Moduli spaces for jets of Riemannian metrics at a point

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ABSTRACT

We construct the moduli space of r-jets of Riemannian metrics at a point on a smooth manifold. The construction is closely related to the problem of classification of metric jets via scalar differential invariants.

The moduli space is proved to be a differentiable space which admits a finite canonical stratification into smooth manifolds. A complete study on the stratification of moduli spaces is carried out for metrics in dimension n = 2.

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0. Introduction

Let X be an n-dimensional smooth manifold. Fixed a point x₀ ∈ X and an integer r ≥ 0, we will denote by Jᵣₓ₀ M the smooth manifold of r-jets of Riemannian metrics at x₀. On the manifold Jᵣₓ₀ M, there exists a natural action of the group Diffₓ₀ of germs at x₀ of local diffeomorphisms leaving x₀ fixed, so it yields an equivalence relation on Jᵣₓ₀ M:

\[ jr_x₀ \gamma \equiv jr_x₀ \bar{\gamma} \iff jr_x₀ (\tau^* \gamma) = jr_x₀ \bar{\gamma}, \quad \text{for some} \ \tau \in \text{Diff}_x₀. \]

The quotient space \( Mᵣⁿ := Jᵣₓ₀ M / \text{Diff}_x₀ \) is called moduli space for r-jets of Riemannian metrics in dimension n. It depends neither on the point x₀ nor on the n-dimensional manifold X chosen.

The purpose of this paper is to study the structure of moduli spaces \( Mᵣⁿ \).

Moduli spaces \( Mᵣⁿ \) have been studied in the literature through their function algebras \( C∞(Mᵣⁿ) := C∞(Jᵣₓ₀ M) / \text{Diff}_x₀ \). This function algebra \( C∞(Mᵣⁿ) \) is nothing but the algebra of scalar differential invariants of order ≤ r of Riemannian metrics. Muñoz and Valdés [8,9] prove that it is an essentially finitely-generated algebra and they determine the number of its functionally independent generators. In a more general setting (homogeneous geometric structures), Vinogradov [14] has proved that an open subset of the moduli space of \( \infty \)-jets has the structure of a diffiety, and he has pointed out that the cohomology classes of this diffiety are the characteristic classes of the geometric structure considered. (See also [13].)

Let us also mention that explicit moduli spaces for jets of geometric structures have been obtained for linear frames (see [4]) and for webs in dimension 2 (see [15]).

Apart from some trivial exceptions, moduli spaces \( Mᵣⁿ \) of jets of metrics are not smooth manifolds, but they possess a differentiable structure in a more general sense: that of a differentiable space. (The typical example of differentiable space...
is a closed subset \( Y \subseteq \mathbb{R}^m \) where a function \( f : Y \to \mathbb{R} \) is said to be differentiable if it is the restriction to \( Y \) of a smooth function on \( \mathbb{R}^m \), see [10].

In addition, the differentiable structure of \( M^r_n \) is not too far from a smooth structure, since it admits a stratification by a finite number of smooth submanifolds. Our results can be summed up in the following

**Theorem 0.1.** Every moduli space \( M^r_n \) is a differentiable space and it admits a finite canonical stratification

\[
M^r_n = S^r_{[H_0]} \sqcup \cdots \sqcup S^r_{[H_1]},
\]

for locally closed subspaces \( S^r_{[H_1]} \) which are smooth manifolds. Moreover, one of them is an open connected dense subset of \( M^r_n \).

Each stratum of this decomposition of the space \( M^r_n \) consists of those metric jets having essentially the same group of automorphisms. To be more precise, let us denote by \([H]\) the conjugacy class of a closed subgroup \( H \) of the orthogonal group \( O(n) \). Then \( S^r_{[H]} \) is the set of equivalence classes of jets \( f^r_{s0} g \) whose group of automorphisms \( \text{Aut}(f^r_{s0}) \) is conjugate to \( H \), viewing \( \text{Aut}(f^r_{s0}) \) as a subgroup of the orthogonal group \( O(T_{s0}X, g_{s0}) \cong O(n) \).

It is convenient to notice that Theorem 0.1 is not valid for pseudo-Riemannian metrics. For metrics of any signature, the problem lies on the existence of non-closed orbits for the action of \( \text{Diff}_{s0} \) on the space \( f^r_{s0} M \) of \( r \)-jets of such metrics, which means that the corresponding moduli space \( f^r_{s0} M / \text{Diff}_{s0} \) is not a \( T_1 \) topological space, and consequently, it does not admit a structure of differentiable space either. Moreover, the above stratification in terms of the groups of automorphisms does not produce smooth strata.

In dimension \( n = 2 \), we improve the above theorem by determining exactly all the strata which appear in the decomposition of each moduli space \( M^r_{n=2} \). Let us consider the only, up to conjugacy, closed subgroups of the orthogonal group \( O(2) \): the finite group \( K_m \) of rotations of order \( m (m \geq 1) \), the dihedral group \( D_m \) of order \( 2m (m \geq 1) \), the special orthogonal group \( SO(2) \) and \( O(2) \) itself. The stratification of \( M^r_2 \) is determined by the following

**Theorem 0.2.** The strata in the moduli space \( M^r_{n=2} \) correspond exactly to the following conjugacy classes: \([O(2)], [D_1], \ldots, [D_{r−2}], [K_1], \ldots, [K_{r−4}]. \) (And also \([K_1], \) if \( r = 4 \).)

Finally, we include Appendix A where we analyze the equivalence problem for infinite-order jets of Riemannian metrics.

1. Preliminaries

1.1. Quotient spaces

Throughout this paper, we are going to handle geometric objects of a more general nature than smooth manifolds, which appear when one considers the quotient of a smooth manifold by the action of a Lie group.

**Definition 1.1.** Let \( X \) be a topological space. A **sheaf of continuous functions** on \( X \) is a map \( \mathcal{O}_X \) which assigns a subalgebra \( \mathcal{O}_X(U) \subseteq \mathbb{C}(U, \mathbb{R}) \) to every open subset \( U \subseteq X \), with the following condition:

For every open subset \( U \subseteq X \), every open cover \( U = \bigcup U_i \) and every function \( f : U \to \mathbb{R} \), it is verified

\[
f \in \mathcal{O}_X(U) \iff f|_{U_i} \in \mathcal{O}_X(U_i), \quad \forall i.
\]

In particular, if \( V \subseteq U \) are open subsets in \( X \), then it is verified

\[
f \in \mathcal{O}_X(U) \implies f|_V \in \mathcal{O}_X(V).
\]

**Definition 1.2.** We will call **ringed space** the pair \((X, \mathcal{O}_X)\) formed by a topological space \( X \) and a sheaf of continuous functions \( \mathcal{O}_X \) on \( X \).

Although the concept of ringed space in the literature, specially in that concerning Algebraic Geometry, is much broader, the previous definition is good enough for our purposes.

Every open subset \( U \) of a ringed space \((X, \mathcal{O}_X)\) is itself, in a very natural way, a ringed space, if we define \( \mathcal{O}_U(V) := \mathcal{O}_X(V) \) for every open subset \( V \subseteq U \).

Hereinafter, a ringed space \((X, \mathcal{O}_X)\) will usually be denoted just by \( X \), dropping the sheaf of functions.

**Definition 1.3.** Given two ringed spaces \( X \) and \( Y \), a **morphism of ringed spaces** \( \varphi : X \to Y \) is a continuous map such that, for every open subset \( V \subseteq Y \), the following condition is held:

\[
f \in \mathcal{O}_Y(V) \implies f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V)).
\]
A morphism of ringed spaces $\phi : X \to Y$ is said to be an **isomorphism** if it has an inverse morphism, that is, there exists a morphism of ringed spaces $\psi : Y \to X$ verifying $\phi \circ \psi = \text{Id}_Y$, $\psi \circ \phi = \text{Id}_X$.

**Example 1.4 (Smooth manifolds).** The space $\mathbb{R}^n$, endowed with the sheaf $C^\infty_{\mathbb{R}^n}$ of smooth functions, is an example of ringed space. An $n$-smooth manifold is precisely a ringed space in which every point has an open neighborhood isomorphic to $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$. Smooth maps between smooth manifolds are nothing but morphisms of ringed spaces.

**Example 1.5 (Quotients by the action of a Lie group).** Let $G \times X \to X$ be a smooth action of a Lie group $G$ on a smooth manifold $X$, and let $\pi : X \to X/G$ be the canonical quotient map.

We will consider on the quotient topological space $X/G$ the following sheaf $C^\infty_{X/G}$ of “differentiable” functions:

For every open subset $V \subseteq X/G$, $C^\infty_{X/G}(V)$ is defined to be

$$C^\infty_{X/G}(V) := \{ f : V \to \mathbb{R} : f \circ \pi \in C^\infty(\pi^{-1}(V)) \}. $$

Note that there exists a canonical $\mathbb{R}$-algebra isomorphism:

$$C^\infty_{X/G}(V) \xrightarrow{\cong} C^\infty(\pi^{-1}(V))^G, $$

$$f \mapsto f \circ \pi.$$ 

The pair $(X/G, C^\infty_{X/G})$ is an example of ringed space, which we will call **quotient ringed space** of the action of $G$ on $X$.

As it would be expected, this space verifies the **universal quotient property**: Every morphism of ringed spaces $\varphi : X \to Y$, which is constant on every orbit of the action of $G$ on $X$, factors uniquely through the quotient map $\pi : X \to X/G$, that is, there exists a unique morphism of ringed spaces $\tilde{\varphi} : X/G \to Y$ verifying $\varphi = \tilde{\varphi} \circ \pi$.

