Some results on extending and sharpening the Weierstrass product inequalities

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Abstract

In this paper, we establish two extensions of Weierstrass’s inequality involving symmetric functions by means of the theory of majorization, and give an interesting sharpness of Weierstrass’s inequality by using the arithmetic–geometric mean inequality. Furthermore, we apply these results to improve a well-known inequality and deduce some new inequalities.

Keywords: Weierstrass’s inequality; Majorization; Schur-concave function; Elementary symmetric function; Arithmetic–geometric mean inequality

1. Introduction

Let \(0 < x_i < 1, \ i = 1, 2, \ldots, n, \ n \geq 2\). The following product inequalities

\[
\prod_{i=1}^{n} (1 + x_i) > 1 + \sum_{i=1}^{n} x_i, \quad \prod_{i=1}^{n} (1 - x_i) > 1 - \sum_{i=1}^{n} x_i,
\]

are known in the literature as Weierstrass’s inequality [1].

Weierstrass’s inequality (1) is one of the most important inequalities concerning product polynomials, it has stimulated the interest of many researchers, a number of papers
have been written on this inequality involving new proofs, noteworthy generalizations and numerous applications (see [2–7]). In this paper, there are two purposes. The first is to establish some new extensions of Weierstrass’s inequality by means of the theory of majorization. The second is to sharpen Weierstrass’s inequality by using the arithmetic–geometric mean inequality.

In what follows, $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and positive integers, respectively, $I$ is an interval, $x = (x_1, x_2, \ldots, x_n)$ denotes a $n$-tuple ($n$-dimensional real vector), the set of vectors can be written as

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n): x_i \in \mathbb{R}, \ i = 1, 2, \ldots, n\}.$$  

$$\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n): x_i > 0, \ i = 1, 2, \ldots, n\}.$$  

$$I^n = \{(x_1, x_2, \ldots, x_n): x_i \in I, \ i = 1, 2, \ldots, n\}.$$  

As usual, we denote by $\sigma_1, \sigma_2, \ldots, \sigma_n$ the elementary symmetric function of the variables $x_1, x_2, \ldots, x_n$. In addition, we define an analogous elementary symmetric function of variables $x_1, x_2, \ldots, x_n$, i.e.,

**Definition 1.** Let $x \in \mathbb{R}^n$, we define the $k$th symmetric function as follows:

$$\sigma_k(x) = \sigma_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}, \ k = 1, 2, \ldots, n.$$  

$$\tau_k(x) = \tau_k(x_1, x_2, \ldots, x_n) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}, \ k = 1, 2, \ldots, n.$$  

The following definitions was introduced by I. Schur [8].

**Definition 2.** For any $x = (x_1, x_2, \ldots, x_n), \ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, let $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ denote the components of $x$ and $y$ in decreasing order, respectively. The $n$-tuple $y$ is said to majorize $x$ (or $x$ is to be majorized by $y$) in symbols $x \prec y$, if

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]} \quad \text{holds for } k = 1, 2, \ldots, n - 1, \quad \text{and} \quad \sum_{i=1}^n x_i = \sum_{i=1}^n y_i. \quad (2)$$  

**Definition 3.** A real-valued function $\varphi$ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-concave on $\mathcal{A}$, if

$$x \prec y \quad \text{on } \mathcal{A} \quad \Rightarrow \quad \varphi(x) \geq \varphi(y), \quad (3)$$  

and $\varphi$ is strictly Schur-concave on $\mathcal{A}$ if strict inequality $\varphi(x) > \varphi(y)$ holds when $x$ is not a permutation of $y$. 
2. Lemmas

The following lemma is called Schur’s condition (see [8, p. 57], [9, p. 259]). It provides an approach for testing whether a vector valued function is Schur-concave or not.

**Lemma 1.** Let \( \varphi(x) = \varphi(x_1, x_2, \ldots, x_n) \) be symmetric and have continuous partial derivatives on \( I^n \), where \( I \) is an open interval. Then \( \varphi : I^n \to \mathbb{R} \) is Schur-concave if and only if

\[
(x_i - x_j) \left( \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j} \right) \leq 0 \quad \text{on} \quad I^n. \tag{4}
\]

It is strictly Schur-concave on \( I^n \) if (4) is a strict inequality for \( x_i \neq x_j \), \( 1 \leq i, j \leq n \).

Since \( \varphi(x) \) is symmetric, Schur-concave’s condition can be reduced to

\[
(x_1 - x_2) \left( \frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2} \right) \leq 0 \quad \text{on} \quad I^n, \tag{5}
\]

and \( \varphi \) is strictly Schur-concave on \( I^n \) if (5) is a strict inequality for \( x_1 \neq x_2 \).

In Schur’s condition, the domain of \( \varphi(x) \) does not have to be a Cartesian product \( I^n \).

