NORTH-HOLLAND

The Geometry of Basic, Approximate, and Minimum-Norm Solutions of Linear Equations

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ABSTRACT

The *basic solutions* of the linear equations $A\mathbf{x} = \mathbf{b}$ are the solutions of subsystems corresponding to maximal nonsingular submatrices of A. The convex hull of the basic solutions is denoted by $\mathcal{C} = \mathcal{C}(A, \mathbf{b})$. Given $1 \leq p \leq \infty$, the *ℓ_p-approximate solutions* of $A\mathbf{x} = \mathbf{b}$, denoted $\mathbf{x}^{\{p\}}$, are minimizers of $\|A\mathbf{x} - \mathbf{b}\|_p$. Given $M \in \mathcal{D}_m$, the set of positive diagonal $m \times m$ matrices, the solutions of $\min_{\mathbf{x}} \|M(A\mathbf{x} - \mathbf{b})\|_p$ are called *scaled ℓ_p-approximate solutions*. For $1 \leq p_1, p_2 \leq \infty$, the *minimum-$\ell_{p_2}$*-norm *$\ell_{p1}$-approximate solutions* are denoted $\mathbf{x}^{\{p_1\}}_{\{p_2\}}$. Main results:

(1) If $A \in \mathbb{R}^{m\times n}_m$, then $\mathcal{C}$ contains all [some] minimum ℓ_p-norm solutions, for $1 \leq p < \infty$ [$p = \infty$].

(2) For general A and any $1 \leq p_1, p_2 < \infty$ the set $\mathcal{C}$ contains all $\mathbf{x}^{\{p_1\}}_{\{p_2\}}$.

(3) The set of scaled ℓ_p-approximate solutions, with M ranging over $\mathcal{D}_m$, is the same for all $1 < p < \infty$.

(4) The set of scaled least-squares solutions has the same closure as the set of solutions of $\min_{\mathbf{x}} f\left(|A\mathbf{x} - \mathbf{b}|\right)$, where $f : \mathbb{R}^m_+ \to \mathbb{R}$ ranges over all strictly isotone functions.

1. INTRODUCTION

Given $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, consider the linear equation

$$A\mathbf{x} = \mathbf{b}. \tag{1.1}$$

*Supported by grant NSF-STC91-19999.

LINEAR ALGEBRA AND ITS APPLICATIONS 216:25–41 (1995)

0024-3795/95/$9.50
SSDI 0024-3795(93)00093-F

If (1.1) is inconsistent, we often settle for an approximate solution minimizing a norm of the residual $\mathbf{r}(\mathbf{x}) := A\mathbf{x} - \mathbf{b}$. Using the *$\ell_p$-norms*, defined for $1 \leq p \leq \infty$ and $\mathbf{u} = (u_j) \in \mathbb{R}^m$ by

$$\|\mathbf{u}\|_p := \begin{cases} \left(\sum_{j=1}^{m} |u_j|^p\right)^{1/p}, & 1 \leq p < \infty, \\ \max_{1 \leq j \leq m} |u_j|, & p = \infty, \end{cases} \tag{1.2}$$

an *l_p-approximate solution* of (1.1) is a solution of the minimization problem

$$\min\{\|A\mathbf{x} - \mathbf{b}\|_p : \mathbf{x} \in \mathbb{R}^n\}. \tag{1.3}$$

In particular, the l_2-approximate solutions are the *least-squares solutions*.

