

This sum is called the *normal form* of $(D+E)^n$. It is particularly useful, since for example the sum of the coefficients $n_{i,j}$ give an evaluation of $\langle W|(D+E)^n|V\rangle$. If D and E would commute, the expansion of $(D+E)^n$ would be described by binomial coefficients. But in this non-commutative context, the process of expanding and rewriting $(D+E)^n$ in normal form is combinatorially described by permutation tableaux. Then each coefficient $n_{i,j}$ is a generating function for permutation tableaux satisfying certain conditions. Equivalently this can be done with the *alternative tableaux* defined by Viennot [20].

One of the ideas at the origin of this article is the following. From D and E of the matrix Ansatz, we define new operators

$$\hat{D} = \frac{q-1}{q}D + \frac{1}{q} \qquad \text{and} \qquad \hat{E} = \frac{q-1}{q}E + \frac{1}{q}.$$

Some immediate consequences are

$$\hat{D}\hat{E} - q\hat{E}\hat{D} = \frac{1-q}{q^2}, \qquad \langle W|\hat{E} = \langle W|, \quad \text{and} \quad \hat{D}|V\rangle = |V\rangle. \tag{1}$$

This new commutation relation is in a way much more simple than the one satisfied by D and E. It is close to the relation between creation and annihilation operators classically studied in quantum physics. Moreover, from these definitions we have $q(y\hat{D}+\hat{E}) + (1-q)(yD+E) = 1+y$ for some parameter y. By isolating one term of the left-hand side and raising to the n with the binomial rule, we get the following inversion formulas between $(yD+E)^n$ and $(y\hat{D}+\hat{E})^n$:

$$(1-q)^n(yD+E)^n = \sum_{k=0}^{n} \binom{n}{k} (1+y)^{n-k}(-1)^k q^k (y\hat{D}+\hat{E})^k, \tag{2}$$

and

$$q^n(y\hat{D}+\hat{E})^n=\sum_{k=0}^{n}\binom{n}{k}(1+y)^{n-k}(-1)^k(1-q)^k(yD+E)^k. \tag{3}$$

In particular, the first formula means that if we want to compute the coefficients of the normal form of $(yD+E)^n$, it suffices to compute the ones of $(y\hat{D}+\hat{E})^n$ for all n. Notice that taking the normal form is a linear operation.

Up to a factor $-q$, these operators $\hat{D}$ and $\hat{E}$ are also defined in [18,1]. In the first reference, Uchiyama et al. use the new relation between $\hat{D}$ and $\hat{E}$ to find explicit matrix representations of these operators. They derive the eigenvalues and eigenvectors of $\hat{D}+\hat{E}$, and consequently the ones of $D+E$, in terms of orthogonal polynomials. In the second reference, Blythe et al. also use these eigenvalues and obtain an integral form for $\langle W|(D+E)^n|V\rangle$. They also provide an exact integral-free formula of this quantity, although quite complicated since it contains three sum signs and several q-binomial coefficients (however there expression is more general since contain other parameters).

In this article, instead of working on representations of $\hat{D}$ and $\hat{E}$ and their eigenvalues, we study the combinatorics of the rewriting in normal form of $(\hat{D}+\hat{E})^n$, and more generally $(y\hat{D}+\hat{E})^n$ for some parameter y. In the case of $\hat{D}$ and $\hat{E}$, the objects that appear are the *rook placements in Young diagrams*, long-known since the results of Kaplansky, Riordan, Goldman, Foata and Schützenberger (see [15] and the references therein). This method is described in [19], and is the same as the one leading to permutation tableaux or alternative tableaux in the case of D and E.

Definition 1.1. Let λ be a Young diagram. A *rook placement* of shape λ is a partial filling of the cells of λ with rooks (denoted by a circle $\circ$), such that there is at most one rook per row (resp. per column).

For convenience, we distinguish with a cross ($\times$) each cell of the Young diagram that is not below (in the same column) or to the left (in the same row) of a rook. See Fig. 3 further for an example. We will see that the number of crosses is an important statistic on rook placements. It was introduced in [9], as a generalization of the inversion number for permutations. Indeed, if λ is a square of side length n, a rook placements R with n rooks may be seen as the graph of a permutation $\sigma\in\mathfrak{S}_n$, and then the number of crosses in R is the inversion number of σ.

Definition 1.2. The weight of a rook placement R with r rooks and s crosses is $w(R)=p^rq^s$.

The enumeration of rook placements leads to an evaluation of $\langle W|(y\hat{D}+\hat{E})^{n-1}|V\rangle$, hence an evaluation of $\langle W|(yD+E)^{n-1}|V\rangle$ via the inversion formula (2). This is the main result of this article:

Theorem 1.3. *For any $n>0$, we have*

$$\langle W|(yD+E)^{n-1}|V\rangle=\frac{1}{y(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{n-k}y^j\left(\binom{n}{j}\binom{n}{j+k}-\binom{n}{j-1}\binom{n}{j+k+1}\right)\right)\times\left(\sum_{i=0}^{k}y^iq^{i(k+1-i)}\right).$$

The combinatorial interpretation of this polynomial, in terms of permutations, is given in Proposition 6.1. When $y=1$, this can be specialized to:

Theorem 1.4. *For any $n > 0$, we have*

$$\langle W|(D+E)^{n-1}|V\rangle = \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\binom{2n}{n-k}-\binom{2n}{n-k-2}\right)\left(\sum_{i=0}^{k} q^{i(k+1-i)}\right).$$

These two theorems were conjectured by Corteel and Rubey. The earliest conjecture, when $y=1$ and here stated as Theorem 1.4, was first proved by Rubey and Prellberg. The same method also can be used to give an alternative proof of our Theorem 1.3. This alternative proof, as well as the material of this article, is summarized in the extended abstract [5].

This alternative proof relies on a decomposition of weighted Motzkin paths, which gives a combinatorial explanation of the factor $\sum_{j=0}^{n-k} y^j\big(\binom{n}{j}\binom{n}{j+k}-\binom{n}{j-1}\binom{n}{j+k+1}\big)$. But on the other hand, the factor $\sum_{i=0}^{k} y^i q^{i(k+1-i)}$ is obtained by solving a functional equation and this is a completely non-combinatorial step. It may be possible to use the involution principle instead of a functional equation to obtain $\sum_{i=0}^{k} y^i q^{i(k+1-i)}$ but this is still an open problem at the time of writing.

We can see Theorem 1.4 as a variation of the Touchard–Riordan formula [17]. This classical formula gives the q-enumeration of fixed-point-free involutions of size $2n$ with respect to the number of crossings, and it is also the $2n$th moment of the q-Hermite polynomials. This formula is

$$\sum_{I\in \mathrm{Inv}(2n,0)} q^{\mathrm{cr}(I)} = \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)q^{\frac{k(k+1)}{2}}, \tag{4}$$

where $\mathrm{Inv}(2n,0)$ is the set of fixed-point-free involutions on $2n$ elements, and where the number of crossings $\mathrm{cr}(I)$ was defined in [10].

Besides references earlier mentioned, we have to point out the previous results of Williams [21], where Corollary 6.3 gives the coefficients of y^m in $\langle W|(yD+E)^n|V\rangle$. It was obtained by a more direct approach, via the enumeration of Ⅎ-diagrams, and was the only known polynomial formula for the distribution of a permutation pattern of length greater than 2 (see Proposition 6.1). Whereas Williams's work is rather focused on Ⅎ-diagrams, our results give more simple formulas in the case of permutation tableaux and permutations. Moreover Williams's formulas have also been obtained by Kasraoui, Stanton and Zeng in their work on orthogonal polynomials [11].

This article is organised as follows. In Section 2, we describe the link between rook placements and the rewriting of $(\hat{D}+\hat{E})^n$ in normal form. In Sections 3, 4, 5, we obtain enumerative results about rook placements, in particular Section 4 contains the bijective step of this enumeration. In Section 6, we use these results to prove Theorem 1.3, give the combinatorial interpretation of $\langle W|(yD+E)^n|V\rangle$ and some applications of the main theorem. In an Appendix A we give a combinatorial proof of Proposition 5.1, which gives a generalization of the Touchard–Riordan formula.

