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Generalized Bell Numbers and Zeros of Successive Derivatives of an Entire Function

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Six different formulations equivalent to the statement that, for $n \geqslant 2$, the sum $\sum_{k=1}^{n} (-1)^k S(n,k) \neq 0$, where the $S(n,k)$ are Stirling numbers of the second kind, are shown to hold. Using number-theoretic methods, a sufficient condition for the above statement to be true for a set of positive integers n having density 1 is then obtained. It remains open whether it is true for all $n > 2$. The equivalent statements then yield information on the irreducibility of the polynomials $\sum_{k=1}^{n} S(n,k)\, t^{k-1}$ over the rationals, the nonreal zeros for successive derivatives $(d/dz)^n \exp(e^{iz})$, a gap theorem for the nonzero coefficients of $\exp(-e^z)$, and the continuous solution of the differential-difference equation $f(x) = 1$, $0 \leqslant x < 1$, $f'(x) = -\lfloor x \rfloor f(x-1)$, $1 \leqslant x < \infty$, where $\lfloor\ \rfloor$ denotes the greatest integer function.

1. INTRODUCTION

An interesting problem, having arithmetic, combinatorial, and function-theoretic ramifications, is the determinations of the zero coefficients in the Taylor expansion

$$\exp(1 - e^z) = \sum_{n=0}^{\infty} A_n z^n.$$

There is a remarkable variety of different formulations of this problem. One of our main results, which we establish in Section 3, is the following:

THEOREM 1. *For each $n \geqslant 2$, the following are equivalent*:

(a) $\sum_{k=1}^{n} (-1)^k S(n,k) \neq 0$.

(b) *The polynomial* $Q_n(t) = \sum_{k=1}^{n} S(n,k)\, t^{k-1}$ *is irreducible over the rationals.*

(c) *Given that* $\exp(-e^z) = \sum_{k=0}^{\infty} a_k z^k$, *then* $a_n \neq 0$.

(d) *The integral-valued entire function* $A(z)$, *defined by* $A(z) = e \sum_{k=0}^{\infty} (-1)^k k^z / k!$, *does not vanish at* $z = n$.

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(e) *The continuous solution, on* $[0, \infty)$, *of the differential-difference equation*

$$f(x) = 1, \qquad 0 \leqslant x < 1,$$
$$f'(x) = -[x] f(x-1), \qquad 1 < x < \infty,$$

is not zero at $x = n$.

(f) $(d/dz)^n \exp(e^{iz})$ *has no real zeros.*

In (a) and (b), the $S(n, k)$ are the Stirling numbers of the second kind, defined by $S(n, 1) = S(n, n) = 1$ and recursively by $S(n+1, k) = S(n, k-1) - kS(n, k)$. In (d), the terms are to be evaluated on the principal branch and in (e), $[x]$ is the greatest integer function.

It is well known [3] that $\exp(t(e^z - 1))$ is the exponential generating function of the polynomials $S_n(t)$, where

$$S_n(t) = \sum_{k=1}^{n} S(n, k)\, t^k, \tag{1.1}$$

that is,

$$\exp(t(e^z - 1)) = \sum_{n=0}^{\infty} S_n(t)\, z^n/n!. \tag{1.2}$$

Moreover, the $S_n(t)$ generalize [3, 13] the well-known Bell numbers B_n, defined by $B_n = S_n(1)$. For a positive integer m, it can be shown that $S_n(m)$ counts the number of ways in which n distinct objects can be placed in n boxes of m distinct colors.

The $S_n(-1)/e$ are the coefficients of the function given in (e). It can be shown by the methods given in Rota [12] that $S_n(-1)$ counts the difference between the numbers of partitions of a set of n elements into even and odd numbers of congruence classes. The $S_n(-1)$ of (1.1) is the sum $\sum_{k=1}^{n} (-1)^k S(n, k)$ given in (a) of Theorem 1.

The arithmetic (congruence) properties of $S_n(t)$, as well as the asymptotic behavior of $A(z)$ in (b) and the solution of the differential-difference equation in (e), provide a variety of methods for investigating the extent to which the equivalent forms in Theorem 1 are in fact true.

The example $\exp(e^{iz})$ in part (f) is interesting for function-theoretic reasons and (c) can be viewed as a gap theorem on the nonvanishing coefficients for the entire functions $\exp(-e^z)$. We comment further on this in Section 5.