The *set of increasing sequences of r* elements from $\{1, \ldots, m\}$ is

$$Q_{r,m} := \{I = \{i_1, \ldots, i_r\} : 1 \leq i_1 < i_2 < \cdots < i_r \leq m\}.$$

For $A \in \mathbb{R}_r^{m \times n}$, $r > 0$, we denote the index sets

$$\mathcal{I}(A) := \{I \in Q_{r,m} : \operatorname{rank} A_{I*} = r\}$$

of *maximal sets of linearly independent rows*,

$$\mathcal{J}(A) := \{J \in Q_{r,n} : \operatorname{rank} A_{*J} = r\}$$

of *maximal sets of linearly independent columns*,

$$\mathcal{N}(A) := \{(I, J) \in Q_{r,m} \times Q_{r,n} : \operatorname{rank} A_{IJ} = r\}$$

of *maximal nonsingular submatrices*. The index sets $\mathcal{I}(A)$, $\mathcal{J}(A)$, and $\mathcal{N}(A)$ are abbreviated here by $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{N}$ respectively. We have

$$\mathcal{N} = \mathcal{I} \times \mathcal{J} \qquad \text{(see e.g. [2])}. \tag{1.4}$$

The *basic solutions* of the linear equation $A\mathbf{x} = \mathbf{b}$ are the solutions of subsystems corresponding to maximal nonsingular submatrices of A. The basic solutions are, for

$$A \text{ of full column rank:} \qquad \{A_{I*}^{-1}\mathbf{b}_I : I \in \mathcal{I}\}, \tag{1.5}$$

$$A \text{ of full row rank:} \qquad \{\widehat{A_{*J}^{-1}\mathbf{b}} : J \in \mathcal{J}\}, \tag{1.6}$$

$$\text{general } A: \qquad \{\widehat{A_{IJ}^{-1}\mathbf{b}_I} : (I, J) \in \mathcal{N}\}, \tag{1.7}$$

where $\mathbf{b}_I$ is the Ith subvector of $\mathbf{b}$, and $\widehat{\ }$ denotes a vector padded by zeros. The *convex hull of basic solutions* of the given equation $A\mathbf{x} = \mathbf{b}$ is denoted by $\mathcal{C} = \mathcal{C}(A, \mathbf{b})$. The *set of minimizers* [*maximizers*] of a function f is denoted by arg min$f$ [arg max f].

For A of full column rank, Berg [5], proved that the least-squares solution is in the convex hull of basic solutions (1.5). For general A, the least-squares solution of minimal (euclidean) norm lies in the convex hull of the basic solutions (1.7),

$$\mathcal{C} := \operatorname{conv}\{\widehat{A_{IJ}^{-1}\mathbf{b}_I} : (I, J) \in \mathcal{N}\}.$$

This is important for establishing convergence of certain iterative methods, since the set $\mathcal{C}$ is compact.

For A of full column rank, Ben-Tal and Teboulle [4] extended Berg's results to isotone functions, of which l_p-norms can be considered a special case. A continuous function $f : \mathbb{R}_+^m \to \mathbb{R}$ is called *isotone* if

$$0 \le \mathbf{x} \le \mathbf{y} \quad \Rightarrow \quad f(\mathbf{x}) \le f(\mathbf{y}), \tag{1.8}$$

and *strictly isotone* if in addition

$$0 \le \mathbf{x} \le \mathbf{y}, \quad f(\mathbf{x}) = f(\mathbf{y}) \quad \Rightarrow \quad \mathbf{x} = \mathbf{y}, \tag{1.9}$$

where inequalities between vectors are interpreted componentwise. For any $1 \le p \le \infty$ [$1 \le p < \infty$], the l_p norm $\|\mathbf{x}\|_p$ is a [strictly] isotone function of the vector $|\mathbf{x}|$ of absolute values,

$$|\mathbf{x}| := (|x_1|, \ldots, |x_n|)^T. \tag{1.10}$$

LEMMA 1.1 [4]. *Let $A \in \mathbb{R}_n^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and let $f : \mathbb{R}_+^m \to \mathbb{R}$ be isotone. Then the problem*

$$\min_{\mathbf{x}} f(|Ax - \mathbf{b}|) \tag{1.11}$$

has a solution in $\mathcal{C}$. Moreover, if f is strictly isotone, then every solution of (1.11) *lies in $\mathcal{C}$.* ■

These results are extended here along the following lines:

(1) Geometrical properties of scaled l_p-approximate solutions are studied in Section 2 for A of full column rank. We show that for $1 < p < \infty$, the set of scaled

l_p-approximate solutions is the same as the set of scaled least-squares solutions. The set of scaled least-squares solutions is also compared with the set of solutions of

$$\min_{\mathbf{x}} f(|A\mathbf{x} - \mathbf{b}|),$$

where $f : \mathbb{R}^m_+ \to \mathbb{R}$ runs over all strictly isotone functions. The closures of the two sets are the same.