Notations and conventions

We denote by $\mathrm{Par}(n-k,k)$ the set of Young diagrams with exactly k rows and $n-k$ columns, allowing empty rows and columns. The integer n is the half-perimeter of the diagram $\lambda\in\mathrm{Par}(n-k,k)$, and we can see λ as an integer partition $(\lambda_1,\dots,\lambda_k)$ with $n-k \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0$. We use the French convention. We denote by $|\lambda|$ the number of cells in λ, which is also $\sum \lambda_i$.

The North–East boundary of $\lambda\in\mathrm{Par}(n-k,k)$ is a path of n steps, k of them being vertical and $n-k$ horizontal. Reciprocally, for any word w of length n in $\hat{D}$ and $\hat{E}$, with k occurrences of $\hat{E}$, we define $\lambda(w)\in\mathrm{Par}(n-k,k)$ by the following rule: we read w from left to right, and draw one step East for each factor $\hat{D}$, and one step South for each factor $\hat{E}$.

We denote by $\mathrm{Inv}(n,k)$ the set of involutions on $\{1,\dots,n\}$ with k fixed points.

We use the classical q-analogs of integers, factorials, and binomial coefficients, $[n]_q=\frac{1-q^n}{1-q}$, and $[n]_q! = \prod_{i=1}^{n}[i]_q$, and $\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}$. We recall [15] that $\begin{bmatrix} n\\ k\end{bmatrix}_q=\sum q^{|\lambda|}$ where we sum over $\lambda\in\mathrm{Par}(n-k,k)$, and that $q^{k(k+1)/2}\begin{bmatrix} n\\ k\end{bmatrix}_q=\sum q^{|\lambda|}$ where we sum over $\lambda\in\mathrm{Par}(n,k)$ with distinct parts.

Definition 1.5. For any $k, n \geqslant 0$, the Delannoy numbers are defined by

$$\left\{ n \atop k \right\} = \binom{n}{k} - \binom{n}{k-1}.$$

Proposition 1.6. *When $2k \leqslant n$, the number $\left\{ n \atop k \right\}$ counts the left factors of Dyck paths of n steps ending at height $n-2k$. In particular, $\left\{ 2n \atop n \right\}$ is the nth Catalan number. They satisfy the relations:*

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}, \qquad \left\{ n \atop n+1-k \right\} = -\left\{ n \atop k \right\},$$
$$\left\{ 0 \atop 0 \right\} = \left\{ 0 \atop 1 \right\} = 1, \quad \text{and} \quad \left\{ n \atop k \right\} = 0 \quad \text{if } k \notin \{0, \dots, n+1\}.$$

Proof. The number of left factors of Dyck paths of n steps ending at height $n-2k$ is easily seen to satisfy the same relations as $\left\{ n \atop k \right\}$: we just have to distinguish two cases whether the last step is going up or down. □

2. From operator relations to rook placements

In this section, we make the link between the coefficients of the normal form of $(\hat{D}+\hat{E})^n$, and rook placements in Young diagrams. This is done via a combinatorial description of the rewriting in normal form. When $q=1$, we can view it as a combinatorial statement of a classical result in statistical physics, called Wick's theorem. The principle of this method is the same as the one described in the introduction, making the link between D and E and permutation tableaux. Moreover the results of these section are presented in [19] in a slightly different form.

From now on we assume that $\hat{D}$ and $\hat{E}$ are such that $\hat{D}\hat{E} - q\hat{E}\hat{D} = p$ for some parameter p, which is a slight generalization of the relation (1). As in the case of D and E, any word w in $\hat{D}$ and $\hat{E}$ can be uniquely written in normal form:

$$w = \sum_{i,j \geqslant 0} c_{i,j}(w) \hat{E}^i \hat{D}^j,$$

where $c_{i,j}(w) \in \mathbb{Z}[p,q]$. We have

$$\langle W|w|V\rangle = \sum_{i,j \geqslant 0} c_{i,j}(w).$$

The combinatorial interpretation of this polynomial is given by the following proposition:

Proposition 2.1. *Let w be a word in $\hat{E}$ and $\hat{D}$. Then $\langle W|w|V\rangle$ is the sum of weights of rook placements of shape $\lambda(w)$.*

Proof. In the particular case where $p=1$, this is a consequence of [19, Theorem 6.3]. It is possible to adapt results in this reference to obtain this slightly more general case. □

Since $(\hat{D}+\hat{E})^n$ expands into the sum of all words of length n in $\hat{D}$ and $\hat{E}$, we also obtain:

Proposition 2.2. *For any n, $\langle W|(\hat{D}+\hat{E})^n|V\rangle$ is the sum of weights of rook placements of half-perimeter n.*

$T_{0,1,3} = pq + p + p,$ $\qquad T_{1,1,3} = 1 + q + q^2,$ $\qquad T_2 = 1 + (1 + q + p)y + y^2.$

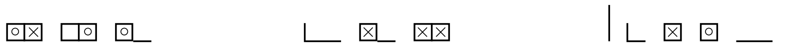

Fig. 1. Some small values of $T_{j,k,n}$ and T_n, together with the rook placements corresponding to each term.

We can also expand $(y\hat{D} + \hat{E})^n$ and get the sum over all words of length n in $\hat{D}$ and $\hat{E}$. But in this case each word w has a coefficient y^m, where m is the number of occurrences of $\hat{D}$ in w. Via the correspondence between words and Young diagrams, the number of occurrences of $\hat{D}$ in w is the number of columns in $\lambda(w)$. This leads to a refined version of the previous proposition.

Proposition 2.3. *For any n, $\langle W|(y\hat{D} + \hat{E})^n|V\rangle$ is the generating function for rook placements of half-perimeter n, the parameter y counting the number of columns.*

3. Basic results about rook placements

In this section we introduce the recurrence relation which will be used in the enumeration of rook placements, and we present two simple examples of enumeration. These two examples involve the q-binomial coefficients and the Delannoy numbers defined at the end of the introduction, and they introduce the more general formulas we will show later.

Definition 3.1. Let $T_{j,k,n}(p,q)$ be the sum of weights of rook placements of half-perimeter n, with k rows, and with j rows containing no rook (or equivalently, with $k-j$ rooks). We also define

$$T_{k,n}(p,q) = \sum_{j=0}^{k} T_{j,k,n}(p,q), \quad \text{and} \quad T_n(p,q,y) = \sum_{k=0}^{n} y^k T_{k,n}(p,q).$$

So $T_{k,n}(p,q)$ is the sum of weights of rook placements of half-perimeter n with k rows, and $T_n(p,q,y)$ is the generating function of rook placements of half-perimeter n, the parameter y counting the number of rows.

Since there is an obvious transposition-symmetry, we can also view the parameter y as counting the number of columns. These are polynomials in the variables p, q and y, so we will sometimes omit the arguments. From Proposition 2.3 we know that $T_n(p,q,y)$ is equal to $\langle W|(y\hat{D} + \hat{E})^n|V\rangle$. In Fig. 1 we give some examples of these polynomials.

Proposition 3.2. *We have the following recurrence relation*:

$$T_{j,k,n} = T_{j-1,k-1,n-1} + q^j T_{j,k,n-1} + p[j+1]_q T_{j+1,k,n-1}. \tag{5}$$

Proof. This is Remark 5.3 in [19] when $p = q = 1$. The proof of this particular case can be adapted. We distinguish three kinds of rook placements enumerated by $T_{j,k,n}$ (see Fig. 2):

- the first column is of size strictly less than k,
- the first column is of size k and contains no rook,
- or the first column is of size k and contains exactly one rook.

We show that these three types respectively lead to the three terms of the recurrence relation.