**Example 1.6 (Inverse limit of smooth manifolds).** Sometimes we will consider an inverse system

$$\cdots \to X_{r+1} \to X_r \to \cdots \to X_1$$

of smooth mappings between smooth manifolds (or, with some more generality, an inverse system of ringed spaces).

The inverse limit $\lim \leftarrow X_r$ is a ringed space in the following natural way. On $\lim \leftarrow X_r$, it is considered the inverse limit topology, that is, the initial topology induced by the evident projections $p_i : \lim \leftarrow X_r \to X_i$. A real function on an open subset of $\lim \leftarrow X_r$ is said to be “differentiable” if it locally coincides with the composition of a projection $p_i : \lim \leftarrow X_r \to X_i$ and a smooth function on $X_i$.

The topological space $\lim \leftarrow X_r$ endowed with the above sheaf of differentiable functions is a ringed space satisfying the suitable universal property:

For every ringed space $Z$, there exists the bijection

$$\text{Hom}(Z, \lim \leftarrow X_r) \xrightarrow{\cong} \lim \text{Hom}(Z, X_r)$$

$$\varphi \mapsto (\cdots, p_r \circ \varphi, \cdots).$$

**Example 1.7.** Let $Z$ be a locally closed subspace of $\mathbb{R}^n$. We define the sheaf $C^\infty_Z$ of differentiable functions on $Z$ to be the sheaf of functions locally coinciding with restrictions of smooth functions on $\mathbb{R}^n$. The pair $(Z, C^\infty_Z)$ is another example of ringed space.

**Definition 1.8.** A (reduced) **differentiable space** is a ringed space in which every point has an open neighborhood isomorphic to a certain locally closed subspace $(Z, C^\infty_Z)$ in some $\mathbb{R}^n$.

A map between differentiable spaces is called **differentiable** if it is a morphism of ringed spaces.

**Theorem 1.9.** (See Schwarz [11,10, Th. 1.14].) Let $G \to GL(V)$ be a finite-dimensional linear representation of a compact Lie group $G$. The quotient space $V/G$ is a differentiable space.

More precisely: Let $p_1, \ldots, p_s$ be a finite set of generators for the $\mathbb{R}$-algebra of $G$-invariant polynomials on $V$; these invariants define an isomorphism of ringed spaces

$$(p_1, \ldots, p_s) : V/G \xrightarrow{\cong} Z \subseteq \mathbb{R}^s,$$

$Z$ being a closed subspace of $\mathbb{R}^s$. 
1.2. Normal tensors

Let \( X \) be an \( n \)-dimensional smooth manifold. Fix a point \( x_0 \in X \) and a pseudo-Riemannian metric \( g \) on \( X \) of fixed signature \((p, q)\), with \( n = p + q \). Let us briefly recall some definitions and results:

**Definition 1.10.** A chart \((z_1, \ldots, z_n)\) in a neighborhood of \( x_0 \) is said to be a normal system for \( g \) at the point \( x_0 \) if the geodesics passing through \( x_0 \) at \( t = 0 \) are precisely the “straight lines” \( \{z_1(t) = \lambda_1 t, \ldots, z_n(t) = \lambda_n t\} \), where \( \lambda_i \in \mathbb{R} \).

As it is well known, via the exponential map \( \exp_{x_0} : T_{x_0} X \to X \), normal systems on \( X \) correspond bijectively to linear systems on \( T_{x_0} X \). Therefore, two normal systems differ in a linear transformation.

**Proposition 1.11.** Let \( g, \bar{g} \) be two pseudo-Riemannian metrics on \( X \). Let us also consider their corresponding exponential maps \( \exp_g, \exp_{\bar{g}} : T_{x_0} X \to X \). For every \( r \geq 0 \) it is verified:

\[
\frac{d^r}{dt^r} \big|_{t=0} g = \frac{d^r}{dt^r} \big|_{t=0} \bar{g} \quad \implies \quad j^{r+1}_0 (\exp_g) = j^{r+1}_0 (\exp_{\bar{g}}).
\]

As a consequence of Proposition 1.11, whose proof is routine, normal systems at \( x_0 \) for a metric \( g \) are determined up to the order \( r + 1 \) by the jet \( j^r_{x_0} g \). This fact will be used later on with no more explicit mention.

**Definition 1.12.** Let \( r \geq 1 \) be a fixed integer and let \( x_0 \in X \). The space of normal tensors of order \( r \) at \( x_0 \), which we will denote by \( N_r \), is the vector space of \((r + 2)\)-covariant tensors \( T \) at \( x_0 \) having the following symmetries:

- \( T \) is symmetric in the first two and last \( r \) indices:
  \[ T_{ijk_1 \ldots kr} = T_{jik_1 \ldots kr}, \quad T_{ijk_1 \ldots kr} = T_{ij(k_1 \ldots k_r)}, \quad \forall \sigma \in S_r; \]
- the cyclic sum over the last \( r + 1 \) indices is zero:
  \[ T_{ijk_1 \ldots kr} + T_{ikr jk_1 \ldots kr-1} + \cdots + T_{jk_1 \ldots kr} = 0. \]

If \( r = 0 \), we will assume \( N_0 \) to be the set of pseudo-Riemannian metrics at \( x_0 \) of a fixed signature \((p, q)\) (which is an open subset of \( S^2 T^*_{x_0} X \), but not a vector subspace).

To show how a pseudo-Riemannian metric \( g \) produces a sequence of normal tensors \( g_{x_0}^r \) at \( x_0 \), let us recall this classical result:

**Lemma 1.13 (Gauss Lemma).** Let \((z_1, \ldots, z_n)\) be germs of a chart centred at \( x_0 \in X \). This chart is a normal system for the germ of a pseudo-Riemannian metric \( g \) if and only if the metric coefficients \( g_{ij} \) verify the equations

\[
\sum_j g_{ij} z_j = \sum_j g_{ij}(x_0) z_j.
\]

Let \((z_1, \ldots, z_n)\) be a normal system for \( g \) at \( x_0 \in X \) and let us denote:

\[
g_{ij, k_1 \ldots kr} := \frac{\partial^r g_{ij}}{\partial z_{k_1} \cdots \partial z_{kr}}(x_0).
\]

If we differentiate \( r + 1 \) times the identity of the Gauss Lemma, we obtain:

\[
g_{ik_0, k_1 \ldots kr} + g_{ik_1, k_2 \ldots kr} + \cdots + g_{ik_r, k_{r+1}} = 0.
\]

This property, together with the obvious fact that the coefficients \( g_{ij, k_1 \ldots kr} \) are symmetric in the first two and in the last \( r \) indices, allows to prove that the tensor

\[
g_{x_0}^r := \sum_{ij k_1 \ldots kr} g_{ij, k_1 \ldots kr} dz_i \otimes dz_j \otimes dz_{k_1} \otimes \cdots \otimes dz_{kr}
\]

is a normal tensor of order \( r \) at \( x_0 \in X \). This construction does not depend on the choice of the normal system \((z_1, \ldots, z_n)\).

**Definition 1.14.** The tensor \( g_{x_0}^r \) is called the \( r \)-th normal tensor of the metric \( g \) at the point \( x_0 \).

A simple computation shows that \( N_1 = 0 \), and, as a consequence, the first normal tensor of a metric \( g \) is always zero, \( g_{x_0}^1 = 0 \).
The normal tensors associated to a metric were first introduced by Thomas [12]. The sequence \((g_{x_0}, g^2_{x_0}, g^3_{x_0}, \ldots, g^r_{x_0})\) of normal tensors of the metric \(g\) at a point \(x_0\) totally determines the sequence \(\{g_{x_0}, R_{x_0}, \nabla_{x_0} R, \ldots, \nabla^{r-2} R\}\) of covariant derivatives at \(x_0\) of the curvature tensor \(R\) of \(g\) and vice versa (see [12]). The main advantage of using normal tensors is the possibility of expressing the symmetries of each \(g^r_{x_0}\) without using the other normal tensors, whereas the symmetries of \(\nabla_{x_0} R\) depend on \(R\) (recall the Ricci identities).

**Remark 1.15.** Using the exact sequence

\[
0 \rightarrow N_r \rightarrow S^r T^*_x X \otimes S^r T^*_x X \rightarrow T^*_x X \otimes S^{r+1} T^*_x X \rightarrow 0,
\]

where \(s\) stands for the symmetrization on the last \((r+1)-\)indices, we obtain

\[
\dim N_r = \binom{n + 1}{2} \binom{n + r - 1}{r} - n \binom{n + r}{r + 1}.
\]

**2. Scalar differential invariants of metrics**

In the remainder of the paper, \(X\) will always be an \(n\)-dimensional smooth manifold.

Let us denote by \(J^r M \rightarrow X\) the fiber bundle of \(r\)-jets of pseudo-Riemannian metrics on \(X\) of fixed signature \((p, q)\), with \(n = p + q\). Its fiber over a point \(x_0 \in X\) will be denoted \(J^r_{x_0} M\).