**Lemma 1** remains true if we replace \( I^n \) by a set \( A \subset \mathbb{R}^n \) with following properties:

(I) \( A \) is convex and has a nonempty interior.

(II) \( A \) is symmetric in the sense that \( x \in A \) implies \( Px \in A \) for any \( n \times n \) permutation matrix \( P \).

**Lemma 2.** If \( 1 < k \leq n \), then \( \sigma_k(x) \) is strictly Schur-concave on \( \mathbb{R}^{n+} \). If \( 1 \leq k < n \), then \( \tau_k(x) \) is strictly Schur-concave on \( \mathbb{R}^{n+} \).

It is known that \( \sigma_k(x) (1 < k \leq n) \) is strictly Schur-concave on \( \mathbb{R}^{n+} \) [8,10]. Now, we prove the second proposition in Lemma 2.

**Proof.** Apparently, \( \tau_k(x) \) is symmetric and has continuous partial derivatives on \( \mathbb{R}^{n+} \).

According to Lemma 1, we only need to prove

\[
(x_1 - x_2) \left( \frac{\partial \tau_k}{\partial x_1} - \frac{\partial \tau_k}{\partial x_2} \right) < 0.
\]

By using the logarithmic algorithm, we have

\[
\log \tau_k(x) = \log \left[ \tau_k(x_2, x_3, \ldots, x_n) \prod_{2 \leq i_1 < \cdots < i_{k-1} \leq n} \left( x_1 + \sum_{j=1}^{k-1} x_{i_j} \right) \right] = \log \tau_k(x_2, x_3, \ldots, x_n) + \sum_{2 \leq i_1 < \cdots < i_{k-1} \leq n} \log \left( x_1 + \sum_{j=1}^{k-1} x_{i_j} \right).
\]

Differentiating \( \log \tau_k(x) \) with respect to \( x_1 \), we deduce that
\[
\frac{\partial \tau_k(x)}{\partial x_1} = \tau_k(x) \sum_{2 \leq i_1 < \cdots < i_{k-1} \leq n} \left( x_1 + \sum_{j=1}^{k-1} x_{i_j} \right)^{-1} \\
= \tau_k(x) \left[ \sum_{3 \leq i_1 < \cdots < i_{k-1} \leq n} \left( x_1 + \sum_{j=1}^{k-1} x_{i_j} \right)^{-1} + \sum_{3 \leq i_1 < \cdots < i_{k-2} \leq n} \left( x_1 + x_2 + \sum_{j=1}^{k-2} x_{i_j} \right)^{-1} \right].
\]

Similarly to the above, we obtain
\[
\frac{\partial \tau_k(x)}{\partial x_2} = \tau_k(x) \left[ \sum_{3 \leq i_1 < \cdots < i_{k-1} \leq n} \left( x_2 + \sum_{j=1}^{k-1} x_{i_j} \right)^{-1} + \sum_{3 \leq i_1 < \cdots < i_{k-2} \leq n} \left( x_1 + x_2 + \sum_{j=1}^{k-2} x_{i_j} \right)^{-1} \right].
\]

Thus
\[
(x_1 - x_2) \left( \frac{\partial \tau_k(x)}{\partial x_1} - \frac{\partial \tau_k(x)}{\partial x_2} \right) \\
= (x_1 - x_2) \tau_k(x) \left[ \sum_{3 \leq i_1 < \cdots < i_{k-1} \leq n} \left( x_1 + \sum_{j=1}^{k-1} x_{i_j} \right)^{-1} - \sum_{3 \leq i_1 < \cdots < i_{k-2} \leq n} \left( x_2 + \sum_{j=1}^{k-2} x_{i_j} \right)^{-1} \right] \\
= (x_1 - x_2)^2 \tau_k(x) \sum_{3 \leq i_1 < \cdots < i_{k-1} \leq n} \left[ \left( x_1 + \sum_{j=1}^{k-1} x_{i_j} \right)^{-1} - \left( x_2 + \sum_{j=1}^{k-1} x_{i_j} \right)^{-1} \right].
\]

When \(2 \leq k < n\), by \(x_i > 0, i = 1, 2, \ldots, n\), and the above equality, we deduce
\[
(x_1 - x_2) \left( \frac{\partial \tau_k(x)}{\partial x_1} - \frac{\partial \tau_k(x)}{\partial x_2} \right) < 0
\]
for \(x_1 \neq x_2\). When \(k = 1\), it follows directly that
\[
(x_1 - x_2) \left( \frac{\partial \tau_1(x)}{\partial x_1} - \frac{\partial \tau_1(x)}{\partial x_2} \right) = -(x_1 - x_2)^2 \tau_1(x)(x_1 x_2)^{-1} < 0
\]
for \(x_1 \neq x_2\), because \(\tau_1(x) = x_1 x_2 \cdots x_n\). The proof is complete. \(\square\)
Lemma 3. Let $\alpha$ be a positive real number, $x > -1$. Then for $k \leq \alpha$ and $k \in \mathbb{N}$, we have

$$(1 + x)^\alpha \geq 1 + \frac{k}{\alpha} \sum_{i=1}^{k} \binom{k}{i} x^i,$$  \hspace{1cm} (6)

with equality holds if and only if $x = 0$ or $k = \alpha$.