The first case is the situation where the first step of the North–East boundary is a step down, or equivalently the first row is of size 0. Removing this step (or row) is a bijection between these first-type rook placements, and the ones enumerated by $T_{j-1,k-1,n-1}$, the first term of (5).

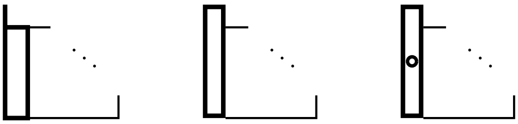

Fig. 2. The three kinds of rook placements we distinguish for proving Proposition 3.2.

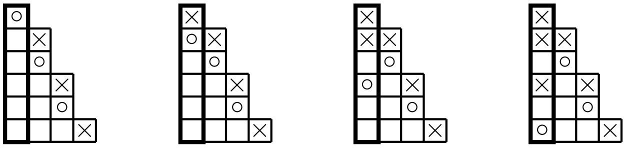

Fig. 3. We have here four rook placements of the third type, which are equal when we remove the first column. Here we have $n=10$, $k=6$, $j=3$, and the sum of their weights is $(p+pq+pq^2+pq^3)p^2q^3 = p[j+1]_q(p^2q^3)$. This illustrates the third term of (5).

In the second case, the first column contains exactly j crosses, one per row without rook. So removing the first column is a bijection between the second-type rook placements, and the ones enumerated by $T_{j,k,n-1}$, and this bijection changes the weight by a factor q^j. This explains the second term of (5).

In the third case, removing the first column is not a bijection since there are several possibilities for the position of the rook in this column. But this map has the property that for any R enumerated by $T_{j+1,k,n-1}$, the preimage set of R contains $j+1$ elements, and their weights are $pw(R)$, $pqw(R), \dots, pq^jw(R)$. See Fig. 3 for an example. This shows that the sum of weights of the third-type rook placements is the third term of (5), and completes the proof. □

Proposition 3.3. *For any k, n, $T_{k,k,n}$ is the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Proof. We are counting rook placements without any rook, *i.e.* such that all cells contain a cross. So this follows from the interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ in terms of partitions. □

This proposition is illustrated for example in Fig. 1 where we see that $T_{1,1,3} = 1+q+q^2 = [3]_q$. The second example of this section is more subtle and we begin with the following lemma.

Lemma 3.4. *Given a Young diagram λ, the number of rook placements of shape λ having no cross and exactly one rook per row is either* 0 *or* 1. *It is* 1 *in the case where the North–East boundary is a Dyck path* (*which means that the ith row of λ starting from the top contains at least i cells, for any i between* 1 *and the number of rows*)*.*

Proof. Suppose that R is a rook placement with no cross and exactly one rook per row. Then the i first rows contain i rooks, which are necessarly in i different columns. So the ith row contains at least i cells. This is true for any i, so the North–East boundary is a Dyck path.

It remains to prove that there is a unique such rook placement in the case where the North–East boundary of a Young diagram λ is a Dyck path. We show that there is only one way to build this rook placement starting from an empty diagram λ. First, notice that each corner of the diagram must contain a rook (as we saw in previous section, the general statement is that each corner contains either a rook or a cross). Then, if we consider the subdiagram of cells that are not in the same row or column of these rooks (see Fig. 4), again all corners of this subdiagram must contain a rook by the same argument. We can even say that his North–East boundary is also a Dyck path: indeed, the boundary of the subdiagram is obtained from the boundary of the diagram by removing each occurrence of a step right followed by a step down. So we can conclude by recurrence. □

Fig. 4. Example of Young diagram whose North–East boundary is a Dyck path, *i.e.* doesn't go below the dotted line. The number of rows is $k=6$, the half-perimeter is $n=14$, and the path ends at height $n-2k=2$.

Proposition 3.5. *If* $2k<n$, *we have* $T_{0,k,n}(p,0)=p^k\left\{ {n \atop k} \right\}$.

Proof. We are counting rook placements with no cross (since $q=0$ here) and exactly k rooks. Each of these rook placements has weight p^k, so we just have to prove that there are $\left\{ {n \atop k} \right\}$ such rook placements. Knowing that $\left\{ {n \atop k} \right\}$ is the number of left factors of Dyck paths of n steps ending at height $n-2k$, this is a consequence of the previous lemma. □

4. The factorization property

In this section we use a factorization property from [19]. Indeed, the recurrence (5) is rather complicated to be solved directly. But the factorization property is a simple relation between $T_{j,k,n}$ and $T_{0,k-j,n}$, and we derive a recurrence relation satisfied by $T_{0,k,n}$.

Indeed, by using a bijection with involutions and examining separately the contribution of rows and columns without rooks, it is possible to prove the following.

Proposition 4.1. *For any* j,k,n, *we have*

$$T_{j,k,n}=\begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q T_{0,k-j,n}. \tag{6}$$

Proof. The main reference is Theorem 6.6 from [19], or more precisely, some ideas in the proof of it. We will not give details about this, to avoid being too long on a subject that has already been treated elsewhere. □

Thanks to this factorisation property of $T_{j,k,n}$, our problem is reduced to the evaluation of $T_{0,k,n}$. But this factorisation property also gives a recurrence relation satisfied by $T_{0,k,n}$.

Corollary 4.2. *We have the following recurrence relation*:

$$T_{0,k,n}=T_{0,k,n-1}+p[n+1-2k]_q T_{0,k-1,n-1}. \tag{7}$$

Proof. When $j=0$, the relation (5) gives $T_{0,k,n}=T_{0,k,n-1}+pT_{1,k,n-1}$. Applying the previous corollary to the second term of this sum gives the desired equality. □

5. Enumeration of rook placements

In this section we solve the recurrence (7), and we obtain an expression for $T_{0,k,n}$ involving both q-binomials and Delannoy numbers, generalizing the two examples of Section 3. Using the factorisation property of $T_{j,k,n}$ and summing over j, we obtain an expression for

$$T_{k,n}=\sum_{j=0}^{k} T_{j,k,n},$$

i.e. for the sum of weights of rook placements of half-perimeter n with k rows. This expression is rather lengthy, with a sum over three indices, but for certain values of p we can simplify it with the q-binomial identities of Lemma 5.2. So in these particular specializations we get expressions for $T_{k,n}$ and T_n without q-binomials.

Proposition 5.1. *When $p = 1-q$, we have*

$$T_{0,k,n}(1-q,q) = \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \begin{Bmatrix} n \\ k-i \end{Bmatrix}. \tag{8}$$

Proof. We give a recursive proof. In Appendix A we give an alternative proof, which is much more combinatorial.

Let us denote by $f(k,n)$ the right-hand side of (8). The initial condition is $f(k,0) = T_{0,k,0} = \delta_{0k}$ so it remains to check relation (7) when $p = 1-q$. Let us define

$$A = \begin{bmatrix} n-1-2k+i \\ i \end{bmatrix}_q, \qquad B = q^{n-2k} \begin{bmatrix} n-1-2k+i \\ i-1 \end{bmatrix}_q,$$
$$C = \begin{Bmatrix} n-1 \\ k-i \end{Bmatrix}, \qquad D = \begin{Bmatrix} n-1 \\ k-i-1 \end{Bmatrix},$$

so that we have

$$f(k,n) = \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} (A+B)(C+D) = \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \big(AC + BC + (A+B)D\big).$$

After expanding this sum, the second term gives

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} BC = -\sum_{i=0}^{k-1} (-1)^i q^{\frac{(i+1)(i+2)}{2}} q^{n-2k} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \begin{Bmatrix} n-1 \\ k-i-1 \end{Bmatrix},$$

where the sum is reindexed such that i becomes $i+1$. And the third term gives

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} (A+B)D = \sum_{i=0}^{k-1} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \begin{Bmatrix} n-1 \\ k-i-1 \end{Bmatrix},$$

after noticing that the term where $i = k$ is 0. Adding the previous two identities yields