It has recently been shown [5] that in an asymptotic density sense, (e) of Theorem 1 is true for "almost" all n, by using rather deep methods in analytic function theory. We provide in Section 4 a sufficient condition using elementary number-theoretic arguments. Some of the other equivalent conditions in (a)–(f) may provide easy alternative proofs. It remains an open question whether or not any of the equivalent statements (a)–(f) are in fact true for all $n > 2$. We conjecture that these statements are true. Further comments about this will be given in Section 7, the concluding remarks.

2. Preliminaries

In Sections 3 and 4 we require results on the congruence properties modulo primes for the polynomials $S_n(u)$, for u an integer, as given by Touchard [13] and Chinthayama and Gandhi [2]. They are

$$S_{n+p}(u) \equiv S_{n+1}(u) + u^p S_n(u) \qquad (\text{mod } p), \tag{2.1}$$

$$S_{kp^t+n}(u) \equiv S_n(u)(p(u) + u^p t)^k \qquad (\text{mod } p), \tag{2.2}$$

where by the right-hand side of this expression we mean to expand by the binomial theorem and drop superscripts to subscripts,

$$S_{\sum_r k_r p^r}(u) \equiv \prod_r (S(u) + ru^p)^{k_r} \qquad (\text{mod } p), \tag{2.3}$$

where the index r runs over the same set of positive integers, and

$$S_{n+(p^p-1)/(p-1)}(u) \equiv u^p S_n(u) \qquad (\text{mod } p). \tag{2.4}$$

Letting $u = -1$, denoting $S_n(-1)$ by A_n, and defining $A_0 = 1$, we get the obvious reductions

$$A_{n+p} \equiv (A_{n+1} - A_n) \qquad (\text{mod } p), \tag{2.1$'$}$$

$$A_{kp^t+n} \equiv A_n(A - t)^k \qquad (\text{mod } p), \tag{2.2$'$}$$

$$A_{\sum_r k_r p^r} \equiv \prod_r (A - r)^{k_r} \qquad (\text{mod } p), \tag{2.3$'$}$$

and

$$A_{n+(p^p-1)/(p-1)} \equiv -A_n \qquad (\text{mod } p). \tag{2.4$'$}$$

Comtet [3, p. 211] gives the following useful rule for calculating the $S_n(u)$ by using differences

$$uS_n(u) = \Delta^n S_k(u)|_{k=1}, \qquad S_0(u) = 1, \tag{2.5}$$

where Δ^n is the nth difference taken with respect to k. It follows easily that

$$A_{n+1} = -\sum_{k=0}^{n} \binom{n}{k} A_k, \qquad A_0 = 1. \tag{2.5'}$$

Facts (i) and (ii) can be used to establish Lemma 1, which will be used in the proof of Theorem 1.

(i) The polynomial $S_n(t) = \sum_{k=1}^{n} S(n,k)\, t^k$ has only real nonpositive and simple roots [10, Vol. II, Ex. 62.1, p. 44].

(ii) $(d/dz)^n F(e^{iz}) = i^n [S(n,1)\, F'(e^{iz})\, e^{iz} + \cdots + S(n,n)\, F^{(n)}(e^{iz})\, e^{inz}]$ (an analog of [10, Vol. I, Ex. 209, p. 44]).

LEMMA 1. *The roots of the* nth *derivative of* $\exp(e^{iz})$ *are given by the points* $z = -i \log |t_k^{(n)}| + \pi(2l-1)$, $l \in \mathbb{Z}$, *where the* $t_k^{(n)}$, $k = 1,..., n$, *are the zeros of* $S_n(t)$. *(Take* $z = \infty$ *when* $t_1^{(n)} = 0$.)

Using (2.5), we have obtained the values of A_n for $0 \leqslant n \leqslant 110$. We give the following list of beginning A_n:

n	0	1	2	3	4	5	6	7	8	9	10
A_n	1	−1	0	1	1	−2	−9	−9	50	267	413

n	11	12	13	14	15
A_n	−2,180	−17,721	−50,533	110,176	1,966,797

3. PROOF OF THEOREM 1

(a) $\Leftrightarrow$ (b) Since $Q_n(t)$ has leading and constant coefficients $S(n,1) = S(n,n) = 1$, it follows that the only possible rational zeros of Q_n are ± 1. As $S_n(t) = tQ_n(t)$, it again follows from (i), Section 2, that $Q_n(-1) \neq 0$. So we have

$$\begin{aligned} &Q_n \text{ irreducible over the rationals} \\ &\Leftrightarrow Q_n(-1) \neq 0 \\ &\Leftrightarrow \sum_{k=1}^{n} (-1)^k S(n,k) \neq 0 \qquad \text{by (1.1).} \end{aligned}$$