Proof. Define

$$f(x) = (1 + x)^\alpha - 1 - \sum_{i=1}^{k} \frac{k}{\alpha} \binom{k}{i} x^i, \quad x > -1.$$  

Then $f$ is a differentiable function with

$$f'(x) = \alpha(1 + x)^{\alpha-1} - \sum_{i=1}^{k} \frac{k}{\alpha} \binom{k}{i} x^{i-1} = \alpha(1 + x)^{\alpha-1} - \sum_{i=1}^{k} \frac{k(i-1)}{\alpha} x^{i-1} = \alpha(1 + x)^{\alpha-1} - \frac{1}{\alpha} \sum_{i=1}^{k} \binom{k-1}{i-1} x^{i-1}.$$

By $1 \leq k \leq \alpha$, we deduce that $f'(x) \leq 0$ for $x \in (-1, 0)$, and $f'(x) \geq 0$ for $x \in (0, +\infty)$. It shows that $f$ is decreasing on $(-1, 0)$ and increasing on $(0, +\infty)$. Thus we have $f(x) \geq f(0) = 0$ for $x \in (-1, +\infty)$, which leads to inequality (6). \(\square\)

3. Extensions of Weierstrass’s inequality

Theorem 1. Let $1 \leq m_j \leq \alpha_j$, $j = 1, 2, \ldots, n$, $1 < k \leq n$, and $k, m_j \in \mathbb{N}$. Then we have the inequalities

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} (1 + x_{i_j})^{\alpha_{i_j}} > \binom{n}{k} + \binom{n-1}{k-1} \sum_{j=1}^{m_j} \frac{\alpha_j}{m_j} \binom{m_j}{i} x_j^i \hspace{1cm} (7)$$

for $x_j > 0$, $j = 1, 2, \ldots, n$;

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} (1 - x_{i_j})^{\alpha_{i_j}} > \binom{n}{k} + \binom{n-1}{k-1} \sum_{j=1}^{m_j} \frac{\alpha_j}{m_j} \binom{m_j}{i} x_j^i \hspace{1cm} (8)$$

for $0 < x_j < 1$ and $\sum_{j=1}^{n} (-1)^{i-1} \frac{\alpha_j}{m_j} \binom{m_j}{i} x_j^i < 1$, $j = 1, 2, \ldots, n$.