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} (BC + AD + BD)$$
$$= \sum_{i=0}^{k-1} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \begin{Bmatrix} n-1 \\ k-i-1 \end{Bmatrix} \big(1 - q^{n-2k+i+1}\big).$$

But we have $[n-2k+i+1]_q \left[\begin{smallmatrix} n-2k+i \\ i \end{smallmatrix}\right]_q = [n-2k+1]_q \left[\begin{smallmatrix} n-2k+i+1 \\ i \end{smallmatrix}\right]_q$, hence

$$\begin{aligned}
&\sum_{i=0}^{k}(-1)^i q^{\frac{i(i+1)}{2}}(BC+AD+BD)\\
&\quad=\sum_{i=0}^{k-1}(-1)^i q^{\frac{i(i+1)}{2}}\begin{bmatrix} n-2k+i+1\\ i\end{bmatrix}_q \left\{\begin{matrix} n-1\\ k-i-1\end{matrix}\right\}\left(1-q^{n-2k+1}\right)\\
&\quad=\left(1-q^{n-2k+1}\right)f(k-1,n-1).
\end{aligned}$$

Since $\sum_{i=0}^{k}(-1)^i q^{\frac{i(i+1)}{2}}AC$ readily gives $f(k,n-1)$, we get the relation

$$f(k,n)=f(k,n-1)+\left(1-q^{n-2k+1}\right)f(k-1,n-1),$$

which is precisely (7) when $p=1-q$. □

Remark. The rook placements enumerated by $T_{0,k,n}$ contain exactly k rooks, so $T_{0,k,n}(p,q)=p^k T_{0,k,n}(1,q)$. This shows that there is no loss of generality in the assumption $p=1-q$ of the previous proposition. Moreover, the Touchard–Riordan formula (4) mentioned in the introduction is a particular case of (8). Indeed, involutions without fixed points correspond to rook placements with exactly one rook per row and one rook per column (therefore with as many rows as columns). So knowing (8), we directly obtain (4):

$$\begin{aligned}
\sum_{I\in \mathrm{Inv}(2n,0)} q^{\mathrm{cr}(I)} &= T_{0,n,2n}(1,q)=\frac{1}{(1-q)^n}T_{0,n,2n}(1-q,q)\\
&=\frac{1}{(1-q)^n}\sum_{i=0}^{n}(-1)^i\left\{\begin{matrix} 2n\\ n-i\end{matrix}\right\}q^{\frac{i(i+1)}{2}}.
\end{aligned}$$

Now using (6) and (8), we have the following equality:

$$T_{j,k,n}(1-q,q)=\begin{bmatrix} n-2k+2j\\ j\end{bmatrix}_q\sum_{i=0}^{k-j}(-1)^i q^{\frac{i(i+1)}{2}}\begin{bmatrix} n-2k+2j+i\\ i\end{bmatrix}_q\left\{\begin{matrix} n\\ k-j-i\end{matrix}\right\}. \tag{9}$$

And as in the previous remark, $T_{j,k,n}(p,q)=p^{k-j}T_{j,k,n}(1,q)$ so that we have a similar expression for any value of p. Summing it over j will give an expression for $T_{k,n}(p,q)$. For certain values of p, it is possible to simplify this sum, and we need the following lemma:

Lemma 5.2. *For any $k,n\geqslant 0$ we have the following q-binomial identities*:

$$\sum_{j=0}^{k}(-1)^j q^{\frac{j(j+1)}{2}}\begin{bmatrix} n-j\\ n-k\end{bmatrix}_q\begin{bmatrix} n-k\\ j\end{bmatrix}_q=1, \tag{10}$$

$$\sum_{j=0}^{k}(-1)^j q^{\frac{j(j-1)}{2}}\begin{bmatrix} n-j\\ n-k\end{bmatrix}_q\begin{bmatrix} n-k\\ j\end{bmatrix}_q=q^{k(n-k)}, \tag{11}$$

$$\sum_{j=0}^{k}(-1)^j q^{\frac{(j-1)(j-2)}{2}}\begin{bmatrix} n-j\\ n-k\end{bmatrix}_q\begin{bmatrix} n-k\\ j\end{bmatrix}_q=\frac{q^{(k+1)(n-k)}-q^{k(n-k)}+q^{k(n+1-k)}-q^{(k+1)(n+1-k)}}{q^{n-1}(1-q)}. \tag{12}$$

Proof. The first two are proved combinatorially. We first prove (11), because it is slightly more simple. It seems that there is no simple combinatorial proof of (12) so we prove it with a recurrence, which is quite similar to the one of Proposition 5.1.

- The left-hand side of (11) counts the pairs $(\lambda, \mu) \in \mathrm{Par}(n-k, k-j) \times \mathrm{Par}(n-k-1, j)$, for some j between 0 and k, signed by $(-1)^j$ and such that μ has distinct parts. More precisely, λ is such that $n-k \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{k-j} \geqslant 0$ and μ is such that $n-k > \mu_1 > \cdots > \mu_j \geqslant 0$. When $k-j > 0$, such a couple (λ, μ) satisfying $\lambda_{k-j} < \mu_j$ or $\mu = (\emptyset)$ can be paired with the couple (λ', μ') such that
$$\lambda' = (\lambda_1, \ldots, \lambda_{k-j-1}), \qquad \mu' = (\mu_1, \ldots, \mu_j, \lambda_{k-j}).$$
This couple satisfies $|\lambda| + |\mu| = |\lambda'| + |\mu'|$ but it has opposite sign. The only couple which is not paired with any other is such that $\lambda_1 = \cdots = \lambda_k = n-k$ and $\mu = (\emptyset)$, it contributes to the sum with a $q^{k(n-k)}$.
- The proof of (10) is quite similar. Here the factor $q^{j(j+1)/2}$ means that we count pairs (λ, μ) as before but such that $n-k \geqslant \mu_1 > \cdots > \mu_j > 0$ (because $j(j+1)/2 = 1 + \cdots + j$). Now the pairing is done by comparing the smallest non-zero part of λ with the smallest part of μ. Depending on the situation, one of these parts is moved from λ to μ, or from μ to λ. The only couple (λ, μ) which is not paired with any other is such that $\lambda_1 = \cdots = \lambda_k = 0$ and $\mu = (\emptyset)$, and it contributes to the sum with a 1.
- When $k = 0$, both sides of (12) are equal to q. Let us denote by $g(n,k)$ the left-hand side of (12). We define
$$A = q^{n-k} \begin{bmatrix} n-j \\ n-k \end{bmatrix}_q, \qquad B = \begin{bmatrix} n-j \\ n-k-1 \end{bmatrix}_q,$$
$$C = \begin{bmatrix} n-k-1 \\ j \end{bmatrix}_q, \qquad D = q^{n-k-j} \begin{bmatrix} n-k-1 \\ j-1 \end{bmatrix}_q,$$
so that $g(n+1, k+1) = \sum_{j=0}^{k+1} (-1)^j q^{(j-1)(j-2)/2} (A+B)(C+D)$. After expanding this product, we get the recurrence relation
$$g(n+1, k+1) = q^{n-k} g(n,k) + g(n, k+1) - q^{n-k} g(n-1, k).$$
In view of the simple expression of the right-hand side of (12), it is straightforward to check that it satisfies the same relation. □

Remark. A referee kindly pointed out that the first two identities of the previous proposition are equivalent via the transformation $q \mapsto 1/q$, moreover both are a special case of the q-Chu–Vandermonde formula.