(2) In Section 3 we consider the problem

$$\min_{\mathbf{x}} \{f\left(|\mathbf{x}|\right) : A\mathbf{x} = \mathbf{b}\},$$

where A is a matrix of full row rank and f is isotone. We show that there is a solution in $\mathcal{C}$. Moreover, if f is strictly isotone then every solution lies in $\mathcal{C}$.

(3) In Section 4 we consider the problem

$$\min_{\mathbf{x}} \left\{f_2\left(|\mathbf{x}|\right) : \mathbf{x} \in \arg\min_{\mathbf{x}} f_1\left(|A\mathbf{x} - \mathbf{b}|\right)\right\}, \tag{1.12}$$

where $A \in \mathbb{R}^{m\times n}_r$. For f_2 isotone and f_1 strictly isotone, $\mathcal{C}$ contains a solution of (1.12). If also f_2 is strictly isotone, then every solution of (1.12) lies in $\mathcal{C}$.

2. A IS OF FULL COLUMN RANK

Notation and terminology: Throughout this section let $A \in \mathbb{R}^{m\times n}_n$ and $\mathbf{b} \in \mathbb{R}^m$. The convex hull of the basic solutions (1.5) is

$$\mathcal{C} := \text{conv}\{A^{-1}_{I*}\mathbf{b}_I : I \in \mathcal{I}\}. \tag{2.1}$$

The *Hadamard product* $\mathbf{u} \circ \mathbf{v}$ of two vectors $\mathbf{u} = (u_j)$ and $\mathbf{v} = (v_j)$ is the vector

$$\mathbf{u} \circ \mathbf{v} := (u_j v_j). \tag{2.2}$$

Let $\mathcal{D}_m$ be the set of all $m \times m$ positive diagonal matrices. For any $1 \leq p \leq \infty$ and $M \in \mathcal{D}_m$, consider the problem

$$\min_{\mathbf{x}} \|M(A\mathbf{x} - \mathbf{b})\|_p, \tag{2.3}$$

whose solution is unique for $1 < p < \infty$. The solutions are called *scaled l_p-approximate solutions*. For $p = 2$, $D \in \mathcal{D}_m$, the *scaled least-squares solution* of $A\mathbf{x} = \mathbf{b}$ is the solution of

$$\min_{\mathbf{x}} \|D^{1/2}(A\mathbf{x} - \mathbf{b})\|_2, \tag{2.4}$$

given by

$$\mathbf{x} = (A^T DA)^{-1} A^T D\mathbf{b} \qquad \text{(see e.g. [3])}. \tag{2.5}$$

Let the set of scaled l_p-approximate solutions be

$$\mathcal{X}^{\{p\}} := \bigcup_{M \in \mathcal{D}_m} \left\{ \arg\min_{\mathbf{x}} \|M(A\mathbf{x} - \mathbf{b})\|_p \right\}, \tag{2.6}$$

and for $p = 2$, by (2.5),

$$\mathcal{X}^{\{2\}} = \left\{ (A^T DA)^{-1} A^T D\mathbf{b} : D \in \mathcal{D}_m \right\}. \tag{2.7}$$

For $1 < p < \infty$, each arg min in (2.6) is a singleton.

THEOREM 2.1. *Let* $A \in \mathbb{R}_n^{m \times n}$, $1 < p < \infty$. *Then* $\mathcal{X}^{\{p\}} = \mathcal{X}^{\{2\}}$.