Proof. By Lemma 3 and the definition of $\sigma_k(x)$, we have

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} (1 + x_{i_j})^{\alpha_{i_j}} = \sigma_k \left( (1 + x_1)^{\alpha_1}, (1 + x_2)^{\alpha_2}, \ldots, (1 + x_n)^{m_n} \right) \geq \sigma_k \left( 1 + \sum_{i=1}^{m_1} \frac{\alpha_1}{m_1} \binom{m_1}{i} x_1^i + \sum_{i=1}^{m_2} \frac{\alpha_2}{m_2} \binom{m_2}{i} x_2^i, \ldots, 1 + \sum_{i=1}^{m_n} \frac{\alpha_n}{m_n} \binom{m_n}{i} x_n^i \right), \hspace{1cm} (9)$$
From Definition 2, it is easy to verify that
\[
1 + \sum_{i=1}^{m_1} \frac{\alpha_1}{m_1} \binom{m_1}{i} x_1^i, 1 + \sum_{i=1}^{m_2} \frac{\alpha_2}{m_2} \binom{m_2}{i} x_2^i, \ldots, 1 + \sum_{i=1}^{m_n} \frac{\alpha_n}{m_n} \binom{m_n}{i} x_n^i
\]
\[\prec\]
\[
1 + \sum_{j=1}^{n} \sum_{i=1}^{m_j} \frac{\alpha_j}{m_j} \binom{m_j}{i} x_j^i, 1, \ldots, 1.
\]
and then using Lemma 2 and Definition 3, we get
\[
\sigma_k \left( 1 + \sum_{i=1}^{m_1} \frac{\alpha_1}{m_1} \binom{m_1}{i} x_1^i, 1 + \sum_{i=1}^{m_2} \frac{\alpha_2}{m_2} \binom{m_2}{i} x_2^i, \ldots, 1 + \sum_{i=1}^{m_n} \frac{\alpha_n}{m_n} \binom{m_n}{i} x_n^i \right)
\]
\[>\]
\[
\sigma_k \left( 1 + \sum_{j=1}^{n} \sum_{i=1}^{m_j} \frac{\alpha_j}{m_j} \binom{m_j}{i} x_j^i, 1, \ldots, 1 \right)
\]
\[=\]
\[
\left( \frac{n}{k} \right) + \left( \frac{n-1}{k-1} \right) \sum_{j=1}^{n} \sum_{i=1}^{m_j} \frac{\alpha_j}{m_j} \binom{m_j}{i} x_j^i.
\]
Combining inequalities (9) and (10), we deduce inequality (7). On the other hand, we note that the conditions \(0 < x_j < 1\) and \(\sum_{j=1}^{n} (\frac{-1}{m_j})^j \frac{\alpha_j}{m_j} \binom{m_j}{i} x_j^i < 1\), such that
\[
1 + \sum_{j=1}^{n} \sum_{i=1}^{m_j} \frac{(\frac{-1}{m_j})^j \alpha_j}{m_j} \binom{m_j}{i} x_j^i > 0 \quad \text{and}
\]
\[
1 + \sum_{i=1}^{m_j} \frac{(\frac{-1}{m_j})^j \alpha_j}{m_j} \binom{m_j}{i} x_j^i > 0, \quad j = 1, 2, \ldots, n.
\]
By
\[
1 + \sum_{i=1}^{m_1} \frac{(\frac{-1}{m_1})^i \alpha_1}{m_1} \binom{m_1}{i} x_1^i, 1 + \sum_{i=1}^{m_2} \frac{(\frac{-1}{m_2})^i \alpha_2}{m_2} \binom{m_2}{i} x_2^i, \ldots,
\]
\[
1 + \sum_{i=1}^{m_n} \frac{(\frac{-1}{m_n})^i \alpha_n}{m_n} \binom{m_n}{i} x_n^i < \left( 1 + \sum_{j=1}^{n} \sum_{i=1}^{m_j} \frac{(\frac{-1}{m_j})^j \alpha_j}{m_j} \binom{m_j}{i} x_j^i, 1, \ldots, 1 \right),
\]
and Lemma 2 and Definition 3, we can prove inequality (8) in a similar way as in the proof of inequality (7). We omit the details. \(\square\)

Now, we present some direct consequences from Theorem 1.
Let \(m_1 = m_2 = \cdots = m_n = 1\). Then from Theorem 1 we obtain the following results.

**Corollary 1.** Assume \(\alpha_j \geq 1, j = 1, 2, \ldots, n, 1 < k \leq n, k \in \mathbb{N}\). Then we have the inequalities
$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} (1 + x_{i_j})^{a_{i_j}} > \binom{n}{k} + \binom{n-1}{k-1} \sum_{j=1}^{n} a_j x_j$$

(11)

for $x_j > 0$, $j = 1, 2, \ldots, n$;

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} (1 - x_{i_j})^{a_{i_j}} > \binom{n}{k} - \binom{n-1}{k-1} \sum_{j=1}^{n} a_j x_j$$

(12)

for $0 < x_j < 1$ and $\sum_{j=1}^{n} a_j x_j < 1$, $j = 1, 2, \ldots, n$.

Choosing $k = n$ in Corollary 1, we get

**Corollary 2.** Assume $\alpha_j \geq 1$, $j = 1, 2, \ldots, n$. Then we have

$$\prod_{j=1}^{n} (1 + x_j)^{a_j} > 1 + \sum_{j=1}^{n} a_j x_j$$

for $x_j > 0$, $j = 1, 2, \ldots, n$;

$$\prod_{j=1}^{n} (1 - x_j)^{a_j} > 1 - \sum_{j=1}^{n} a_j x_j$$

for $0 < x_j < 1$ and $\sum_{j=1}^{n} a_j x_j < 1$, $j = 1, 2, \ldots, n$.

Taking $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$ in Corollary 2, we immediately obtain Weierstrass’s inequality.

**Theorem 2.** Let $1 \leq m_j \leq \alpha_j$, $j = 1, 2, \ldots, n$, $1 \leq k < n$, and $k, m_j \in \mathbb{N}$. Then we have the inequalities

$$\prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} (1 + x_{i_j})^{a_{i_j}} > \binom{k}{i_1} \left[ k + \sum_{j=1}^{m_j} \frac{a_j}{m_j} \binom{m_j}{i_1} x_{i_j} \right]^{\binom{j-1}{i_1}}$$

(15)

for $x_j > 0$, $j = 1, 2, \ldots, n$;

$$\prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} (1 - x_{i_j})^{a_{i_j}} > \binom{k}{i_1} \left[ k + \sum_{j=1}^{m_j} \frac{(-1)^i \alpha_j}{m_j} \binom{m_j}{i_1} x_{i_j} \right]^{\binom{j-1}{i_1}}$$

(16)

for $0 < x_j < 1$ and $\sum_{j=1}^{n} \sum_{i=1}^{m_j} (-1)^{i-1} \frac{a_j}{m_j} \binom{m_j}{i_1} x_{i_j} < 1$, $j = 1, 2, \ldots, n$.