Proposition 5.3.

$$T_{k,n}(1-q, q) = \binom{n}{k}, \qquad T_{k,n}\left(\frac{1-q}{q}, q\right) = \sum_{j=0}^{k} \left\{ {n \atop j} \right\} q^{(k-j)(n-k-j)-j}, \tag{13}$$

$$T_{k,n}\left(\frac{1-q}{q^2},q\right) = \sum_{j=0}^{k}\left\{ n \atop j \right\}\left(\frac{q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)}}{(1-q)q^n}\right). \tag{14}$$

Proof. The three identities of this proposition respectively come from (10), (11) and (12). We prove only the last one, because it is the most important case. The two others are proved similarly but more simply. Multiplying (9) by q^{2j-2k} and summing over j gives

$$\begin{aligned} T_{k,n}\left(\frac{1-q}{q^2},q\right) &= \sum_{j=0}^{k} q^{2j-2k}\begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q \sum_{i=0}^{k-j}(-1)^i q^{\frac{i(i+1)}{2}}\begin{bmatrix} n-2k+2j+i \\ i \end{bmatrix}_q \left\{ n \atop k-j-i \right\} \\ &= \sum_{\substack{0\leqslant i,j \\ i+j\leqslant k}} \left\{ n \atop k-j-i \right\} q^{2j-2k}\begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q (-1)^i q^{\frac{i(i+1)}{2}}\begin{bmatrix} n-2k+2j+i \\ i \end{bmatrix}_q. \end{aligned}$$

Introducing $l = k-j-i$, we get

$$T_{k,n}\left(\frac{1-q}{q^2},q\right) = \sum_{l=0}^{k}\left\{ n \atop l \right\}\sum_{j=0}^{k-l} q^{2j-2k}\begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q (-1)^{k-j-l} q^{\frac{(k-j-l)(k-j-l+1)}{2}}\begin{bmatrix} n-k+j-l \\ k-j-l \end{bmatrix}_q,$$

and after replacing j with $k-l-j$ we also have

$$\begin{aligned} T_{k,n}\left(\frac{1-q}{q^2},q\right) &= \sum_{l=0}^{k}\left\{ n \atop l \right\}\sum_{j=0}^{k-l} q^{-2j-2l}\begin{bmatrix} n-2l-2j \\ k-l-j \end{bmatrix}_q (-1)^j q^{\frac{j(j+1)}{2}}\begin{bmatrix} n-2l-j \\ j \end{bmatrix}_q \\ &= \sum_{l=0}^{k}\left\{ n \atop l \right\}\sum_{j=0}^{k-l}(-1)^j q^{\frac{(j-1)(j-2)}{2}-1-2l}\frac{[n-2l-j]_q!}{[j]_q![k-l-j]_q![n-k-l-j]_q!} \\ &= \sum_{l=0}^{k}\left\{ n \atop l \right\} q^{-1-2l}\sum_{j=0}^{k-l}(-1)^j q^{\frac{(j-1)(j-2)}{2}}\begin{bmatrix} n-2l-j \\ n-l-k \end{bmatrix}_q\begin{bmatrix} n-l-k \\ j \end{bmatrix}_q. \end{aligned}$$

At this point we can apply (11) with $n' = n-2l$ and $k' = k-l$, and get (14). □

Remark. By an obvious argument of symmetry by transposition, we have $T_{k,n} = T_{n-k,n}$, and this can be directly seen in (14). The summand $q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)}$ is unchanged when j is replaced with $n+1-j$. Besides, we have $\left\{ n \atop j \right\} = -\left\{ n \atop n+1-j \right\}$, so

$$\sum_{j=k+1}^{n-k}\left\{ n \atop j \right\}\left(\frac{q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)}}{(1-q)q^n}\right) = 0.$$

A consequence is that in (14) instead of summing over j between 0 and k, we can sum over j between 0 and $\min(k, n-k)$. This is also true for the second identity of (13). In this form, it is clear that $T_{k,n} = T_{n-k,n}$.

The last step of this section is the summing over k to get an expression for $T_n(\frac{1-q}{q^2}, q, y)$.

Proposition 5.4.

$$(1-q)q^n T_n\left(\frac{1-q}{q^2}, q, y\right) = (1+y)G(n) - G(n+1),$$

$$\textit{where } G(n) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left\{ {n \atop j} \right\} \sum_{i=0}^{n-2j} y^{i+j-1} q^{i(n+1-2j-i)}. \tag{15}$$

Proof. First we define $P_k = \sum_{i=0}^{k} y^i q^{i(k+1-i)}$. We have to multiply (14) by y^k, and sum over k between 0 and n. This gives

$$\begin{aligned}
&(1-q)q^n T_n\left(\frac{1-q}{q^2}, q, y\right) \\
&\quad = \sum_{0 \leqslant j \leqslant k \leqslant n} y^k \left\{ {n \atop j} \right\} \left(q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)}\right) \\
&\quad = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} \left(\sum_{k=j}^{n} y^k q^{(k+1-j)(n-k-j)} - \sum_{k=j}^{n} y^k q^{(k-j)(n-k-j)} \right. \\
&\qquad \left. + \sum_{k=j}^{n} y^k q^{(k-j)(n+1-k-j)} - \sum_{k=j}^{n} y^k q^{(k+1-j)(n+1-k-j)} \right) \\
&\quad = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} \left(\sum_{i=1}^{n+1-j} y^{i+j-1} q^{i(n+1-2j-i)} - \sum_{i=0}^{n-j} y^{i+j} q^{i(n-2j-i)} \right. \\
&\qquad \left. + \sum_{i=0}^{n-j} y^{i+j} q^{i(n+1-2j-i)} - \sum_{i=1}^{n+1-j} y^{i+j-1} q^{i(n+2-2j-i)} \right),
\end{aligned}$$

after a reindexing of the second and third sums with $i = k - j$, and of the first and fourth sums with $i = k + 1 - j$. Since $(1-q)q^n T_n$ is a polynomial, we can discard all negative powers of q appearing in these sums. Modulo non-positive powers of q, these four sums are respectively equal to $y^{j-1} P_{n-2j}$, $y^j P_{n-1-2j}$, $y^j P_{n-2j}$, $y^{j-1} P_{n+1-2j}$. But we have to be careful when it comes to the constant terms in q. These constant terms are respectively:

$$[q^0] \sum_{i=1}^{n+1-j} y^{i+j-1} q^{i(n+1-2j-i)} = y^{n-j} \chi_{\{1 \leqslant n+1-2j \leqslant n-j+1\}},$$

$$[q^0] \sum_{i=0}^{n-j} y^{i+j} q^{i(n-2j-i)} = 1 + y^{n-j} \chi_{\{0 \leqslant n-2j \leqslant n-j\}},$$

$$[q^0] \sum_{i=0}^{n-j} y^{i+j} q^{i(n+1-2j-i)} = 1 + y^{n+1-j} \chi_{\{0 \leqslant n+1-2j \leqslant n-j\}},$$

$$[q^0]\sum_{i=1}^{n+1-j} y^{i+j-1}q^{i(n+2-2j-i)} = y^{n+1-j}\chi_{\{1\leqslant n+2-2j\leqslant n+1-j\}},$$

where χ_P is either 0 or 1 whether the property P is false or true. We see that these constant terms in q actually cancel two-by-two, so that it remains

$$\begin{aligned}(1-q)q^nT_n\left(\frac{1-q}{q^2},q,y\right) &= \sum_{j=0}^{n}\left\{ {n \atop j} \right\}\left(\left(y^j+y^{j-1}\right)P_{n-2j}-y^jP_{n-1-2j}-y^{j-1}P_{n+1-2j}\right)\\ &= (1+y)\sum_{j=0}^{n}\left\{ {n \atop j} \right\}y^{j-1}P_{n-2j}-\sum_{j=0}^{n+1}\left(\left\{ {n \atop j-1} \right\}+\left\{ {n \atop j} \right\}\right)y^{j-1}P_{n+1-2j}\\ &= (1+y)G(n)-G(n+1),\quad \text{where } G(n)=\sum_{j=0}^{n}\left\{ {n \atop j} \right\}y^{j-1}P_{n-2j}.\end{aligned}$$

Since the polynomial P_{n-2j} is zero when $n-2j<0$, we can sum over j between 0 and $\lfloor n/2\rfloor$ in the definition of $G(n)$, so that we get (15). □

6. Application to permutation enumeration

In the previous section we have computed T_n, which is also equal to $\langle W|(y\hat{D}+\hat{E})^n|V\rangle$ thanks to the results of Section 2. Now, using the inversion formula (2), we can compute $\langle W|(yD+E)^n|V\rangle$ and prove Theorem 1.3. At the beginning of this section we describe the combinatorial interpretation of this polynomial in terms of permutations and permutation tableaux. Then we prove Theorem 1.3 and Theorem 1.4, and give some applications.