Let Diff\(_{\mathbf{0}}\) be the group of germs of local diffeomorphisms of \(X\) leaving \(x_0\) fixed, and let Diff\(_{\mathbf{0}}\) be the Lie group of \(r\)-jets at \(x_0\) of local diffeomorphisms of \(X\) leaving \(x_0\) fixed. We have the following exact group sequence:

\[
0 \rightarrow H^r_{x_0} \rightarrow \text{Diff}_{\mathbf{0}} \rightarrow \text{Diff}_{x_0} \rightarrow 0,
\]

where \(H^r_{x_0}\) being the subgroup of Diff\(_{\mathbf{0}}\) made up of those diffeomorphisms whose \(r\)-jet at \(x_0\) coincides with that of the identity.

The group Diff\(_{\mathbf{0}}\) acts in an obvious way on \(J^r_{x_0} M\). Note that the subgroup \(H^{r+1}_{x_0}\) acts trivially, so the action of Diff\(_{\mathbf{0}}\) on \(J^r_{x_0} M\) factors through an action of Diff\(_{\mathbf{0}}\)\(^{r+1}\).

**Definition 2.1.** Two \(r\)-jets \(j^r_{x_0} g, j^r_{x_0} \bar{g} \in J^r_{x_0} M\) are said to be **equivalent** if there exists a local diffeomorphism \(\tau \in \text{Diff}_{x_0}\) such that \(j^r_{x_0} \bar{g} = j^r_{x_0} (\tau \cdot g)\).

Equivalence classes of \(r\)-jets of metrics constitute a ringed space. To be precise:

**Definition 2.2.** We call **moduli space** of \(r\)-jets of pseudo-Riemannian metrics of signature \((p, q)\) the quotient ringed space

\[
\mathbb{M}^r_{p,q} := J^r_{x_0} M / \text{Diff}_{x_0} = J^r_{x_0} M / \text{Diff}_{x_0}^{r+1}.
\]

In the case of Riemannian metrics, that is \(p = n, q = 0\), the moduli space will be denoted \(\mathbb{M}^r_n\).

It is important to observe that the moduli space depends neither on the point \(x_0\) nor on the chosen \(n\)-dimensional manifold:

Given a point \(\bar{x}_0\) in another \(n\)-dimensional manifold \(\bar{X}\), let us consider an arbitrary diffeomorphism

\[
X \ni U_{\bar{x}_0} \xrightarrow{\varphi} U_{\bar{x}_0} \subseteq \bar{X}
\]

between corresponding neighborhoods of \(x_0\) and \(\bar{x}_0\), verifying \(\varphi(x_0) = \bar{x}_0\). Such a diffeomorphism induces an isomorphism of ringed spaces between the corresponding moduli spaces,

\[
\begin{align*}
J^r_{x_0} \bar{M} / \text{Diff}_{\bar{x}_0} & \cong J^r_{\bar{x}_0} M / \text{Diff}_{\bar{x}_0} \\
\left[ j^r_{\bar{x}_0} \bar{g} \right] & \mapsto \left[ j^r_{\bar{x}_0} \varphi \# \bar{g} \right],
\end{align*}
\]

which is independent of the choice of the diffeomorphism \(\varphi\). So both moduli spaces are canonically identified.

Let us now consider the quotient morphism

\[
J^r_{x_0} M \xrightarrow{\pi} J^r_{x_0} M / \text{Diff}_{x_0} = \mathbb{M}^r_{p,q}.
\]

Recall that a function \(f\) defined on an open subset \(U \subseteq \mathbb{M}^r_{p,q}\) is said to be **differentiable** if \(f \circ \pi\) is a smooth function on \(\pi^{-1}(U)\), that is,

\[
C^\infty(U) = C^\infty(\pi^{-1}(U))^{\text{Diff}_{x_0}}.
\]
Every pseudo-Riemannian metric $g$ on $X$ of signature $(p, q)$ defines a map

$$
\begin{align*}
X & \overset{m_g}{\longrightarrow} M^r_{p,q} \\
x & \longmapsto \{ j^r_x g \},
\end{align*}
$$

which is “differentiable”, that is, it is a morphism of ringed spaces.

**Definition 2.3.** A (scalar) differential invariant of order $\leq r$ of pseudo-Riemannian metrics of signature $(p, q)$ is defined to be a global differentiable function on $M^r_{p,q}$.

Taking into account the ringed space structure of $M^r_{p,q}$, we can simply write:

$$
\{ \text{Differential invariants of order } \leq r \} = C^\infty (M^r_{p,q}) = C^\infty (j^r_x M)^{\text{Diff}_{x_0}}.
$$

A differential invariant $h : M^r_{p,q} \to \mathbb{R}$ associates with every pseudo-Riemannian metric $g$ on $X$ a smooth function on $X$, denoted by $h(g)$, through the formula $h(g) := h \circ m_g$, that is,

$$
h(g)(x) = h(\{ j^r_x g \}).
$$

In any chart, $h(g)$ is a function smoothly depending on the coefficients of the metric and their subsequent partial derivatives up to the order $r$,

$$
h(g)(x) = h\left( g_{ij}(x), \frac{\partial g_{ij}}{\partial x_k}(x), \ldots, \frac{\partial^r g_{ij}}{\partial x_{k_1} \ldots \partial x_{k_r}}(x) \right),
$$

which is equivariant with respect to the action of local diffeomorphisms,

$$
h(\tau^* g) = \tau^* (h(g)).
$$

For a general discussion on the concept of differential invariant, see [5].

### 3. A Fundamental Lemma

The aim of this section is to construct a slice of the action of $\text{Diff}_{x_0}$ on $j^r_{x_0} M$, which is isomorphic to a certain finite-dimensional linear representation $V^r$ of the orthogonal group $O(p, q)$. From this fact, we will obtain an isomorphism

$$
M^r_{p,q} = V^r / O(p, q).
$$

This bijection is already known at a set-theoretic level (see [2] and also [7] for $G$-structures which possess a linear connection). We just add the fact that this bijection is an isomorphism of ringed spaces.

Let us fix for this entire section a chart $(x_1, \ldots, x_n)$ centred at $x_0$.

We will denote by $N^r_{x_0}$ the smooth manifold of $j^r_{x_0} M$ formed by r-jets at $x_0$ of metrics of signature $(p, q)$ for which $(x_1, \ldots, x_n)$ is a normal system (that is, Taylor expansions of the coefficients of such metrics with respect to $(x_1, \ldots, x_n)$ satisfy the equations of the Gauss Lemma up to the order $r$).

Let us denote by $G_{ln}$ the general linear group in dimension $n$. Considering every matrix in $G_{ln}$ as a linear transformation of the chart $(x_1, \ldots, x_n)$, we can think of $G_{ln}$ as a subgroup of $\text{Diff}_{x_0}$.

Via the action of the group $\text{Diff}_{x_0}$ on $j^r_{x_0} M$, the subgroup $G_{ln}$, for its part, acts leaving the submanifold $N^r_{x_0}$ stable. Let us describe this submanifold in terms of normal tensors:

**Lemma 3.1.** The $G_{ln}$-equivariant map

$$
N^r_{x_0} \longrightarrow N_0 \times N_2 \times \cdots \times N_r, \quad j^r_{x_0} g \longmapsto (g_{x_0}, g^2_{x_0}, \ldots, g^r_{x_0})
$$

is a diffeomorphism.

**Proof.** The inverse map is defined in the obvious way:

Given $(T^0, T^1, \ldots, T^r) \in N_0 \times N_2 \times \cdots \times N_r$, consider the metric jet $j^r_{x_0} g$ which is determined by the identities

$$
\frac{\partial^s g_{ij}}{\partial x_{k_1} \ldots \partial x_{k_s}}(x_0) := T^r_{ij k_1 \ldots k_s}, \quad s = 0, \ldots, r.
$$

The symmetries of tensors $T^i$ guarantee that the coefficients $g_{ij}$ of the metric $g$ verify the equations of the Gauss Lemma up to the order $r$, that is, $j^r_{x_0} g \in N^r_{x_0}$. □
Consider the subgroup of Diff$_{x_0}$

\[ H^1_{x_0} := \{ \tau \in \text{Diff}_{x_0} : j^1_{x_0} \tau = j^1_{x_0} (\text{Id}) \}. \]

Note the following exact group sequence:

\[ 0 \longrightarrow H^1_{x_0} \longrightarrow \text{Diff}_{x_0} \longrightarrow \text{Gl}(T_{x_0}X) \longrightarrow 0, \]

where the epimorphism Diff$_{x_0} \rightarrow \text{Gl}(T_{x_0}X)$ takes every diffeomorphism to its linear tangent map at $x_0$.

**Lemma 3.2.** The inclusion $\mathcal{N}^{\tau}_{x_0} \hookrightarrow J^r_{x_0} M$ admits a smooth retraction $\varphi : J^r_{x_0} M \to \mathcal{N}^{\tau}_{x_0}$ whose fibers are the orbits of the action of $H^1_{x_0}$.

As a consequence, we have an isomorphism of ringed spaces

\[ \mathcal{N}^{\tau}_{x_0} \cong J^r_{x_0} M/H^1_{x_0}. \]

**Proof.** Given a metric jet $j^r_{x_0} g \in J^r_{x_0} M$, consider a metric $g$ representing it. Let $(\bar{z}_1, \ldots, \bar{z}_n)$ be the only normal system centred at $x_0$ with respect to $g$ which satisfies $d_{x_0} \bar{z}_i = d_{x_0} z_i$.