(a) $\Leftrightarrow$ (c) From (1.2) we have $\exp(1 - e^z) = \sum_{n=0}^{\infty} S_n(-1)\, z^n/n!$. Thus (d) $a_n = e^{-1} S_n(-1)/n!$. The desired result follows immediately, since $S_n(-1) = \sum_{k=1}^{n} (-1)^k S(n,k)$ by (1.1).

(a) $\Leftrightarrow$ (d) We expand $\exp(1-e^z)$ and compare the coefficients of $z^n/n!$. On the one hand, $\exp(1-e^z)=\sum_{k=0}^{\infty} S_k(-1)\,z^k/k!$. On the other hand,

$$\begin{aligned}\exp(1-e^z) &= e\sum_{n=0}^{\infty}(-1)^n e^{nz}/n! \\ &= e\sum_{n=0}^{\infty}\left((-1)^n/n!\sum_{k=0}^{\infty}\frac{n^k}{k!}z^k\right) \\ &= e\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}(-1)^n n^k/n!\right)z^k/k!.\end{aligned}$$

Therefore $A_k=S_k(-1)=e\sum_{n=0}^{\infty}(-1)^n n^k/n!$.

In addition, for z complex, we define the function $A(z)$ by $A(z)=e\sum_{n=0}^{\infty}(-1)^n n^z/n!$. Then A is an entire function of z and $A(k)=A_k$ for k a positive integer. Since A_k is an integer, for k a positive integer, $A(z)$ is integer valued for z a positive integer. Thus (a) and (d) are equivalent.

(a) $\Leftrightarrow$ (e) Clearly, the continuous function satisfying

$$\begin{aligned} f(x)&=1 && \text{for}\quad x\in[0,1], \\ f'(x)&=-[x]f(x-1) && \text{for}\quad 1\leqslant x<\infty \end{aligned} \tag{3.1}$$

is piecewise polynomial. We define a sequence of polynomials $\{q_n(y)\}$ on $[0,1]$ by the relationship

$$q_n(y-n)=(-1)^n f(y) \qquad \text{for}\quad n\leqslant y<n+1. \tag{3.2}$$

The transformation $y-n=x$ leads in a straightforward manner to the conditions

$$q_0(x)=1, \qquad 0\leqslant x\leqslant 1, \tag{3.3}$$

$$q_n(0)=-q_{n-1}(1), \tag{3.4}$$

$$q_n'(x)=nq_{n-1}(x), \qquad 0\leqslant x\leqslant 1, \tag{3.5}$$

where we have made use of the continuity of $f(x)$ and where $q_n'(0)$ and $q_n'(1)$ are to be interpreted as right- and left-hand derivatives, respectively.

The unique solution of (3.3)–(3.5) can be shown, by direct verification, to be given by

$$q_n(x)=\sum_{x=0}^{n}\binom{n}{k}q_k(0)\,x^{n-k},$$

where $q_n(0) = -q_{n-1}(1)$. It then follows that

$$q_{n+1}(0) = -\sum_{k=0}^{n} \binom{n}{k} q_k(0), \tag{3.6}$$

with $q_0(0) = 1$, from (3.3).

By comparing (3.6) and (2.5′), it is clear that $q_n(0) = A_n$ or, equivalently, by (3.2) and (1.1), that $f(n) = (-1)^n \sum_{k=1}^{n} (-1)^k S(n, k)$.

(f) ⇔ (b). By Lemma 1, a zero of $(d/dz)^n \exp(e^{iz})$ is real if and only if $+1$ or -1 is a zero of $Q_n(t)$. But by (i) in Section 2, all zeros of $Q_n(t)$ are negative.