Proposition 6.1. *(See [4,6,7,11,16,20].) For any $n \geqslant 1$ the following polynomials are equal:*

- $\langle W|y(yD+E)^{n-1}|V\rangle$,
- *the generating function for permutation tableaux of size n, the number of lines counted by y and the number of superfluous 1's counted by q,*
- *the generating function for permutations of size n, the number of ascents plus 1 counted by y and the occurrences of the pattern 13-2 counted by q,*
- *the generating function for permutations of size n, the number of weak exceedances counted by y and the number of crossings counted by q,*
- *the nth moment of the q-Laguerre polynomials.*

Proof. All this material is present in the references. See also the references for definitions. In particular there are several possible definitions for the q-Laguerre polynomials: the one we mention is defined as a rescaled version of the Al-Salam-Chihara polynomials as in [11]. We recall that the nth moment of these q-Laguerre polynomials is the sum of weights of *Laguerre histories* of n steps. This is also present in [4].

Definition 6.2. A Laguerre history is a weighted Motzkin path such that:

- the weight of a horizontal step at height h is q^i for some $i \in \{0,\dots,h-1\}$ or yq^i for some $i \in \{0,\dots,h\}$,
- the weight of a North–East step starting at height h is q^i for some $i \in \{0,\dots,h\}$,
- the weight of a South–East step starting at height h is yq^i for some $i \in \{0,\dots,h-1\}$.

The bijections between permutations and Laguerre histories, namely the Françon–Viennot and Foata–Zeilberger bijections (see [4]), give the equality of the last three items in the list of Proposition 6.1.

The link between the operators D and E of the matrix Ansatz and the permutation tableaux was given by Corteel and Williams in [7]. This shows the equality of the first two items in the list. See also [20].

To end this proof we can use a bijection between permutation tableaux and permutations from [6]: the number of columns in permutation tableaux corresponds to the number of ascents in permutations, and the number of superfluous 1's corresponds to the number of occurrences of the pattern 13-2. We also have to mention the previous results of Postnikov, who made the link between Ⅎ-diagrams, which generalize the permutation tableaux, and alignments in decorated permutations [14,21]. □

We now give the formula for the polynomials of Proposition 6.1. This is Theorem 1.3 stated in the introduction.

Theorem 6.3. *For any $n \geqslant 1$, we have*

$$\langle W|(yD+E)^{n-1}|V\rangle = \frac{1}{y(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{n-k} y^j\left(\binom{n}{j}\binom{n}{j+k}-\binom{n}{j-1}\binom{n}{j+k+1}\right)\right) \times\left(\sum_{i=0}^{k} y^i q^{i(k+1-i)}\right).$$

Proof. Using the main result of the previous section (15) and the inversion formula (2), we obtain

$$\begin{aligned}
&\langle W|(1-q)^n(yD+E)^{n-1}|V\rangle \\
&= (1-q)\sum_{k=0}^{n-1}\binom{n-1}{k}(1+y)^{n-1-k}(-q)^k\langle W|(y\hat{D}+\hat{E})^k|V\rangle \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}(1+y)^{n-1-k}(-1)^k\big((1+y)G(k)-G(k+1)\big) \\
&= \sum_{k=0}^{n}\binom{n}{k}(1+y)^{n-k}(-1)^k G(k) \\
&= \sum_{\substack{0\leqslant i\leqslant k\leqslant n\\ i\equiv k \bmod 2}}\binom{n}{k}(1+y)^{n-k}(-1)^k\left\{\begin{matrix}k\\ \frac{k-i}{2}\end{matrix}\right\}y^{(k-i)/2-1}P_i \\
&= \frac{1}{y}\sum_{i=0}^{n}(-1)^i\left(\sum_{k=0}^{\lfloor\frac{n-i}{2}\rfloor}\binom{n}{2k+i}(1+y)^{n-2k-i}\left\{\begin{matrix}2k+i\\ k\end{matrix}\right\}y^k\right)P_i,
\end{aligned}$$

after a reindexing such that k becomes $2k+i$. It remains to simplify the sum between parentheses. After expanding the power of $1+y$, this sum is

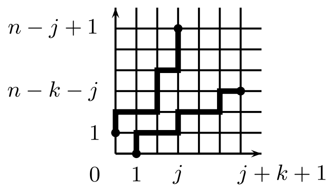

Fig. 5. Interpretation of a determinant of binomials in terms of lattice paths. In this example, we have $n=8$, $j=3$, $k=2$.

$$\begin{aligned}
&\sum_{k=0}^{\lfloor\frac{n-i}{2}\rfloor}\sum_{j=0}^{n-2k-i}\binom{n}{2k+i}\binom{n-2k-i}{j}\left\{ {2k+i \atop k} \right\} y^{k+j}\\
&\quad=\sum_{0\leqslant k,j}\frac{n!}{j!(n-2k-i-j)!}\left(\frac{1}{k!(k+i)!}-\frac{1}{(k-1)!(k+i+1)!}\right)y^{k+j}\\
&\quad=\sum_{0\leqslant k\leqslant m}\frac{n!}{(m-k)!(n-m-k-i)!}\left(\frac{1}{k!(k+i)!}-\frac{1}{(k-1)!(k+i+1)!}\right)y^{m}\\
&\quad=\sum_{m=0}^{n-i}y^m\left(\binom{n}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{n-m}{k+i}-\binom{n}{m-1}\sum_{k=0}^{m}\binom{m-1}{k-1}\binom{n-m+1}{k+i+1}\right).
\end{aligned}$$

But thanks to the Vandermonde identity, the two sums over k may be simplified: $\sum_{k=0}^{m}\binom{m}{k}\binom{n-m}{k+i}=\binom{n}{m+i}$, and $\sum_{k=0}^{m}\binom{m-1}{k-1}\binom{n-m+1}{k+i+1}=\binom{n}{m+i+1}$, and this completes the proof. □

Remark. The number $\binom{n}{j}\binom{n}{j+k}-\binom{n}{j-1}\binom{n}{j+k+1}$ may be seen as the determinant of a (2×2)-matrix of binomial coefficients. The Lindström–Gessel–Viennot lemma gives a combinatorial interpretation of this quantity in terms of lattice paths: it is the number of pairs of non-intersecting paths with starting points $(1,0)$ and $(0,1)$, with end points $(j, n-j+1)$ and $(j+k+1, n-k-j)$, and only with unit steps going North or East, as in Fig. 5. In particular when $k=0$, this is the Narayana number $N(n+1, j+1)$.

Proposition 6.4. *The coefficient of y^m in $\langle W|(yD+E)^{n-1}|V\rangle$ is given by:*

$$\begin{aligned}
\big[y^m\big]\langle W|(yD+E)^{n-1}|V\rangle &= \frac{1}{(1-q)^n}\sum_{k=0}^{n}\sum_{j=m-k}^{m}(-1)^k q^{(m-j)(k+j+1-m)}\\
&\quad\times\left(\binom{n}{j}\binom{n}{j+k}-\binom{n}{j-1}\binom{n}{j+k+1}\right).
\end{aligned}$$

Proof. We just have to expand the products in the equality of Theorem 1.3, since each of the factors between parentheses is a polynomial in y and their coefficients are explicit. □

In [21], Williams provides a different formula for the same polynomial, indeed $[y^k]\langle W|(yD+E)^{n-1}|V\rangle$ is also equal to

$$\sum_{i=0}^{k-1}(-1)^i[k-i]^n q^{ki-k^2}\left(\binom{n}{i}q^{k-i}+\binom{n}{i-1}\right).$$

She shows that this polynomial is a q-analog of Eulerian numbers that interpolates between Narayana number (when $q=0$), binomial coefficients (when $q=-1$), and of course Eulerian numbers (when $q=1$).