Let $\tau$ be the local diffeomorphism which transforms one chart into another: $\tau^*(\bar{z}_i) = z_i$. The condition $d_{x_0} \bar{z}_i = d_{x_0} z_i$ implies that the linear tangent map of $\tau$ at $x_0$ is the identity, i.e. $\tau \in H^1_{x_0}$.

As $(\bar{z}_1, \ldots, \bar{z}_n)$ is a normal system for $g$, $(z_1 = \tau^*(\bar{z}_1), \ldots, z_n = \tau^*(\bar{z}_n))$ is a normal system for $\tau^* g$; that is, $j^r_{x_0} (\tau^* g) \in \mathcal{N}^{\tau}_{x_0}$.

Therefore, the retraction we were looking for is the following map:

\[ J^r_{x_0} M \xrightarrow{\varphi} \mathcal{N}^{\tau}_{x_0} \]

\[ j^r_{x_0} g \mapsto j^r_{x_0} (\tau^* g), \]

with $\tau$ depending on $g$.

Let us now see that $\varphi$ is constant on each orbit of the action of $H^1_{x_0}$. Let $j^r_{x_0} g'$ be another point in the same orbit as $j^r_{x_0} g$, so we can write $g' = \sigma^* g$ for some $\sigma \in H^1_{x_0}$.

Since $(\bar{z}_1, \ldots, \bar{z}_n)$ is a normal system for $g$, $(z'_1 = \sigma^* (\bar{z}_1), \ldots, z'_n = \sigma^* (\bar{z}_n))$ is a normal system for $g' = \sigma^* g$. Then $z_i = \tau^*(\bar{z}_i) = \tau^*(\sigma^* (\bar{z}_i))$, and, if we apply the definition of $\varphi$, we get

\[ \varphi(j^r_{x_0} g') = j^r_{x_0} (\tau^* \sigma^* (\bar{z}_i) g') = j^r_{x_0} (\tau^* g) = \varphi(j^r_{x_0} g). \]

As $\varphi$ is constant on each orbit of the action of $H^1_{x_0}$, it induces, according to the universal quotient property, a morphism of ringed spaces:

\[ J^r_{x_0} M/H^1_{x_0} \longrightarrow \mathcal{N}^{\tau}_{x_0}. \]

This map is indeed an isomorphism of ringed spaces, because it has an obvious inverse morphism, which is the following composition:

\[ \mathcal{N}^{\tau}_{x_0} \hookrightarrow J^r_{x_0} M \to J^r_{x_0} M/H^1_{x_0}. \]

**Corollary 3.3.** There exists an isomorphism of ringed spaces

\[ \mathcal{N}^{\tau}_{x_0}/\text{Gl}_n \cong J^r_{x_0} M/\text{Diff}_{x_0} = \mathbb{M}^{\tau}_{p,q}. \]

**Proof.** Via the epimorphism

\[ \text{Diff}_{x_0} \longrightarrow \text{Diff}_{x_0}/H^1_{x_0} = \text{Gl}(T_{x_0}X), \]

the subgroup $\text{Gl}_n$ gets identified with $\text{Gl}(T_{x_0}X)$. Consequently, the subgroups $H^1_{x_0}$ and $\text{Gl}_n$ generate $\text{Diff}_{x_0}$.

If we consider the isomorphism

\[ \mathcal{N}^{\tau}_{x_0} \cong J^r_{x_0} M/H^1_{x_0} \]

of Lemma 3.2 and take quotient with respect to the action of $\text{Gl}_n$, we get the desired isomorphism:

\[ \mathcal{N}^{\tau}_{x_0}/\text{Gl}_n \cong (J^r_{x_0} M/H^1_{x_0})/\text{Gl}_n = J^r_{x_0} M/\text{Diff}_{x_0}. \]

Let us denote by $\delta_{x_0}$ the metric of signature $(p, q)$ on $T_{x_0}X$, whose matrix is diagonal $(1, \ldots, 1, -1, \ldots, -1)$ with respect to the basis $(\partial \bar{z}_1, \ldots, \partial \bar{z}_n)$.

We consider the orthogonal group $O(p, q) \subset \text{Gl}_n \subset \text{Diff}_{x_0}$ as a group of linear transformations of $(z_1, \ldots, z_n)$.
Notation. Let us denote
\[ \mathcal{V}_x^r := \{ f_{x_0}^r g \in \mathcal{N}_x^r : g_{x_0} = \delta_{x_0} \}. \]
that is, \( \mathcal{V}_x^r \) is the smooth submanifold of \( J^r_{x_0} M \) formed by \( r \)-jets of metrics of signature \( (p, q) \) for which \( (z_1, \ldots, z_n) \) is a normal system and \( (\partial_{z_1}, \ldots, \partial_{z_n}) \) is an orthonormal basis at \( x_0 \).

Via the action of the group \( \text{Diff}_{x_0} \), the subgroup \( O(p, q) \) acts on \( J^r_{x_0} M \) leaving the submanifold \( \mathcal{V}_x^r \) stable. By restricting Lemma 3.1 to the submanifold \( \mathcal{V}_x^r \subset \mathcal{N}_x^r \) we obtain a \( O(p, q) \)-equivariant diffeomorphism
\[ \mathcal{V}_x^r \xrightarrow{\sim} \delta_{x_0} \times N \times \cdots \times N_r, \quad f_{x_0}^r g \mapsto (\delta_{x_0}, g_{x_0}^2, \ldots, g_{x_0}^r). \]

Moreover, since \( \text{Gl}_n \) acts transitively on \( N_0 \), and \( O(p, q) \) is the stabilizer subgroup of \( \delta_{x_0} \in N_0 \), we have an isomorphism
\[ (N \times \cdots \times N_r)/O(p, q) = (N_0 \times N_2 \times \cdots \times N_r)/\text{Gl}_n \xrightarrow{\sim} \mathcal{N}_x^r/\text{Gl}_n \cong \mathbb{M}^r_{p, q}. \]

To sum up, we can state the main result of this section:

**Lemma 3.4 (Fundamental Lemma).** There exists a slice \( \mathcal{V}_x^r \) of the action of the group \( \text{Diff}_{x_0} \) on \( J^r_{x_0} M \) such that:

(a) \( \mathcal{V}_x^r \) is isomorphic to a linear representation of the group \( O(p, q) \),
\[ \mathcal{V}_x^r \xrightarrow{\sim} N_2 \times \cdots \times N_r, \quad f_{x_0}^r g \mapsto (g_{x_0}^2, \ldots, g_{x_0}^r). \]

(b) There exists an isomorphism of ringed spaces
\[ \mathcal{V}_x^r/O(p, q) \xrightarrow{\sim} (J^r_{x_0} M)/\text{Diff}_{x_0} \cong \mathbb{M}^r_{p, q}. \]

In conclusion, the moduli space \( \mathbb{M}^r_{p, q} \) is isomorphic to the quotient space of a linear representation of the group \( O(p, q) \),
\[ (N_2 \times \cdots \times N_r)/O(p, q) \xrightarrow{\sim} \mathbb{M}^r_{p, q}. \]

4. Structure of the moduli spaces

Let \( V \) be a finite-dimensional linear representation of a reductive Lie group \( G \). The \( \mathbb{R} \)-algebra of \( G \)-invariant polynomials on \( V \) is finitely generated (Hilbert–Nagata theorem, see [3]). Let \( p_1, \ldots, p_s \) be a finite set of generators for that algebra; by a result of Luna [6], every smooth \( G \)-invariant function \( f \) on \( V \) can be written as \( f = F(p_1, \ldots, p_s) \), for some smooth function \( F \in C^\infty(\mathbb{R}^s) \).

**Theorem 4.1 (Finiteness of differential invariants).** There exists a finite number \( p_1, \ldots, p_s \in C^\infty(\mathbb{M}^r_{p, q}) \) of scalar differential invariants of order \( \leq r \) such that any other differential invariant \( f \) of order \( \leq r \) is a smooth function of the former ones, i.e. \( f = F(p_1, \ldots, p_s) \), for a certain \( F \in C^\infty(\mathbb{R}^s) \).

**Proof.** By the Fundamental Lemma 3.4,
\[ C^\infty(\mathbb{M}^r_{p, q}) = C^\infty((N_2 \times \cdots \times N_r)/O(p, q)), \]
and we can conclude by applying the above theorem by Luna to the linear representation \( N_2 \times \cdots \times N_r \) of the orthogonal group \( O(p, q) \). \( \square \)

This theorem is stated in [8] for the Riemannian case.

**Remark 4.2.** Using the theory of invariants for the orthogonal group and the fact that the sequence of normal tensors \( \{ g_{x_0}, g_{x_0}^2, g_{x_0}^3, \ldots, g_{x_0}^r \} \) is equivalent to the sequence \( \{ g_{x_0}, R_{x_0}, \nabla_{x_0} R, \ldots, \nabla_{x_0}^r R \} \), it can be proved that the generators \( p_1, \ldots, p_s \) of Theorem 4.1 can be chosen to be Weyl invariants, that is, scalar quantities constructed from the sequence \( \{ g_{x_0}, R_{x_0}, \nabla_{x_0} R, \ldots, \nabla_{x_0}^r R \} \) by reiteration of the following operations: tensor products, raising and lowering indices, and contractions.

**Theorem 4.3.** Scalar differential invariants of order \( \leq r \) separate points in the moduli space \( \mathbb{M}^r_{p, q} \).
Consequently, scalar differential invariants of order \( \leq r \) classify \( r \)-jets of Riemannian metrics (at a point).
Proof. For positive definite metrics, the orthogonal group \( O(n) \) is compact. It is a well-known fact that, if \( V \) is a linear representation of a compact Lie group \( G \), then smooth \( G \)-invariant functions on \( V \) separate the orbits of the action of \( G \), or, in other words, the algebra \( C^\infty(V/G) \) separates the points in \( V/G \).