**Proof.** By Lemma 3 and the definition of $\tau_k(x)$, we get

$$\prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} (1 + x_{i_j})^{a_{i_j}}$$

$$= \tau_k \left( (1 + x_1)^{a_1}, (1 + x_2)^{a_2}, \ldots, (1 + x_n)^{a_n} \right)$$

$$\geq \tau_k \left( 1 + \sum_{i=1}^{m_1} \frac{a_1}{m_1} \binom{m_1}{i_1} x_{i_1}, 1 + \sum_{i=1}^{m_2} \frac{a_2}{m_2} \binom{m_2}{i_2} x_{i_2}, \ldots, 1 + \sum_{i=1}^{m_n} \frac{a_n}{m_n} \binom{m_n}{i_n} x_{i_n} \right),$$

(17)
According to Definition 2, we have
\[
\left( 1 + \sum_{i=1}^{m_1} \frac{\alpha_1}{m_1} m_1^i x_1^i, 1 + \sum_{i=1}^{m_2} \frac{\alpha_2}{m_2} m_2^i x_2^i, \ldots, 1 + \sum_{i=1}^{m_n} \frac{\alpha_n}{m_n} m_n^i x_n^i \right)
\prec \left( 1 + \sum_{j=1}^{n} \sum_{i=1}^{m_j} \frac{\alpha_j}{m_j} m_j^i x_j^i, 1, \ldots, 1 \right),
\]
and then applying Lemma 2 and Definition 3, we get
\[
\tau_k \left( 1 + \sum_{i=1}^{m_1} \frac{\alpha_1}{m_1} m_1^i x_1^i, 1 + \sum_{i=1}^{m_2} \frac{\alpha_2}{m_2} m_2^i x_2^i, \ldots, 1 + \sum_{i=1}^{m_n} \frac{\alpha_n}{m_n} m_n^i x_n^i \right)
> \tau_k \left( 1 + \sum_{j=1}^{n} \sum_{i=1}^{m_j} \frac{\alpha_j}{m_j} m_j^i x_j^i, 1, \ldots, 1 \right)
= k^{(n-1)} \left[ k + \sum_{j=1}^{n} \sum_{i=1}^{m_j} \frac{\alpha_j}{m_j} m_j^i x_j^i \right]^{\left(\frac{n-1}{k-1}\right)}. \quad (18)
\]
Combining inequalities (17) and (18) leads to inequality (15). Similarly, we can prove inequality (16), we omit the details. \(\Box\)

If we take \(m_1 = m_2 = \cdots = m_n = 1\) in Theorem 2, we obtain the following corollary.

**Corollary 3.** Assume \(\alpha_j \geq 1, j = 1, 2, \ldots, n, 1 \leq k < n, k \in \mathbb{N}\). Then we have the inequalities
\[
\prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} (1 + x_{i_j})^{\alpha_{i_j} x_{i_j}} > k^{\left(\frac{n-1}{k-1}\right)} \left( k + \sum_{j=1}^{n} \alpha_j x_j \right)^{\left(\frac{n-1}{k-1}\right)} \quad (19)
\]
for \(x_j > 0, j = 1, 2, \ldots, n;\)
\[
\prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} (1 - x_{i_j})^{\alpha_{i_j} x_{i_j}} > k^{\left(\frac{n-1}{k-1}\right)} \left( k - \sum_{j=1}^{n} \alpha_j x_j \right)^{\left(\frac{n-1}{k-1}\right)} \quad (20)
\]
for \(0 < x_j < 1\) and \(\sum_{j=1}^{n} \alpha_j x_j < 1, j = 1, 2, \ldots, n;\).

It is obvious that Weierstrass’s inequality can follow from Corollary 3 with \(\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1\) and \(k = 1\).

4. Sharpness of Weierstrass’s inequality

In this section we give an interesting sharpness of Weierstrass’s inequality.
Theorem 3. Let \( x_i > 0 \), \( i = 1, 2, \ldots, n \), \( n \geq 2 \), \( n \in \mathbb{N} \). Then we have
\[
\prod_{i=1}^{n} (1 + x_i) \geq 1 + \sum_{i=1}^{n} x_i + (2^n - n - 1) \left( \prod_{i=1}^{n} x_i \right)^{\frac{2^{n-1} - n^n}{2^n - n^n - 1}}, \tag{21}
\]
with equality holds if and only if \( x_1 = x_2 = \cdots = x_n = 1 \) or \( n = 2 \).