PROOF. The result is trivially true if $\mathbf{b} \in R(A)$, the *range* of A.

Let $\mathbf{b} \notin R(A)$. The function $f(\mathbf{x}) := \|M(A\mathbf{x} - \mathbf{b})\|_p$ is convex and differentiable, and a point $\mathbf{x}^*$ is the optimal solution of (2.3) if and only if

$$\nabla f(\mathbf{x}^*) = \mathbf{0}, \tag{2.8}$$

that is,

$$\overline{A}^T \left(\overline{\mathbf{r}} \left(\mathbf{x}^* \right) \circ |\overline{\mathbf{r}} \left(\mathbf{x}^* \right)|^{p-2} \right) = \mathbf{0}, \tag{2.9}$$

where

$$\overline{A} := MA, \qquad \overline{\mathbf{b}} := M\mathbf{b}, \qquad \overline{\mathbf{r}}(\mathbf{x}^*) := \overline{A}\mathbf{x}^* - \overline{\mathbf{b}}. \tag{2.10}$$

$\mathcal{X}^{\{p\}} \subset \mathcal{X}^{\{2\}}$: Let $\mathbf{x}^*$ be the solution of (2.3), and let the diagonal matrix $\overline{M} = \operatorname{diag}(\overline{m}_j)$ be defined by

$$\overline{m}_j := \begin{cases} |\overline{\mathbf{r}}_j(\mathbf{x}^*)|^{p-2} & \text{if} \quad \overline{\mathbf{r}}_j(\mathbf{x}^*) \neq \mathbf{0}, \\ 1 & \text{otherwise}. \end{cases} \tag{2.11}$$

Then (2.9) gives

$$\overline{A}^T \overline{M} (\overline{A}\mathbf{x}^* - \overline{\mathbf{b}}) = \mathbf{0}. \tag{2.12}$$

Therefore

$$\mathbf{x}^* = \left(A^T DA \right)^{-1} A^T D\mathbf{b} \in \mathcal{X}^{\{2\}}, \qquad \text{where} \quad D := M\overline{M}M.$$

$\mathcal{X}^{\{2\}} \subset \mathcal{X}^{\{p\}}$: Let $\mathbf{x}^*$ be any scaled least-squares solution, i.e., $\mathbf{x}^*$ satisfies

$$A^T D(A\mathbf{x}^* - \mathbf{b}) = \mathbf{0} \qquad \text{for some} \quad D = \operatorname{diag}(d_j) \in \mathcal{D}_m. \tag{2.13}$$

Let $\mathbf{r}(\mathbf{x}^*) = A\mathbf{x}^* - \mathbf{b}$, and define the matrix $M = \operatorname{diag}(m_j)$ by

$$m_j := \begin{cases} \sqrt[p]{\dfrac{d_j}{|r_j(\mathbf{x}^*)|^{p-2}}} & \text{if} \quad r_j(\mathbf{x}^*) \neq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{2.14}$$

Then (2.13) gives (2.9). ■

REMARK 2.1. Theorem 2.1 states that every l_p-approximate solution is a scaled least-squares solution. This implies that l_p-approximation problems can be solved as a sequence of scaled least-squares problems, adjusting the scale at each iteration. Indeed, equations (2.9) and (2.12) are the basis of the well-known IRLS (iterative reweighted least squares) algorithm for solving l_p-approximation problems, $1 < p < \infty$; see e.g. [9; 11, p. 250].

REMARK 2.2. We can prove now that $\mathcal{X}^{\{2\}} \subset \mathcal{X}^{\{p\}}$ for $p = 1$ and $p = \infty$ by imitating the proof of $\mathcal{X}^{\{2\}} \subset \mathcal{X}^{\{p\}}$ in Theorem 2.1. As there, let $\mathbf{x}^*$ be any scaled least-squares solution. $\mathcal{X}^{\{2\}} \subset \mathcal{X}^{\{1\}}$: For $p = 1$, with M given by (2.14), (2.13) gives

$$\overline{A}^T \operatorname{sign} \overline{\mathbf{r}}(\mathbf{x}^*) = \mathbf{0},$$

where $\operatorname{sign} \overline{\mathbf{r}}(\mathbf{x}^*) = (\operatorname{sign} \overline{\mathbf{r}}_i(\mathbf{x}^*))$, the signum vector. We conclude that $\mathbf{x}^*$ is a solution of (2.3) for $p = 1$; see for example [11, p. 130].