This, together with the Fundamental Lemma, proves the statement. \( \square \)

Neither assertion in Theorem 4.3 is valid for pseudo-Riemannian metrics. See note in Section 5.2 for a counterexample. For such metrics, moduli spaces \( \mathcal{M}_{p,q}^n \) are generally pathological in a topological sense, since they have non-closed points (they are not \( T_1 \) topological spaces).

In the Riemannian case, Schwarz Theorem 1.9 and the Fundamental Lemma directly provide the following

**Theorem 4.4.** Moduli spaces \( \mathcal{M}_{n}^r \) are differentiable spaces.

More precisely: Let \( p_1, \ldots, p_s \) be the basis of differential invariants of order \( \leq r \) mentioned in Theorem 4.1. These invariants induce an isomorphism of differentiable spaces

\[
(p_1, \ldots, p_s) : \mathcal{M}_{n}^r \cong Z \subseteq \mathbb{R}^s,
\]

\( Z \) being a closed subspace of \( \mathbb{R}^s \).

Although the differentiable space \( \mathcal{M}_{n}^r \) is not in general a smooth manifold, its structure is not so deficient as it could seem at first sight, since we are going to prove that it admits a finite stratification by certain smooth submanifolds.

**Definition 4.5.** Let us consider \( V_n = \mathbb{R}^n \) endowed with its standard inner product \( \delta \), and the corresponding orthogonal group \( O(n) := O(V_n, \delta) \). We will denote by \( \mathcal{T} \) the set of conjugacy classes of closed subgroups in \( O(n) \).

Given another \( n \)-dimensional vector space \( \overline{V}_n \) with an inner product \( \tilde{\delta} \), we can also consider the set \( \mathcal{\bar{T}} \) of conjugacy classes of closed subgroups in \( O(\overline{V}_n, \tilde{\delta}) \).

Observe that there exists a canonical identification

\[
\mathcal{T} \longrightarrow \mathcal{\bar{T}}, \quad [H] \longmapsto [\varphi \circ H \circ \varphi^{-1}],
\]

where \( \varphi \) stands for any isometry \( \varphi : V_n \to \overline{V}_n \).

As the identification does not depend on the choice of the isometry \( \varphi \), from now on we will suppose that the set \( \mathcal{T} \) is just “the same” for every pair \((V_n, \delta)\).

Note that \( \mathcal{T} \) possesses a partial order relation: \( [H] \leq [H'] \), if there exist some representatives \( H \) and \( H' \) of \( [H] \) and \( [H'] \) respectively, such that \( H \subseteq H' \).

**Definition 4.6.** The **group of automorphisms** of a Riemannian metric jet \( j_{x_0}^r g \) is defined to be the stabilizer subgroup \( \text{Aut}(j_{x_0}^r g) \subseteq \text{Diff}_{x_0}^{r+1} \) of \( j_{x_0}^r g \):

\[
\text{Aut}(j_{x_0}^r g) := \{ j_{x_0}^{r+1} \tau \in \text{Diff}_{x_0}^{r+1} : j_{x_0}^r (\tau^* g) = j_{x_0}^r g \}.
\]

Given \( \tau \in \text{Diff}_{x_0} \), let us denote by \( \tau_{x_0} : T_{x_0} X \to T_{x_0} X \) the linear tangent map of \( \tau \) at \( x_0 \).

**Lemma 4.7.** The group morphism

\[
\text{Aut}(j_{x_0}^r g) \longrightarrow O(T_{x_0} X, g_{x_0}) \simeq O(n)
\]

\[
j_{x_0}^{r+1} \tau \longmapsto \tau_{x_0, x_0}
\]

is injective.

**Proof.** For any \( \tau \in \text{Diff}_{x_0} \) and any metric \( g \) on \( X \) we have the following commutative diagram of local diffeomorphisms:

\[
\begin{array}{ccc}
T_{x_0} X & \xrightarrow{\exp_{x_0} \tau} & X \\
\downarrow \tau & & \downarrow \\
T_{x_0} X & \xrightarrow{\exp_g} & X
\end{array}
\]

If \( j_{x_0}^{r+1} \tau \in \text{Aut}(j_{x_0}^r g) \), that is, \( j_{x_0}^r (\tau^* g) = j_{x_0}^r g \), then \( j_{x_0}^{r+1} (\exp_{x_0} \gamma) = j_{x_0}^{r+1} (\exp_g) \) because of Proposition 1.11.
Now, taking \((r + 1)\)-jets in the above diagram, we obtain:
\[
j_{x_0}^{r+1} \tau = j_{y_0}^{r+1}(\exp_g) \circ j_{y_0}^{r+1} \tau \circ j_{x_0}^{r+1}(\exp_{g^{-1}}).
\]
hence \(j_{x_0}^{r+1} \tau\) is determined by its linear part \(\tau_+\).

By the previous lemma, the group \(\text{Aut}(j_{x_0}^r g)\) can be viewed as a subgroup (determined up to conjugacy) of the orthogonal group \(O(n)\).

**Definition 4.8.** The type map is defined to be the map
\[
t : \mathcal{M}_n^r \to \mathcal{T}, \quad [j_{x_0}^r g] \mapsto [\text{Aut}(j_{x_0}^r g)].
\]
For each \([H] \in \mathcal{T}\), the stratum of type \([H]\) is said to be the subset \(S[H] \subseteq \mathcal{M}_n^r\) of those points of type \([H]\).

**Theorem 4.9** (Stratification of the moduli space). The type map \(t : \mathcal{M}_n^r \to \mathcal{T}\) verifies the following properties:

1. \(t\) takes a finite number of values \([H_0], \ldots, [H_k]\), one of which, say \([H_0]\), is minimum.
2. Semicontinuity: For every type \([H] \in \mathcal{T}\), the set of points in \(\mathcal{M}_n^r\) of type \(\leq [H]\) is an open subset of \(\mathcal{M}_n^r\). In particular, every stratum \(S[H_0]\) is a locally closed subspace of \(\mathcal{M}_n^r\).
3. Every stratum \(S[H_1]\) is a smooth submanifold of \(\mathcal{M}_n^r\).
4. The (also called generic) stratum \(S[H_0]\) of minimum type is a dense connected open subset of \(\mathcal{M}_n^r\).

**Proof.** Fix a positive definite metric \(\delta_{x_0}\) on \(T_{x_0}X\) and denote by \(O(n)\) its orthogonal group. The Fundamental Lemma 3.4 tells us that there exists an isomorphism
\[
\mathcal{M}_n^r \cong (N_2 \times \cdots \times N_r)/O(n).
\]
This isomorphism takes every class \([j_{x_0}^r g] \in \mathcal{M}_n^r\) with \(g_{x_0} = \delta_{x_0}\), to the sequence of normal tensors \([g_{x_0}^2, \ldots, g_{x_0}^r] \in (N_2 \times \cdots \times N_r)/O(n)\).

Let us check that the subgroup \(\text{Aut}(j_{x_0}^r g) \hookrightarrow O(n), j_{x_0}^{r+1} \tau \mapsto \tau_+\) coincides with the subgroup
\[
\text{Aut}(g_{x_0}^2, \ldots, g_{x_0}^r) := \{ \sigma \in O(n) : \sigma^*(g_{x_0}^k) = g_{x_0}^k, \forall k \leq r\}.
\]
It is clear that if an automorphism \(j_{x_0}^{r+1} \tau\) leaves \(j_{x_0}^r g\) fixed, then the sequence of its normal tensors must also remain fixed by the automorphism: \(\tau^*(g_{x_0}^k) = g_{x_0}^k\).

Conversely, given an automorphism \(\sigma : T_{x_0}X \to T_{x_0}X\) of the sequence of normal tensors \((g_{x_0}^2, \ldots, g_{x_0}^r)\), let us consider a normal system \((z_1, \ldots, z_r)\) for \(g\) at \(x_0\).
Via the identification provided by the exponential map \(\exp_g : T_{x_0}X \to X\), the map \(\sigma\) can be viewed as a diffeomorphism of \(X\) (a linear transformation of normal systems).

In normal coordinates, the expression of the normal tensor \(g_{x_0}^k\) corresponds to the expression of the homogeneous part of degree \(k\) of the jet \(j_{x_0}^r g\). Hence it is an immediate consequence that the linear transformation \(\sigma\) leaves \(j_{x_0}^r g\) fixed, i.e. \(j_{x_0}^{r+1} \sigma \in \text{Aut}(j_{x_0}^r g)\).

The identity \(\text{Aut}(j_{x_0}^r g) = \text{Aut}(g_{x_0}^2, \ldots, g_{x_0}^r)\) implies that the following diagram is commutative:
\[
\begin{array}{ccc}
\mathcal{M}_n^r & \xrightarrow{t} & \mathcal{T} \\
\downarrow & & \downarrow \\
(N_2 \times \cdots \times N_r)/O(n) & \xrightarrow{t} & \mathcal{T} \\
\left[g_{x_0}^2, \ldots, g_{x_0}^r\right] & \mapsto & [\text{Aut}(g_{x_0}^2, \ldots, g_{x_0}^r)].
\end{array}
\]

Therefore, our theorem has come down to the case of a linear representation \(V (= N_2 \times \cdots \times N_r)\) of a compact Lie group \(G (= O(n))\) and the corresponding type map:
\[
V/G \xrightarrow{\tau} \mathcal{T} = \{\text{conjugacy classes of closed subgroups of } G\}
\]
\([v] \mapsto [\text{Stabilizer subgroup of } v].
\]

For this type map, the analogous properties to 1–4 in the statement are well known (see [1, Chap. IX, Section 9, Th. 2 and Exer. 9]).
Remark 4.10. Except for trivial cases, the generic stratum has type \( H_0 = [0] \).