4. Number-Theoretic Results

The modular recurrences (2.1′)-(2.4′) can be used to obtain a lower bound on the relative number of nonzero A_n $(= S_n(-1))$. Using (2.4′), the antiperiods, modulo 2, 3, 5, and 7, are found to be 3, 13, 781, and 137, 257, respectively. These numbers have no common divisors; thus the antiperiod of the sequence of 4-tuples of residues mod 2, 2, 5, and 7, simultaneously, is the product of the antiperiods, which is 4,180,710,963. Within one antiperiod, the number of zero residues was found, by calculation, to be 1, 4, 156, and 19,608 for $p = 2$, 3, 5, and 7, respectively. From these facts and the relative primacy of the antiperiods, we determine that there are 12,235,382 simultaneous zero residues mod 2, 3, 5, and 7 in one antiperiod, resulting in a relative frequency of 0.0029266. We restate this result as follows: In Theorem 1, (a)–(f) are true for more than 99.7 percent of integers $n \geqslant 0$.

After a previous version of this paper was written, Edrei [5] informed one of the authors that he had shown that for $h(z) \exp(-e^z) = \sum_{n=0}^{\infty} a_n z^n$, where h is an entire function of finite order with further suitable restrictions, $a_n \neq 0$ for 100% of the values of n with a possible exceptional set, the exceptional set being asymptotically determined. His methods were entirely function-theoretic in nature. For the special case $h(z) = 1$, we give, following a preliminary lemma, a purely arithmetic sufficient condition based on the congruence properties stated in Section 2 for the A_n defined by (1.2) for $t = -1$.

LEMMA 2. *Let p be a prime for which the minimum period of the A_n (mod p) is $2(p^p - 1)/(p - 1)$. Then all solutions of*

$$y_{n+p} \equiv y_{n+1} - y_n \pmod{p} \tag{4.1}$$

contain the same number $2(p^{p-1} - 1)/(p - 1)$ of zero residues in one period.

Proof. By (2.1′), one solution of (4.1) is given by the sequence A_n. It has recently been shown by one of us [8] that under the stated hypothesis this sequence contains in one period exactly two runs of $p-1$ consecutive zeros of the form $0, 0, \ldots, 0, a$ and $0, 0, \ldots, 0, -a$, one delayed $(p^p - 1)/(p-1)$ after the other. By linearity, a solution y_n of y_n of (4.1) is given by

$$y_n \equiv bA_n \qquad (\text{mod } p), \tag{4.2}$$

where $b \in \{1, 2, 3, \ldots, p-1\}$. When b ranges over $\{1, 2, 3, \ldots, (p-1)/2\}$, $(p-1)/2$ translation-distinct solutions of (4.1) are generated. Together, these solutions contain $[(p-1)/2][2(p^p - 1)/(p-1)] = p^p - 1$ distinct p-tuples. This is exactly the number of distinct p-tuples of integers (mod p), excluding the zero p-tuple $0, 0, \ldots, 0$, so all solutions of (4.1) must be of the form (4.2). Therefore all solutions must contain the same number of zeros in one period. Since the zero p-tuple is missing, one period of each of the $(p-1)/2$ translation-distinct solutions must together contain p^{p-1} occurrences of each nonzero residue and $p^{p-1} - 1$ zero residues. Thus any one solution contains $2(p^{p-1} - 1)/(p-1)$ zeros in one period.

THEOREM 2. *Let A_n be the sequence defined by*

$$\exp(1 - e^z) = \sum_{n=0}^{\infty} A_n \frac{z^n}{n!},$$

and let $z(N)$ denote the number of zero A_n for $n = 1, 2, \ldots, N$. If there exist arbitrarily large primes p for which the minimal period of A_n (mod p) *is $2(p^p - 1)/(p-1)$, then*

$$\lim_{N \to \infty} \frac{z(N)}{N} = 0.$$

Proof. By Lemma 2, there are $2(p^{p-1} - 1)/(p-1)$ zero residues A_n (mod p) in one minimal period of length $2(p^p - 1)/(p-1)$. Denoting this minimal period by l_p, we have, for each $k = 1, 2, \ldots,$

$$\frac{z(kl_p)}{kl_p} < \frac{1}{p}.$$

It follows that taking N and p sufficiently large we may make $z(N)/N$ arbitrarily small, completing the proof.

5. Connections with Complex Function Theory

We define two classes of entire functions as follows:

(R) functions f for which f, f', and f'' have only real zeros.

(L) $f(z) = Az^m \exp(-az^2 + bz) \prod_n (1 - z/z_n) \exp(z/z_n)$, where A is a constant, $a \geqslant 0$, b and the z_n are real, and $\sum_n |z_n|^{-2} < \infty$.

The class (L) is the Laguerre–Pólya class.