$\mathcal{X}^{\{2\}} \subset \mathcal{X}^{\{\infty\}}$: Let $p = \infty$, and define the matrix $M = \operatorname{diag}(m_j)$ by

$$m_j := \begin{cases} \dfrac{\sum_{i=1}^m d_i |r_i(\mathbf{x}^*)|^2}{r_j(\mathbf{x}^*)|} & \text{if} \quad r_j(\mathbf{x}^*) \neq \mathbf{0}, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$-M^{-1} D\mathbf{r}(\mathbf{x}^*) \in \operatorname{conv}\{(\operatorname{sign} \overline{r}_j(\mathbf{x}^*))\mathbf{e}_j : |\overline{r}_j(\mathbf{x}^*)| = \|\overline{\mathbf{r}}(\mathbf{x}^*)\|_\infty\} \cap N(\overline{A}^T),$$

where $N(\cdot)$ denotes the null space. By the theorem in [6, p. 35], $\mathbf{x}^*$ is a solution of (2.3) for $p = \infty$. ■

If the norm $\|\cdot\|$ is not isotone (isotone norms are also called *monotone*), then the solutions of $\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|$ may lie outside $\mathcal{C}$.

EXAMPLE 2.1. Let

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The basic solutions are

$$\mathbf{x}_1 = -1, \qquad \mathbf{x}_2 = 1,$$

and their convex hull is the interval

$$\mathcal{C} = [-1, 1].$$

For

$$W = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix},$$

the norm $\|\mathbf{x}\|_W := \|W^{1/2}\mathbf{x}\|_2$ is not isotone. The solution of $\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_W$ is

$$\begin{aligned} \mathbf{x} &= (A^T W A)^{-1} A^T W \mathbf{b} \\ &= 2 \notin \mathcal{C}. \end{aligned}$$

The following example shows that in general $\mathcal{X}^{\{2\}} \neq \mathcal{X}^{\{\infty\}}$.

EXAMPLE 2.2. (Based on [7, Example 5.2]). Let

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -2 & 2 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ -2 \\ 2 \end{pmatrix}.$$

The left plot of Figure 1 shows $\mathcal{X}^{\{2\}}$, which consists of the interiors of the two shaded triangles and their common point $\mathbf{x} = (2, 0)$. The l_∞-approximate solutions are on the line segment X. Finally, the set $\mathcal{X}^{\{\infty\}}$ consists of all points between the two lines L_1, L_2 (excluding L_1, L_2).

Ben-Tal and Teboulle proved $\mathcal{X}^{\{2\}} \subset \mathcal{C}$. Recently Hanke and Neumann [7] showed $\mathcal{X}^{\{2\}}$ to be a union of finitely many polytopes, in general not convex, and

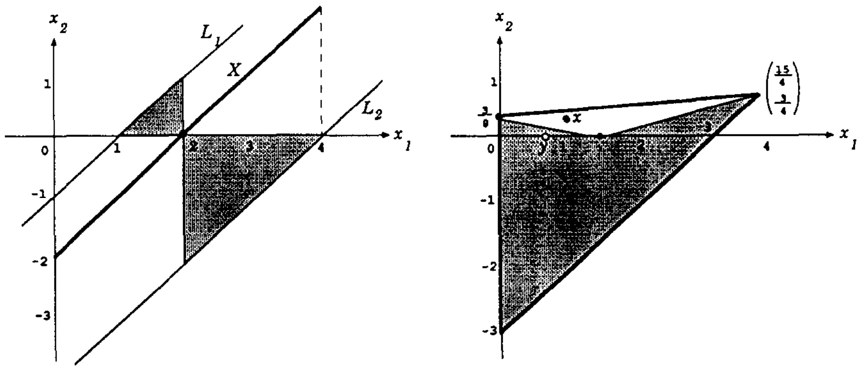

FIG. 1. Illustration of Example 2.2 and 2.3.