We can also obtain these results from the expression of Proposition 6.4. For example, if we set $q=0$ in the previous equality, it gives that the number of permutations avoiding the pattern 13-2 and with m ascents is

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\left(\binom{n}{m}\binom{n}{m+k}-\binom{n}{m-1}\binom{n}{m+k+1}\right)\\
&\quad=\binom{n}{m}^2+\sum_{k=1}^{n}(-1)^k\binom{n}{m}\binom{n}{m+k}+\sum_{k=1}^{n+1}(-1)^k\binom{n}{m-1}\binom{n}{m+k}\\
&\quad=\binom{n}{m}^2+\sum_{k=1}^{n}(-1)^k\binom{n+1}{m}\binom{n}{m+k}=\binom{n}{m}^2-\binom{n+1}{m}\sum_{k=0}^{m}(-1)^{k+m}\binom{n}{k}.
\end{aligned}$$

This alternating sum of binomials is also the binomial $\binom{n-1}{m}$. So the number we get is $\binom{n}{m}^2-\binom{n+1}{m}\binom{n-1}{m}$. Although it is not the most common way to define it, this number is the Narayana number $N(n,m)$, as can be combinatorially seen using again the Lindström–Gessel–Viennot lemma.

We now give the specialization when $y=1$. This is Theorem 1.4 stated in the introduction.

Theorem 6.5. *For any $n \geqslant 1$, we have*

$$\langle W|(D+E)^{n-1}|V\rangle=\frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\binom{2n}{n-k}-\binom{2n}{n-k-2}\right)\left(\sum_{i=0}^{k}q^{i(k+1-i)}\right). \tag{16}$$

Proof. We just have to substitute $y=1$ into the equality of Theorem 1.3. We can simplify the resulting expression using again the Vandermonde identity, indeed we have $\sum_{j=0}^{n-k}\binom{n}{j}\binom{n}{j+k}=\binom{2n}{n-k}$, and $\sum_{j=0}^{n-k}\binom{n}{j-1}\binom{n}{j+k+1}=\binom{2n}{n-k-2}$, and the result follows. □

Among the several objects of the list in Proposition 6.1, the most studied are probably permutations and the pattern 13-2, see for example [3,6,16,12]. In particular in [3,12] we can find methods for obtaining, as a function of n for a given k, the number of permutations of size n with exactly k occurrences of the pattern 13-2. By taking the Taylor series of (1.4), we obtain direct and quick proofs for these previous results. As an illustration we give the formulas for $k \leqslant 3$ in the following proposition.

Proposition 6.6. *The order-3 Taylor series of $\langle W|(D+E)^{n-1}|V\rangle$ is*

$$\langle W|(D+E)^{n-1}|V\rangle=C_n+\binom{2n}{n-3}q+\frac{n}{2}\binom{2n}{n-4}q^2+\frac{(n+1)(n+2)}{6}\binom{2n}{n-5}q^3+O\left(q^4\right),$$

where C_n is the nth Catalan number.

Proof. On one side, we have $(1-q)^{-n} = 1 + nq + \binom{n+1}{2}q^2 + \binom{n+2}{3}q^3 + O(q^4)$. On the other side, we have $\sum_{i=0}^{k} q^{i(k+1-i)} = 1 + q\delta_{1k} + 2q^2\delta_{2k} + 2q^3\delta_{3k} + O(q^4)$. The constant term is

$$\sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-2}\right) = \binom{2n}{n} - \binom{2n}{n-1} = C_n.$$

So this Taylor series is

$$\left(1 + nq + \binom{n+1}{2}q^2 + \binom{n+2}{3}q^3\right)\left(C_n - \left(\binom{2n}{n-1} - \binom{2n}{n-3}\right)q + \left(\binom{2n}{n-2} - \binom{2n}{n-4}\right)q^2 - \left(\binom{2n}{n-3} - \binom{2n}{n-5}\right)q^3\right).$$

After expanding the product, all coefficients can be seen as the product of $\binom{2n}{n}$ and a rational fraction of n. So the simplification is just a matter of simplifying rational fractions of n, which is straightforward. □

More generally, a computer algebra system can provide higher order terms, for example it takes no more than a few seconds to obtain the following closed formula for $[q^{10}]\langle W|(D+E)^{n-1}|V\rangle$:

$$\frac{(2n)!}{10!(n+12)!(n-8)!}\big(n^{13} + 70n^{12} + 2093n^{11} + 32\,354n^{10} + 228\,543n^9 - 318\,990n^8 - 17\,493\,961n^7 - 104\,051\,458n^6 - 6\,828\,164n^5 + 2\,022\,876\,520n^4 + 6\,310\,831\,968n^3 + 5\,832\,578\,304n^2 + 14\,397\,419\,520n + 5\,748\,019\,200\big),$$

which is quite an improvement when compared to the methods of [12]. Besides these exact formulas, the following proposition gives the asymptotic for permutations with a given fixed number of occurrences of the pattern 13-2.

Theorem 6.7. *For any $m \geqslant 0$ we have the following asymptotic when n goes to infinity*:

$$\big[q^m\big]\langle W|(D+E)^{n-1}|V\rangle \sim \frac{4^n n^{m-\frac{3}{2}}}{\sqrt{\pi}\,m!}.$$

Proof. When n goes to infinity, the numbers $\binom{2n}{n-k} - \binom{2n}{n-k-2}$ are dominated by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. It implies that in $(1-q)^n\langle W|(D+E)^{n-1}|V\rangle$, each higher order term grows at most as fast as the constant term C_n. On the other side, the coefficient of q^m in $(1-q)^{-n}$ is equivalent to $n^m/m!$. So we have the asymptotic

$$\big[q^m\big]\langle W|(D+E)^{n-1}|V\rangle \sim \frac{C_n n^m}{m!}.$$

Knowing the asymptotic of the Catalan numbers, we can conclude the proof. □

Since any occurrence of the pattern 13-2 in a permutation is also an occurrence of the pattern 1-3-2, a permutation with k occurrences of the pattern 1-3-2 has at most k occurrences of the pattern 13-2. So we get the following corollary.

Corollary 6.8. *Let $\psi_k(n)$ be the number of permutations in $\mathfrak{S}_n$ with at most k occurrences of the pattern* 1-3-2. *For any constant $C > 1$ and $k \geqslant 0$, we have*

$$\psi_k(n) \leqslant C \frac{4^n n^{k-\frac{3}{2}}}{\sqrt{\pi} k!}$$

when n is sufficiently large.

Proof. By the previous remark we have

$$\psi_k(n) \leqslant \sum_{i=0}^{k} [q^i] \langle W|(D+E)^{n-1}|V\rangle,$$

so this is a consequence of Theorem 6.7, which gives the asymptotics of each of these terms. □

So far we have mainly used Theorem 1.4. Now we illustrate what we can do with the refined formula given in Theorem 1.3. We already mentioned that we get Narayana numbers when $q = 0$, but we can also get the coefficients of higher degree in q. For example it is conjectured in [21] that the coefficient of qy^m in $\langle W|y(yD+E)^{n-1}|V\rangle$ is equal to $\binom{n}{m+1}\binom{n}{m-2}$. With our results we can prove:

Proposition 6.9. *The coefficients of qy^m and q^2y^m in $\langle W|y(yD+E)^{n-1}|V\rangle$ are respectively*

$$\binom{n}{m+1}\binom{n}{m-2} \quad \text{and} \quad \binom{n+1}{m-2}\binom{n+1}{m+2}\frac{nm+m-m^2-4}{2(n+1)}.$$

Proof. A naive expansion of the Taylor series in q gives a lengthy formula, which is simplified straightforwardly after noticing it is the product of $\binom{n}{m}^2$ and a rational fraction of n and m. □

Appendix A

We give here a combinatorial proof of Proposition 5.1. As noticed earlier, this result is a generalization of the Touchard–Riordan formula (4), and this combinatorial proof is a generalization of Penaud's combinatorial proof [13] of (4). We follow very closely this reference, even in some notations. Moreover the ideas of this proof were inspired by the alternative proof of Theorem 1.3 mentioned in the introduction (see [5]).