Remark 4.11. The dimension of the moduli space \( M_n^r \) (or rather that of its generic stratum) can be deduced directly from the Fundamental Lemma and the formulae giving the dimensions of spaces \( N_r \) of normal tensors which were presented in Section 1.

The result (due, in a different language, to J. Muñoz and A. Valdés [9]) is as follows:

\[
\dim M_n^0 = \dim M_n^1 = 0, \quad \forall n \geq 1,
\]
\[
\dim M_n^1 = 0, \quad \forall r \geq 0,
\]
\[
\dim M_n^2 = 1, \quad \dim M_n^r = \frac{1}{2} (r + 1)(r - 2), \quad \forall r \geq 3,
\]
\[
\dim M_n^r = n + \frac{(r - 1)n^2 - (r + 1)n}{2(r + 1)} \left( \frac{n + r}{r} \right), \quad \forall n \geq 3, \quad r \geq 2.
\]

Remark 4.12. Theorem 4.9 is not valid for pseudo-Riemannian metrics. A detailed analysis of the moduli space of 2-jets of metrics of signature \( (p = 2, q = 1) \) shows the following facts:

The type map is not semicontinuous, some strata are neither smooth manifolds nor separated spaces, and the stratum of the minimal type is neither an open nor a dense subset. Therefore, in the pseudo-Riemannian case, fixing the type is not sufficient to obtain smooth strata.

5. Moduli spaces in dimension \( n = 2 \)

5.1. Stratification

We are going to determine the stratification of moduli spaces \( M_n^r \) of \( r \)-jets of Riemannian metrics in dimension \( n = 2 \). Let us consider the vector space \( \mathbb{R}^2 = \mathbb{C} \), endowed with the standard Euclidean metric, and its corresponding orthogonal group \( O(2) \). We will denote by \((x, y)\) the Cartesian coordinates and by \( z = x + iy \) the complex coordinate.

Let us denote by \( \sigma_m : \mathbb{C} \to \mathbb{C} \) the rotation of angle \( 2\pi / m \) (that is, \( \sigma_m(z) = \epsilon_m z \), with \( \epsilon_m = \cos(2\pi / m) + i \sin(2\pi / m) \) a primitive \( m \)-th root of unity) and by \( \tau : \mathbb{C} \to \mathbb{C} \), \( \tau(z) = \bar{z} \) the complex conjugation.

The only (up to conjugacy) closed subgroups of \( O(2) \) are the following ones:

\[
SO(2) := \{ \varphi \in O(2) : \det \varphi = 1 \} \quad (\text{special orthogonal group}),
\]
\[
K_m := \langle \sigma_m \rangle \quad (\text{group of rotations of order } m \) \text{ } (m \geq 1),
\]
\[
D_m := \langle \sigma_m, \tau \rangle \quad (\text{dihedral group of order } 2m \) \text{ } (m \geq 1),
\]

and \( O(2) \) itself. All these subgroups are normal but the dihedral \( D_m \).

The subgroup \( SO(2) \) of rotations is identified with the multiplicative group \( S_1 \subset \mathbb{C} \) of complex numbers of modulus 1,

\[
S_1 \longrightarrow SO(2), \quad \alpha \longmapsto \rho_\alpha, \quad \rho_\alpha(z) := \alpha z.
\]

Besides, every element in \( O(2) \) is either \( \rho_\alpha \) or \( \tau \rho_\alpha \), for some \( \alpha \in S_1 \).

The action of \( O(2) \) on \( \mathbb{R}^2 \) induces an action on the algebra \( \mathbb{R}[x, y] \) of the polynomials on \( \mathbb{R}^2 \), to be more specific:

\[
\varphi \cdot P(x, y) := P(\varphi^{-1}(x, y)).
\]

The following lemma provides us with the list of all invariant polynomials with respect to each of the subgroups of \( O(2) \) above mentioned:

Lemma 5.1. The following identities hold:

1. \( \mathbb{R}[x, y]^{K_m} = \mathbb{R}[x^2 + y^2, p_m(x, y), q_m(x, y)] \).
2. \( \mathbb{R}[x, y]^{D_m} = \mathbb{R}[x^2 + y^2, p_m(x, y)] \).
3. \( \mathbb{R}[x, y]^{O(2)} = \mathbb{R}[x, y]^{O(2)} = \mathbb{R}[x^2 + y^2] \).

with \( p_m(x, y) = \text{Re}(z^m) \) and \( q_m(x, y) = \text{Im}(z^m) \).

Proof. 1. Let us consider the algebra of polynomials on \( \mathbb{R}^2 \) with complex coefficients,

\[
\mathbb{C}[x, y] = \mathbb{C}[z, \bar{z}] = \bigoplus_{ab} \mathbb{C}z^a \bar{z}^b.
\]
Every summand is stable under the action of $K_m$, since
\[
\sigma_m \cdot (z^a \bar{z}^b) = \frac{1}{\varepsilon_m}\varepsilon_m^{-1} z^a \bar{z}^b = \varepsilon_m^{-1} z^a \bar{z}^b.
\]

This formula also tells us that the monomial $z^a \bar{z}^b$ is invariant by $K_m$ if and only if $b - a \equiv 0 \mod m$, that is, $b - a = \pm km$ for some $k \in \mathbb{N}$. Then invariant monomials are of the form
\[
z^a \bar{z}^b = (z\bar{z})^{2km} \quad \text{or} \quad z^a \bar{z}^b = (z\bar{z})^b z\bar{z}^m,
\]
whence
\[
\mathbb{C}[x, y]^{K_m} = \mathbb{C}[z\bar{z}, z^m, \bar{z}^m].
\]

As $z\bar{z} = x^2 + y^2$, $z^m + \bar{z}^m = 2p_m(x, y)$ and $z^m - \bar{z}^m = 2i q_m(x, y)$, we can conclude that
\[
\mathbb{C}[x, y]^{K_m} = \mathbb{C}[x^2 + y^2, p_m(x, y), q_m(x, y)],
\]
and particularly,
\[
\mathbb{R}[x, y]^{K_m} = \mathbb{R}[x^2 + y^2, p_m(x, y), q_m(x, y)].
\]

2. As $D_m = (K_m, \tau)$, we get
\[
\mathbb{C}[x, y]^{D_m} = \left(\mathbb{C}[x, y]^{K_m}\right)^{\tau}
\]
\[
= \mathbb{C}[z\bar{z}, z^m, \bar{z}^m]^{\tau} = \left[\left(\bigoplus_k \mathbb{C}[z\bar{z}] z^{km}\right) \oplus \left(\bigoplus_k \mathbb{C}[z\bar{z}] \bar{z}^{km}\right)\right]^{\tau}
\]
\[
= \bigoplus_k \mathbb{C}[z\bar{z}] (z^{km} + \bar{z}^{km}) = \mathbb{C}[z\bar{z}, z^m, \bar{z}^m] = \mathbb{C}[x^2 + y^2, p_m(x, y)],
\]
and, in particular,
\[
\mathbb{R}[x, y]^{D_m} = \mathbb{R}[x^2 + y^2, p_m(x, y)].
\]

3. Every summand in the decomposition
\[
\mathbb{C}[z\bar{z}] = \bigoplus_{ab} \mathbb{C} z^a \bar{z}^b
\]
is stable under the action of $SO(2)$, since for every $\rho_u \in SO(2)$ it is satisfied:
\[
\rho_u \cdot (z^a \bar{z}^b) = \frac{1}{a^u b^u} z^a \bar{z}^b.
\]

Moreover, this formula assures that the only monomials $z^a \bar{z}^b$ which are $SO(2)$-invariant are those verifying $a = b$. Then,
\[
\mathbb{C}[x, y]^{SO(2)} = \mathbb{C}[z\bar{z}]^{SO(2)} = \mathbb{C}[z\bar{z}] = \mathbb{C}[x^2 + y^2],
\]
whence
\[
\mathbb{R}[x, y]^{SO(2)} = \mathbb{R}[x^2 + y^2].
\]

Finally, this identity proves that $SO(2)$-invariant polynomials are $O(2)$-invariant too, so the obvious inclusion $\mathbb{R}[x, y]^{SO(2)} \subseteq \mathbb{R}[x, y]^{O(2)}$ is indeed an equality. \(\square\)

**Corollary 5.2.** With the same notations used in the previous lemma, it is verified:

1. $D_m$ is the stabilizer subgroup of the polynomial $p_m(x, y)$, and there exists no polynomial in $\mathbb{R}[x, y]$ of degree $< m$ whose stabilizer subgroup is $D_m$.
2. $K_m (m \geq 2)$ is the stabilizer subgroup of the polynomial $p_m(x, y) + (x^2 + y^2)q_m(x, y)$, and there exists no polynomial in $\mathbb{R}[x, y]$ of degree $< m + 2$ whose stabilizer subgroup is $K_m$. 