Proof. By Maclaurin’s inequality [11],
\[
\left[ \frac{\sigma_1(x_1, x_2, \ldots, x_n)}{(n)} \right] \geq \left[ \frac{\sigma_2(x_1, x_2, \ldots, x_n)}{(n)} \right] \geq \cdots \geq \left[ \frac{\sigma_n(x_1, x_2, \ldots, x_n)}{(n)} \right]^{\frac{1}{n}}, \tag{22}
\]
we obtain
\[
\prod_{i=1}^{n} (1 + x_i) = 1 + \sum_{k=1}^{n} \sigma_k(x_1, x_2, \ldots, x_n)
\geq 1 + \sum_{i=1}^{n} x_i + \sum_{k=2}^{n} {n \choose k} \left[ \sigma_n(x_1, x_2, \ldots, x_n) \right]^{\frac{k}{n}}. \tag{23}
\]
Using arithmetic–geometric mean inequality, we have
\[
\sum_{k=2}^{n} {n \choose k} \left[ \sigma_n(x_1, x_2, \ldots, x_n) \right]^{\frac{k}{n}}
= \sum_{k=2}^{n} \left[ \sigma_n(x_1, \ldots, x_n) \right]^{\frac{k}{n}} + \cdots + \left[ \sigma_n(x_1, \ldots, x_n) \right]^{\frac{n}{n}}
\geq \left( \sum_{k=2}^{n} {n \choose k} \right) \left[ \sigma_n(x_1, \ldots, x_n) \right]^{\frac{1}{n}}
= (2^n - n - 1) \left( \prod_{i=1}^{n} x_i \right)^{\frac{2^{n-1} - 1}{2^n - 1}}.
\]
Combining (23) and the above inequality, we obtain inequality (21). The conditions of equality in (21) follows from Maclaurin’s inequality and arithmetic–geometric mean inequality. The proof is complete. \( \square \)

Theorem 4. Let \( 0 < x_i \leq 1 \), \( i = 1, 2, \ldots, n \), \( n \geq 2 \), \( n \in \mathbb{N} \). Then we have
\[
\prod_{i=1}^{n} (1 - x_i) \geq 1 - \sum_{i=1}^{n} x_i + (n - 1) \left( \prod_{i=1}^{n} x_i \right)^{\frac{2^{n-1}}{2^n - n^n - 1}}, \tag{24}
\]
with equality holds if and only if \( x_1 = x_2 = \cdots = x_n = 1 \) or \( n = 2 \).
Proof. Define
\[
f(k) = 1 - \sum_{i=1}^{k} x_i + (k - 1) \left( \prod_{i=1}^{k} x_i \right)^{\frac{k}{k-1}} - \prod_{i=1}^{k} (1 - x_i), \quad k = 2, 3, \ldots, n.
\]
When \( k \geq 3, \ k \in \mathbb{N}, \) we have
\[
(1 - x_k) f(k - 1) = (1 - x_k) \left[ 1 - \sum_{i=1}^{k-1} x_i + (k - 2) \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} \right] - \prod_{i=1}^{k} (1 - x_i)
\]
\[
= 1 - \sum_{i=1}^{k-1} x_i + (k - 2) \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} - (k - 2) x_k \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}}
\]
\[
+ x_k \sum_{i=1}^{k-1} x_i - \prod_{i=1}^{k} (1 - x_i).
\]

Using arithmetic–geometric mean inequality, we get
\[
(k - 2) \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} - (k - 2) x_k \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} + x_k \sum_{i=1}^{k-1} x_i
\]
\[
\geq (k - 2) \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} - (k - 2) x_k \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} + (k - 1) x_k \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}}
\]
\[
= (k - 2) x_k \left[ \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{1}{k-1}} - \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} \right]
\]
\[
+ \left[ x_k \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{1}{k-1}} + \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{1}{k-2}} + \cdots + \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-3}{k-2}} \right]^{\frac{1}{k-2}}
\]
\[
\geq (k - 2) x_k \left[ \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{1}{k-1}} - \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} \right]
\]
\[
+ (k - 1) \left[ x_k \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{1}{k-1}} \right]^{\frac{1}{k-1}}
\]
\[
= (k - 1) \left( \prod_{i=1}^{k} x_i \right)^{\frac{k}{k-1}} + (k - 2) x_k \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{1}{k-2}} \left[ 1 - \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} \right]
\]
\[
+ (k - 1) x_k \left[ \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{1}{k-1}} \right]^{\frac{1}{k-1}} + \frac{1}{k-2} \left[ 1 - x_k \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k}{k-1}} - \frac{1}{k-2} \left( \prod_{i=1}^{k-1} x_i \right)^{\frac{k-1}{k-2}} \right]
\]
\[ \geq (k - 1) \left( \prod_{i=1}^{k} x_i \right) \frac{k}{k-2}, \]