Proposition 6.10. *There is a bijection between involutions on $\{1,\dots,n\}$ and weighted Motzkin paths of n steps with the following properties*:

- *The weight of an East step at height h is q^h.*
- *The weight of a South–East step starting at height h is q^i for some $i \in \{0,\dots,h-1\}$.*

Moreover the image of an involution I on $\{1,\dots,n\}$ is a weighted Motzkin path with total weight $q^{\mu(I)}$.

Proof. See Theorem 6.6 in [19]. This can obtained via the same methods as the bijection between involution without fixed points and Hermite histories, see [13]. It is also very similar to the Foata–Zeilberger bijection as presented in [4]. See Fig. 6 for an example. □

To compute $T_{0,k,n}(1,q)$, we have to sum the weights of the weighted Motzkin paths having n steps, and $n-2k$ East steps. When we multiply by $(1-q)^k$, there are many cancellations in this sum.

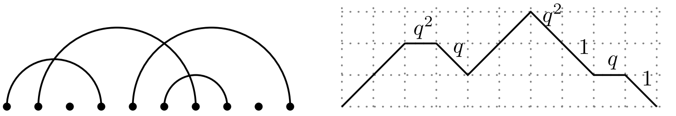

Fig. 6. An involution and the corresponding weighted Motzkin path.

Indeed we easily see that to compute $T_{0,k,n}(1-q,q)$, we have to sum the weights of Motzkin paths of n steps satisfying conditions $(C2)$:

- the weight of an East step at height h is q^h.
- the weight of a South–East step starting at height h is either 1 or $-q^h$.

Now, we give a decomposition of these weighted Motzkin paths.

Proposition 6.11. *There is a weight-preserving bijection between weighted Motzkin paths satisfying $(C2)$ and couples (H_1, H_2) such that for some $i \in \{0, \dots, k\}$,*

- *H_1 is a left factor of a Dyck path, with n steps and ending at height $n-2k+2i$,*
- *H_2 is a weighted Motzkin path of $n-2k+2i$ steps, with $n-2k$ East steps, satisfying conditions $(C2)$ above, and also that any South–East step following a North–East step has weight $-q^h$ (i.e. not 1).*

Proof. This is similar to Lemma 1 in [5]. □

A weighted Motzkin path as H_2 above is called a *core*. The enumeration of left factors of Dyck path is given by Delannoy numbers. On the other hand, to compute the sum of weights of cores we need two other lemmas.

Lemma 6.12. *There is an involution γ_i on cores of length $n-2k+2i$ with $n-2k$ East steps, with the following properties*:

- *if a core and its image are different they have opposite weights,*
- *the fixed points of γ_i are the cores such that*:
  - *the i first steps are North–East, and all following steps are East or South–East,*
  - *a South–East step starting at height h has weight $-q^h$ (i.e. not 1).*

Proof. In this proof we use a word notation for cores: the letters x, z, y, and $\bar{y}$ respectively correspond to North–East steps, East steps, South–East steps weighted by 1, and South–East steps weighted by $-q^h$. For a core c, let $u(c)$ be the length the last sequence of consecutive x's. Let $v(c)$ be the height of the last y if there is no x after this y, and i otherwise. The fixed points of γ_i are the cores such that $u(c) = v(c) = i$.

From now on we assume that c does not satisfy $u(c) = v(c) = i$. The involution γ_i is such that $u(c) \geqslant v(c)$ if and only if $u(\gamma_i(c)) < v(\gamma_i(c))$. Suppose that $u(c) \geqslant v(c)$. Let $\tilde{c}$ be the word obtained from c when we replace the last y with a $\bar{y}$. There is a unique factorization $\tilde{c} = f_1 x^{u(c)} a y^j f_2$ such that:

- a is either z or $\bar{y}$,
- f_2 begins with a $\bar{y}$ and contains no x.

We set

$$\gamma_i(c) = f_1 x^{u(c)-v(c)} a y^j x^{v(c)} f_2.$$

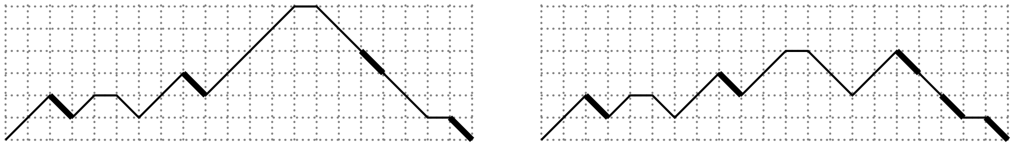

Fig. 7. A core c and its image by γ_i. The thick lines indicate the $\bar{y}$, *i.e.* the South–East steps weighted by $-q^h$. In this example we have $n-2k=3$, $i=9$, $u=4$, $v=2$. We can check that $w(c)=-q^{17}=-w(\gamma_i(c))$.

See Fig. 7 for an example.

Simple arguments of word combinatorics show that:

- c and its image have opposite weights,
- any core c' such that $u(c')<v(c')$ is obtained as a $\gamma_i(c)$ for some c satisfying $u(c)\geqslant v(c)$. Indeed, let $\tilde{c}'$ be the word obtained from c' by replacing the last $\bar{y}$ at height $u(c')$ with a y. There is unique factorization $\tilde{c}'=f_1ay^jx^{u(c')}f_2$, where a is z or $\bar{y}$ and f_2 contains no x. Then $c=f_1x^{u(c')}ay^jf_2$ has the required properties.

These arguments, put together, show that γ_i has the claimed properties. □

Lemma 6.13. *The sum of weights of the fixed points of γ_i is*

$$\sum_{H_2\in Fix(\gamma_i)} w(H_2)=(-1)^i q^{\frac{i(i+1)}{2}}\begin{bmatrix} n-2k+i \\ i\end{bmatrix}_q.$$

Proof. A fixed point of γ_i is fully characterized by the heights $h_1,\dots,h_{n-2k}$ of the $n-2k$ East steps, and these heights can take any values such that $i\geqslant h_1\geqslant\cdots\geqslant h_{n-2k}\geqslant 0$. Such a fixed point of γ_i has weight

$$(-1)^i q^{\frac{i(i+1)}{2}} q^{\sum h_i},$$

indeed the South–East steps have weights $-q^i,\dots,-q^2,-q$ and they correspond to the factor $(-1)^i q^{\frac{i(i+1)}{2}}$. It remains to sum over h_i to conclude. □

Now we can prove Proposition 5.1, which was

$$T_{0,k,n}(1-q,q)=\sum_{i=0}^{k}(-1)^i q^{\frac{i(i+1)}{2}}\begin{bmatrix} n-2k+i \\ i\end{bmatrix}_q \left\{ \begin{matrix} n \\ k-i\end{matrix}\right\}.$$

Proof. The decomposition of weighted Motzkin paths stated in Proposition 6.11 gives

$$T_{0,k,n}(1-q,q)=\sum_{i=0}^{k}\left\{ \begin{matrix} n \\ k-i\end{matrix}\right\}\sum_{H_2} w(H_2),$$

where the second sum is over cores H_2 of $n-2k+2i$ steps with $n-2k$ East steps. Thanks to Lemma 6.12, we can restrict the second sum to the fixed points of the involution γ_i. And thanks to Lemma 6.13, this sum is

$$\sum_{H_2} w(H_2)=(-1)^i q^{\frac{i(i+1)}{2}}\begin{bmatrix} n-2k+i \\ i\end{bmatrix}_q.$$

This completes the proof. □

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