---

3. $K_1 = \{Id\}$ is the stabilizer subgroup of the polynomial $x + xy$, and there exists no polynomial in $\mathbb{R}[x, y]$ of degree $< 2$ whose stabilizer subgroup is $K_1$.

**Proof.** 1. Using that every element in $O(2)$ is either of the form $\rho_\alpha$ or of the form $\rho_\mu \circ \tau$, it is a matter of routine to check that the stabilizer subgroup of the polynomial $p_m(x, y) = \text{Re}(z^m)$ is $D_m$.

If there were another polynomial $\bar{p}(x, y)$ of degree $< m$ with the same property, $\bar{p}(x, y)$ should be a power of $x^2 + y^2$, because of Lemma 5.1(2), and in that case its stabilizer subgroup would be the whole $O(2)$, against our hypothesis.

2. According to Lemma 5.1(1), every $K_m$-invariant polynomial of degree $\leq m$ is of the form $\lambda p_m(x, y) + \mu q_m(x, y)$ (up to addition of a power of $x^2 + y^2$). However, a polynomial of such a form does not have $K_2$ as its stabilizer subgroup, but a larger dihedral group: after multiplying by a scalar, we can indeed assume $\lambda^2 + \mu^2 = 1$; if $\alpha = \lambda - i\mu$, then

$$
\lambda p_m(x, y) + \mu q_m(x, y) = \text{Re}(\alpha z^m) = \text{Re}((\beta z)^m) \quad (\text{with } \beta^m = \alpha)
$$

whose stabilizer subgroup is the dihedral group $\rho_{\beta^{-1}} \cdot D_m \cdot \rho_\beta$, which is conjugate to the stabilizer subgroup $D_m$ of $p_m(x, y)$.

In particular, taking $\lambda = 0$, $\mu = -1$, we get that the stabilizer subgroup of $q_m(x, y)$ is $\rho_{\beta^{-1}} \cdot D_m \cdot \rho_\beta$, for $\beta^m = i$.

As no polynomial of degree $\leq m$ has the desired stabilizer subgroup $K_m$, and there are not any $K_m$-invariant polynomials of degree $m + 1$ (up to a power of $x^2 + y^2$), the following degree to be considered is $m + 2$. The stabilizer subgroup of the polynomial $p_m(x, y) + (z^2 + y^2)q_m(x, y)$, of degree $m + 2$, is the intersection of the stabilizer subgroups of its two homogeneous components, $p_m(x, y)$ and $(z^2 + y^2)q_m(x, y)$, that is,

$$
D_m \cap (\rho_{\beta^{-1}} \cdot D_m \cdot \rho_\beta) = K_m \quad (\beta^m = i).
$$

3. This case is trivial. \(\square\)

**Theorem 5.3.** The strata in the moduli space $M_2^r$ correspond exactly to the following types: $[O(2)], [D_1], \ldots, [D_{r-2}], [K_1], \ldots, [K_{r-4}]$. (And also $[K_1]$ if $r = 4$.)

**Proof.** It is a classical result (see [2]) that in dimension 2 every Riemannian metric can be written in normal coordinates $(x, y)$ (in a unique way up to an orthogonal transformation) as follows:

$$
g = dx^2 + dy^2 + h(x, y)(y \; dx - x \; dy)^2,
$$

for some smooth function $h(x, y)$.

Observe that the stabilizer subgroup of $O(2)$ for the jet $j^k_h$ is the same as that for $j^{k^2}_g$.

If we take $h(x, y) = 0$, we get a metric (the Euclidean one, i.e. $g = dx^2 + dy^2$) whose group of automorphisms (for any jet order) is $O(2)$.

Choosing $h(x, y) = p_m(x, y)$, we obtain an $r$-jet of metrics (with $r \geq m + 2$) whose stabilizer subgroup is $D_m$, because of Corollary 5.2(1).

If we choose $h(x, y) = p_m(x, y) + (z^2 + y^2)q_m(x, y)$, we get an $r$-jet of metrics (with $r \geq m + 4$) whose stabilizer subgroup is $K_m$, by Corollary 5.2(2).

If we make $h(x, y) = x + xy$, then we get an $r$-jet of metrics (with $r \geq 4$) whose stabilizer subgroup is $K_1$, according to Corollary 5.2(3).

Finally, let us note that no $r$-jet of metrics can have $SO(2)$ as its stabilizer subgroup, since such a metric would correspond to a jet $j^r_0 h$ whose stabilizer subgroup should be $SO(2)$, which is impossible, because, by Lemma 5.1(3), every $SO(2)$-invariant polynomial is also $O(2)$-invariant. \(\square\)

**Corollary 5.4.** Every closed subgroup of $O(2)$, except for $SO(2)$, is the group of automorphisms of a metric jet $j^r_0 g$ on $\mathbb{R}^2$ for some order $r$.

**Corollary 5.5.** The number of strata in $M_2^r$ is:

$$
\text{Number of strata in } M_2^r = \begin{cases} 
1 & \text{for } r = 0, 1, 2, \\
2 & \text{for } r = 3, \\
4 & \text{for } r = 4, \\
2r - 5 & \text{for } r \geq 5.
\end{cases}
$$

**5.2. Examples**

Now we describe, without proofs, low order jets in dimension $n = 2$.

For order $r = 0, 1$ (and in any dimension $n$) moduli spaces $M_2^r$ come down to a single point.
The moduli space is a line:

$$M^2_2 \rightarrow \mathbb{R}, \quad [j^2_{x_0}g] \mapsto K_g(x_0).$$

In other words, the curvature classifies 2-jets of Riemannian metrics in dimension \(n = 2\).

In this case there is just one stratum, the generic one, whose type is \([O(2)]\).

**Case \(r = 3\).**

The moduli space is a closed semiplane:

$$M^3_2 \rightarrow \mathbb{R} \times [0, +\infty), \quad [j^3_{x_0}g] \mapsto (K_g(x_0), |\text{grad}_{x_0}K_g|^2).$$

That is to say, the curvature and the square of the modulus of the gradient of the curvature classify 3-jet of metrics in dimension \(n = 2\).

Now we have two different strata:

The generic stratum \(S_{D_1} = \mathbb{R} \times (0, +\infty), \) with type \([D_1]\). This stratum is the set of all classes of jets \(j^3_{x_0}g\) verifying \(\text{grad}_{x_0}K_g \neq 0\) (in this case, the group of automorphisms is the group of order 2 generated by the reflection across the vector \(\text{grad}_{x_0}K_g\)).

The non-generic stratum \(S_{[O(2)]} = \mathbb{R} \times \{0\}, \) with type \([O(2)]\), is the set of all classes of jets \(j^3_{x_0}g\) verifying \(\text{grad}_{x_0}K_g = 0\) (which are invariant with respect to every orthogonal transformation of normal coordinates).

**Note.** If we consider metrics of signature \((+, -)\), instead of Riemannian metrics, then the map

$$M^3_2 \rightarrow \mathbb{R} \times [0, +\infty), \quad [j^3_{x_0}g] \mapsto (K_g(x_0), |\text{grad}_{x_0}K_g|^2)$$

is not injective, that is, differential invariants do not classify 3-jet of metrics of signature \((+, -)\). To illustrate this, consider two metrics \(g, \tilde{g}\) of signature \((+, -)\), such that \(K_g(x_0) = K_{\tilde{g}}(x_0)\), \(\text{grad}_{x_0}K_g = 0\) and \(\text{grad}_{x_0}K_{\tilde{g}}\) is a non-zero isotropic vector with respect to \(\tilde{g}_{x_0}\). Both jets \(j^3_{x_0}g, j^3_{x_0}\tilde{g}\) cannot be equivalent (because the gradient of the curvature at \(x_0\) equals zero for the first metric, whereas it is non-zero for the other one), but its differential invariants coincide: \(K_g(x_0) = K_{\tilde{g}}(x_0)\) and \(|\text{grad}_{x_0}K_g|^2 = |\text{grad}_{x_0}K_{\tilde{g}}|^2 = 0\).

**Case \(r = 4\).**

A set of generators for differential invariants of order 4 is given by the following five functions:

\[
\begin{align*}
p_1(j^4_{x_0}g) &= K_g(x_0), \\
p_2(j^4_{x_0}g) &= |\text{grad}_{x_0}K_g|^2, \\
p_3(j^4_{x_0}g) &= \text{trace}(\text{Hess}_{x_0}K_g), \\
p_4(j^4_{x_0}g) &= \text{det}(\text{Hess}_{x_0}K_g), \\
p_5(j^4_{x_0}g) &= \text{Hess}_{x_0}K_g(\text{grad}_{x_0}K_g, \text{grad}_{x_0}K_g),
\end{align*}
\]

where \(\text{Hess}_{x_0}K_g := (\nabla dK_g)_{x_0}\) stands for the hessian of the curvature function at \(x_0\).