Recently, Hellerstein and Williamson [6, 7] verified an old conjecture of Pólya by proving that the only entire functions real on the real axis which satisfy (R) are those in (L). The requirement that f be real on the real axis is necessary because of the example $g(z) = \exp(e^{iz})$ in (f) of Theorem 1 (see [4, Theorem 3]). It is well known that all the zeros of the successive derivatives of functions of the form (L) are real. In contrast, g satisfies the hypothesis (R) but does not have the form (L).

Hypothesis (c) of Theorem 1 is of interest in connection with the general result of Renyi [11] that at least half of the coefficients of a periodic entire function do not vanish. Edrei's result [5] of course shows that for $\exp(-e^z)$ almost all coefficients do not vanish.

6. The Zeros of $S_n(x)$

We give some general information about the zeros of the polynomials $Q_n(t) = \sum_{k=1}^n S(n, k)\, t^{k-1}$. By use of the elementary symmetric relations and letting $t_n^{(n)} > t_{n-1}^{(n)} > \cdots > t_2^{(n)}$ denote the zeros of $Q_n(t)$, we have

$$S(n, 1) = 1 = \pm t_2^{(n)} \cdots t_n^{(n)}, \tag{6.1}$$

$$-S(n, n-1) = (t_2^{(n)} + \cdots + t_n^{(n)}). \tag{6.2}$$

Noting that $Q_2(t) = 1 + t$, we get from (6.1) that for each $n \geqslant 3$, $Q_n(t)$ has a zero in $(-1, 0)$, since otherwise, the product of all the zeros is not ± 1. Thus $Q_n(t)$ must have zeros in $(-\infty, -1)$ for $n \geqslant 3$, for the same reasons. Furthermore, Comtet [3, p. 208] gives the following formula for $S(n, k)$: Expand $(1 + 2 + \cdots + k)^{n-k}$ by the multinomial theorem and afterwards suppress the multinomial coefficients. Taking $k = n - 1$, we get $S(n, n-1) = (1 + 2 + \cdots + (n-1))^1 = n(n-1)/2 \sim n^2/2$, and hence the average of the zeros goes to $-\infty$ as $n \to \infty$, by (6.2). Moreover, as it is well known [3] that $(d/dt)[e^t S_n(t)] = S_{n+1}(t)\, e^t$, the negative roots of S_n and S_{n+1} interlace.

The above gives some information on the distribution of the roots of each $S_n(t)$. A question arises about the distribution of the set of all zeros of the $P_n(t)$, $n = 1, 2, 3, \ldots$. In order to answer this question, we first state a definition due to Pólya [9]. A point z belongs to the *final set of f* if every neighborhood of z contains zeros of infinitely many derivatives of f.

Very recently, Edrei [5] has shown that the final set for $\exp(-e^z)$ consists of the lines $y = 2\pi l$, $l \in \mathbb{Z}$. Since $e^{iz} = e^{-e^{i(z+\pi)}}$, Edrei's result implies that the final set of e^{iz} consists of the vertical lines $x = (2l+1)\pi$, $l \in \mathbb{Z}$. This result, together with the obvious fact that $S_n(t) = tQ_n(t)$, has the following important consequence by Lemma 1.

THEOREM 3. *The set of all the zeros of the polynomials $S_n(t)$, $n = 1, 2, \ldots$, is dense on* $(-\infty, 0]$.

7. Concluding Remarks

As we have seen, while the set of roots of all the polynomials $S_n(t)$ are dense on $(-\infty, 0]$, by Theorem 3, no rational number $-r$, $r > 0$, $r \neq 1$, can be a zero of any of the $S_n(t)$. Consequently, it is clear for these values of r that all the coefficients of $\exp(r(e^z - 1))$, given in (1.2), are nonzero. So $r = -1$ is the hard case. If (a)–(f) are not true for all $n > 2$, there must be some polynomial $Q_n(t)$, as given in (e) of Theorem 1, that is reducible over the rationals, the existence of which would itself be of interest.

The various equivalent conditions in Theorem 1 may provide yet other approaches to this problem. One possibility is provided by the differential-difference equation of (e) in Theorem 1. In fact, deBruijn [1] has determined the class of continuous solutions of the problem $g'(t) = tg(t-1)$, $t > 0$, related to the Bell numbers $S_n(1)$. A determination of the asymptotic behavior of the continuous solution of the equation $f'(t) = -|t| f(t-1)$ in (e) would be of considerable interest for $S_n(-1)$.

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