where the last inequality follows from the following simple inequalities:

\[
\begin{align*}
0 < x_i & \leq 1, \quad i = 1, 2, \ldots, k, \quad k \geq 3, \\
\frac{k - 1}{2k - 4} - \frac{1}{k - 1} & = \frac{(k - 2)^2 + 1}{(2k - 4)(k - 1)} > 0, \\
\frac{k}{2k - 2} - \frac{1}{(k - 1)^2} - \frac{1}{2} & = \frac{k - 3}{2(k - 1)^2} \geq 0, \\
\frac{k}{2k - 2} - \frac{1}{k - 1} & = \frac{k - 2}{2k - 2} > 0.
\end{align*}
\]

And then we obtain

\[
(1 - x_k) f (k - 1) \geq 1 - \sum_{i=1}^{k} x_i + (k - 1) \left( \prod_{i=1}^{k} x_i \right) \frac{k}{k-2} - \prod_{i=1}^{k} (1 - x_i) = f(k),
\]

that is

\[ f(k) \leq (1 - x_k) f (k - 1), \quad k = 3, 4, \ldots, n. \]

Therefore for \( n \geq 3, \ n \in \mathbb{N}, \) we have

\[
\begin{align*}
f(n) & \leq (1 - x_n) f (n - 1) \leq (1 - x_n)(1 - x_{n-1}) f (n - 2) \leq \cdots \\
& \leq (1 - x_n)(1 - x_{n-1}) \cdots (1 - x_2) f (2).
\end{align*}
\]

By the above inequalities with \( f(2) = 1 - x_1 - x_2 + x_1x_2 - (1 - x_1)(1 - x_2) = 0, \) we get \( f(n) \leq 0 \) for all \( n \geq 2, \ n \in \mathbb{N}, \) which is equivalent to inequality (24). Arithmetic–geometric mean inequality shows that equality in (24) holds if and only if \( x_1 = x_2 = \cdots = x_n = 1 \) or \( n = 2. \) This completes the proof of Theorem 4. \( \square \)

In particular, applying Theorems 3 and 4 with appropriate conditions, we can obtain the following inequalities.

**Corollary 4.** Assume \( x_i > 0 \) and \( \prod_{i=1}^{n} x_i = 1, \ i = 1, 2, \ldots, n, \ n \geq 2, \ n \in \mathbb{N}. \) Then

\[
\prod_{i=1}^{n} (1 + x_i) - \sum_{i=1}^{n} x_i \geq 2^n - n. \tag{25}
\]

**Corollary 5.** Assume \( x_i > 0 \) and \( \sum_{i=1}^{n} x_i = 1, \ i = 1, 2, \ldots, n, \ n \geq 2, \ n \in \mathbb{N}. \) Then

\[
\prod_{i=1}^{n} (1 - x_i)^{n-1} > (n - 1)^{n-1} \left( \prod_{i=1}^{n} x_i \right)^\frac{2}{n}. \tag{26}
\]

**Remark.** It is clear that \( 2^n - n - 1 \) and \( n - 1 \) are positive numbers for \( n \geq 2, \) so Weierstrass’s inequality is weaker than the inequalities in Theorems 3 and 4. Namely, inequalities (21) and (24) have sharpened Weierstrass’s inequality.
5. Some applications

As examples of the applications, we shall extend a well-known inequality and establish a class of new inequalities for simplex by using the above results.

Theorem 5. Assume \( x_j > 1, \alpha_j \geq 1, j = 1, 2, \ldots, n, 1 < k \leq n, k \in \mathbb{N} \), and \( A_k = \min_{i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} \alpha_{ij} \). Then

\[
\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} (1 + x_{i_j})^{\alpha_{ij}} > \frac{2^{A_k}}{1 + A_k} \left( \frac{n - 1}{k - 1} \right) \left( 1 + A_k \right) \frac{n}{k} - A_n + \sum_{j=1}^{n} \alpha_j x_j.
\]

(27)

Proof. From the assumptions of Theorem 5 and using Corollary 1, it is easy to observe that

\[
\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} (1 + x_{i_j})^{\alpha_{ij}} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} 2^{\sum_{j=1}^{k} \alpha_{ij}} \prod_{j=1}^{k} \left( 1 + \frac{x_{i_j} - 1}{2} \right)^{\alpha_{ij}}
\]

\[
\geq 2^{A_k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \left( 1 + \frac{x_{i_j} - 1}{2} \right)^{\alpha_{ij}}
\]

\[
> 2^{A_k} \left[ \binom{n}{k} + \frac{n - 1}{k - 1} \sum_{j=1}^{n} \alpha_j \left( \frac{x_j - 1}{2} \right) \right]
\]

\[
\geq 2^{A_k} \left[ \binom{n}{k} + \frac{n - 1}{k - 1} \sum_{j=1}^{n} \alpha_j \left( \frac{x_j - 1}{1 + A_k} \right) \right]
\]

\[
= \frac{2^{A_k}}{1 + A_k} \frac{n - 1}{k - 1} \left( 1 + A_k \right) \frac{n}{k} - A_n + \sum_{j=1}^{n} \alpha_j x_j.
\]