These above functions satisfy the following inequalities:

\[
p_2 \geq 0, \quad p_2^2 - 4p_4 \geq 0, \quad (2p_5 - p_2p_3)^2 \leq p_2^2(p_3^2 - 4p_4).
\]

To say it in other words, these five differential invariants define an isomorphism of differentiable spaces

\[ (p_1, \ldots, p_5): M^4_2 \rightarrow Y \subseteq \mathbb{R}^5, \]

\(Y\) being the closed subset in \(\mathbb{R}^5\) determined by the inequalities

\[
\begin{align*}
x_2 &\geq 0, \\
x_3^2 - 4x_4 &\geq 0, \\
(2x_5 - x_2x_3)^2 &\leq x_2^2(x_3^2 - 4x_4).
\end{align*}
\]

In this case, the moduli space \(M^6_2\) has the following four strata:
– The generic stratum of all classes of jets $j^4_{x_0}g$ verifying that $\text{grad}_{x_0}K_g$ is not an eigenvector of $\text{Hess}_{x_0}K_g$ (therefore, the eigenvalues of $\text{Hess}_{x_0}K_g$ are different). The type of this stratum (group of automorphisms of its jets) is $[K_1 = \{\text{Id}\}]$ (Fig. 1).

![Fig. 1. Generic stratum.](image)

– The stratum of those classes of metric jets $j^4_{x_0}g$ verifying that $\text{grad}_{x_0}K_g$ is a non-zero eigenvector of $\text{Hess}_{x_0}K_g$. Its type is $[D_1]$: the group of automorphisms of each metric jet is generated by the reflection across the vector $\text{grad}_{x_0}K_g$ (Fig. 2).

![Fig. 2. Stratum $S_{[D_1]}$.](image)

– The stratum composed of those classes of metric jets $j^4_{x_0}g$ with $\text{grad}_{x_0}K_g = 0$ and verifying that the eigenvectors of $\text{Hess}_{x_0}K_g$ are different. The type of this stratum is $[D_2]$: the group of automorphisms of each metric jet is generated by the reflections across either eigenvector of $\text{Hess}_{x_0}K_g$ (Fig. 3).

![Fig. 3. Stratum $S_{[D_2]}$.](image)

– The stratum of all classes of jets $j^4_{x_0}g$ with $\text{grad}_{x_0}K_g = 0$ and verifying that the eigenvectors of $\text{Hess}_{x_0}K_g$ are both equal. The type of the stratum is $[O(2)]$ (Fig. 4).

![Fig. 4. Stratum $S_{[O(2)]}$.](image)
Appendix A. Classification of $\infty$-jets of metrics

In Section 4 we have seen that differential invariants of order $\leq r$ classify $r$-jets of Riemannian metrics at a point (Theorem 4.3). We are now going to generalize this result for infinite-order jets.

In the proof of next lemma we will use the following well-known fact [1, Chap. IX, Section 9, Lemma 6]:

Let $G$ be a compact Lie group. Every decreasing sequence of closed subgroups $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$ stabilizes, that is, there exists an integer $s$ such that $H_s = H_{s+1} = H_{s+2} = \cdots$.

**Lemma A.1.** Let $G$ a compact Lie group and let

$$\cdots \longrightarrow X_{r+1} \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1$$

be an inverse system of smooth $G$-equivariant maps between smooth manifolds endowed with a smooth action of $G$. There exists an isomorphism of ringed spaces:

$$\left(\lim X_r \right)/G \overset{\sim}{\longrightarrow} \lim (X_r / G)$$

$$[(\ldots, x_2, x_1)] \longmapsto (\ldots [x_2], [x_1]).$$

**Proof.** Because of the universal quotient property, compositions of morphisms

$$\lim X_r \longrightarrow X_r \longrightarrow X_r / G$$

$$(\ldots, x_2, x_1) \longmapsto x_r \longmapsto [x_r]$$

induce morphisms of ringed spaces

$$(\lim X_r )/G \longrightarrow (X_r / G)$$

$$[(\ldots, x_2, x_1)] \longmapsto [x_r].$$

which, for their part, because of the universal inverse limit property, define a morphism of ringed spaces

$$\phi : \left(\lim X_r \right)/G \longrightarrow \lim (X_r / G)$$

$$[(\ldots, x_2, x_1)] \longmapsto (\ldots [x_2], [x_1]).$$

It is easy to check that this morphism is surjective. Let us see that it is also injective.

First note that, given a point $[(\ldots, x_2, x_1)] \in \lim X_r$, we can get the decreasing sequence $H_{x_1} \supseteq H_{x_2} \supseteq H_{x_3} \supseteq \cdots$ of closed subgroups of $G$, where $H_{x_k}$ stands for the stabilizer subgroup of $x_k$. This chain stabilizes, since $G$ is compact, so for a certain $s$ it is verified $H_{x_k} = H_{x_{k+1}} = H_{x_{k+2}} = \cdots$.

Let now $[(\ldots, x_2, x_1)]$ and $[(\ldots, x_2, x_1')]$ be two points in $\lim X_r / G$ having the same image through $\phi$, i.e. $[x_k] = [x_k']$, for each $k \geq 0$. Write $x'_s = g \cdot x_s$ for some $g \in G$. As the morphisms $x_s \longrightarrow x_k$ (with $s \geq k$) are $G$-equivariant, it is verified that $x'_s = g \cdot x_k$ for every $k \leq s$.

Let us show that the same happens when $k > s$. As $[x_k] = [x'_k]$, we have $x'_k = g_k \cdot x_k$ for a certain $g_k \in G$; applying that $x_k \longrightarrow x_s$ is equivariant yields $x'_k = g_k \cdot x_s$, and then (comparing with $x'_s = g \cdot x_s$) $g^{-1}g_k \in H_{x_s}$; since $H_{x_s} = H_{x_k}$, it follows that $g^{-1}g_k \in H_{x_k}$, and hence the condition $x'_k = g_k \cdot x_k$ is equivalent to $x'_s = g \cdot x_s$. In conclusion, $x'_k = g \cdot x_k$ for every $k > 0$, and therefore $[(\ldots, x_2, x_1)]$ and $[(\ldots, x_2, x_1')]$ are the same point in $\lim X_r / G$.

Once we have proved that $\phi$ is bijective, it is routine to check that $\phi$ is an isomorphism of ringed spaces. \[\Box\]

**Definition A.2.** Let $x_0 \in X$ and let

$$J_{x_0}^\infty M := \lim_{\leftarrow} J^r_{x_0} M$$

be the ringed space of $\infty$-jets of Riemannian metrics at $x_0$ on $X$. The quotient ringed space

$$M_n^\infty := J_{x_0}^\infty M / \text{Diff}_{x_0}$$

is called moduli space of $\infty$-jets of Riemannian metrics in dimension $n$.

In the same fashion as for finite-order jets, the moduli space $M_n^\infty$ depends neither on the choice of the point $x_0$ nor on that of the $n$-dimensional manifold $X$.

For every integer $r > 0$ we have an evident morphism of ringed spaces

$$M_n^\infty \longrightarrow M_n^r$$

$$[J_{x_0}^\infty g] \longmapsto [J_{x_0}^r g],$$
and these morphisms allow to define another morphism of ringed spaces:

\[ \lim \mathcal{M}_n^\infty \rightarrow \lim \mathcal{M}_n^r \]

\[ \left[ j_0^\infty g \right] \mapsto \left( \ldots \left[ j_0^r g \right] \ldots \right). \]

**Theorem A.3.** There exists an isomorphism of ringed spaces

\[ \mathcal{M}_n^\infty \rightarrow \lim \mathcal{M}_n^r \]

\[ \left[ j_0^\infty g \right] \mapsto \left( \ldots \left[ j_0^r g \right] \ldots \right). \]

**Proof.** Fix a chart \((z_1, \ldots, z_n)\) centered at \(x_0\). With the same notations as in Section 3, let us define

\[ \mathcal{N}^\infty := \lim \mathcal{N}^r. \]

In other words, \(\mathcal{N}^\infty\) is the subspace of \(j_0^\infty M\) formed by all those \(\infty\)-jets at \(x_0\) of Riemannian metrics having \((z_1, \ldots, z_n)\) as a normal system. All lemmas in Section 3, with their corresponding proofs, remain valid when substituting the integer \(r\) for \(\infty\). In particular, our fundamental Lemma 3.4, when \(r = \infty\), gives the desired isomorphism of ringed spaces:

\[ \lim \left( \prod_{k \geq 1} N_k \right) / O(n) = \left( \lim (N_2 \times \cdots \times N_r) / O(n) \right) \]

(by Lemma A.1)

\[ \lim \left( (N_2 \times \cdots \times N_r) / O(n) \right) = \lim \mathcal{M}_n^r. \]

**Remark A.4.** We do not know if the above theorem remains valid in the pseudo-Riemannian case.

**Remark A.5.** \(\mathcal{M}_n^\infty\) is not a mere ringed space. According to a result by Vinogradov [14,13], an open subset of \(\mathcal{M}_n^\infty\) has the structure of a diffiety. The result is obtained not only for metrics but, more generally, for any homogeneous geometric structure which admits \(n\) independent scalar differential invariants. It is the so-called characteristic diffiety of the geometric structure under consideration.

**Corollary A.6.** Scalar differential invariants of finite order classify \(\infty\)-jets of Riemannian metrics: Two metric jets \(j_0^\infty g\) and \(j_0^\infty \bar{g}\) are equivalent if and only if for each finite-order differential invariant \(h\) it is satisfied \(h(\bar{g}),(x_0) = h(g),(x_0)\).

**Proof.** According to Theorem A.3, we get:

\[ j_0^\infty g \equiv j_0^\infty \bar{g} \iff j_0^r g \equiv j_0^r \bar{g}, \quad \forall r \geq 0. \]

To complete our proof, it is sufficient to use the fact that differential invariants of order \(\leq r\) classify \(r\)-jets of metrics (Theorem 4.3).

**References**