The proof is complete. \( \square \)

Remark. In the special case when \( k = n \), the inequality (27) reduces to the following inequality:

\[
\prod_{j=1}^{n} (1 + x_j)^{\alpha_j} > \frac{2^{A_n}}{1 + A_n} \left( 1 + \sum_{j=1}^{n} \alpha_j x_j \right),
\]

(28)

which was proved by Pečarić [3] (see also [2, p. 69]).

Theorem 6. Let \( \{A_1, A_2, \ldots, A_{n+1}\} \) denote the vertex set of \( n \)-dimensional simplex \( \Omega_n \) in \( E^n (n \geq 2) \), \( r \) denotes the inradius of \( \Omega_n \). For \( i = 1, 2, \ldots, n + 1 \), let \( h_i \) be the altitude of \( \Omega_n \) from vertex \( A_i \), let \( r_i \) be the radius of \( i \)th escribed hypersphere of \( \Omega_n \). Then for \( \alpha \geq 1, 1 < k \leq n + 1 \), we have
\[ \sum_{1 \leq i_1 < \cdots < i_k \leq n+1} \prod_{j=1}^{k} \left( \frac{r_{i_j}}{r_{i_j} - r} \right)^{\alpha} > \alpha(n^2 - 1) \binom{n}{k-1} + \binom{n+1}{k}, \quad (29) \]

\[ \sum_{1 \leq i_1 < \cdots < i_k \leq n+1} \prod_{j=1}^{k} \left( \frac{h_{i_j}}{h_{i_j} - r} \right)^{\alpha} > 2\alpha \left( 1 + \frac{1}{n} \right) \binom{n}{k-1} + \binom{n+1}{k}, \quad (30) \]

For \( \alpha \geq 1 \), \( 1 \leq k < n+1 \), we have
\[ \prod_{1 \leq i_1 < \cdots < i_k \leq n+1} \sum_{j=1}^{k} \left( \frac{r_{i_j}}{r_{i_j} - r} \right)^{\alpha} > k^{(\alpha)} \left[ k + \alpha(n^2 - 1) \right]^{(\alpha)_{n-1}}, \quad (31) \]

\[ \prod_{1 \leq i_1 < \cdots < i_k \leq n+1} \sum_{j=1}^{k} \left( \frac{h_{i_j}}{h_{i_j} - r} \right)^{\alpha} > k^{(\alpha)} \left[ k + 2\alpha \left( 1 + \frac{1}{n} \right) \right]^{(\alpha)_{n-1}}. \quad (32) \]

**Proof.** Let \( S_i \) denote the area of the \( i \)th face \( A_1 \cdots A_{i-1}A_{i+1} \cdots A_{n+1} \) of \( \Omega_n \).

Using the identity of simplex [12, p. 463]
\[ r_i = r \left( \sum_{i=1}^{n+1} S_i \right) / \left( -2S_i + \sum_{i=1}^{n+1} S_i \right), \quad h_i = r \left( \sum_{i=1}^{n+1} S_i \right) / S_i, \]

together with Cauchy’s inequality, it follows that
\[ \sum_{i=1}^{n+1} \frac{2r}{r_i} = \left( \sum_{i=1}^{n+1} S_i \right) \left( \sum_{i=1}^{n+1} 1/S_i \right) - 2n - 2 \geq (n+1)^2 - 2n - 2 = n^2 - 1, \]
\[ \sum_{i=1}^{n+1} \frac{2r}{h_i} = \frac{2}{n} \left[ \sum_{i=1}^{n+1} (-S_i + \sum_{i=1}^{n+1} S_i) \left( \sum_{i=1}^{n+1} S_i \right) \right] - 2n - 2 \geq \frac{2(n+1)^2 - 2n - 2}{n} = 2 \left( 1 + \frac{1}{n} \right). \]

It is obvious that \( 2r/(r_i - r) > 0, 2r/(h_i - r) > 0, i = 1, 2, \ldots, n+1 \). Now, substituting \( x_{i_1} = 2r/(r_{i_1} - r) \) and \( x_{i_j} = 2r/(h_{i_j} - r) \) into Corollary 1 respectively, and let \( \alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = \alpha \), we obtain inequalities (29) and (30). Similarly, from Corollary 3 we get inequalities (31) and (32). The proof is complete. \( \square \)

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**References**