cl $\mathcal{X}^{\{2\}} \subset C$, where cl denotes closure. The results of [7] and Theorem 2.1 imply that not all vectors in C are scaled l_p-approximate solutions for $1 < p < \infty$. The next example shows not all vectors in C are solutions of $\min_{\mathbf{x}} f(|A\mathbf{x} - \mathbf{b}|)$ for some strictly isotone function f.

EXAMPLE 2.3. (Based on [7, Example 5.1]). Let

$$A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \\ 2 & 8 \\ 2 & -6 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 3 \\ 3 \end{pmatrix}.$$

The right plot of Figure 1 shows the convex hull C of basic solutions (the triangle bounded by thick lines), and the set cl $\mathcal{X}^{\{2\}}$ (the shaded region).

Consider the points

$$\mathbf{x} = \begin{pmatrix} 1 \\ \frac{3}{8} \end{pmatrix} \in C \backslash \mathrm{cl}\, \mathcal{X}^{\{2\}} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} \in \mathcal{X}^{\{2\}}.$$

Then

$$|A\mathbf{x} - \mathbf{b}| = \begin{pmatrix} \frac{19}{4} \\ 1 \\ 2 \\ \frac{13}{4} \end{pmatrix} > |A\mathbf{y} - \mathbf{b}| = \begin{pmatrix} \frac{14}{3} \\ \frac{2}{3} \\ \frac{5}{3} \\ \frac{5}{3} \end{pmatrix},$$

which implies

$$f(|A\mathbf{x} - \mathbf{b}|) > f(|A\mathbf{y} - \mathbf{b}|)$$

for any strictly isotone function f, showing that the point $\mathbf{x}$ is not a solution of $\min_{\mathbf{x}} f(|A\mathbf{x} - \mathbf{b}|)$.

Let $\mathcal{F}_m$ be the set of all strictly isotone functions on $\mathbb{R}^m$, and let

$$\mathcal{X}^{\{F\}} := \bigcup_{f \in \mathcal{F}_m} \left\{ \mathbf{x} : \mathbf{x} \in \arg\min_{\mathbf{x}} f(|A\mathbf{x} - \mathbf{b}|) \right\}. \tag{2.15}$$

The question

$$\text{cl}\, \mathcal{X}^{\{2\}} \stackrel{?}{=} \text{cl}\, \mathcal{X}^{\{F\}},$$

suggested by Example 2.3, is answered in the affirmative, in Theorem 2.4. First we need the following results. Let $\mathcal{S}$ be a polytope in $\mathbb{R}^m$,

$$\mathcal{S} = \left\{ \mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}^i : \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geq 0, \quad i = 1, \ldots, k \right\}, \tag{2.16}$$

such that $\mathbf{0} \notin \mathcal{S}$. For any $D \in \mathcal{D}_m$, denote

$$\mathbf{x}_D = \arg\min_{\mathbf{x} \in \mathcal{S}} \|D\mathbf{x}\|_2. \tag{2.17}$$

We denote by $\mathbf{x} \lneqq \mathbf{y}$ the fact $\mathbf{x} \leq \mathbf{y}$, $\mathbf{x} \neq \mathbf{y}$. Also denote

$$\mathcal{P} := \{\mathbf{x}_D : D \in \mathcal{D}_m\}, \tag{2.18}$$

$$\mathcal{A} := \{\mathbf{x} \in \mathcal{S} : \nexists \mathbf{y} \in \mathcal{S} \text{ such that } |\mathbf{y}| \lneqq |\mathbf{x}|\}. \tag{2.19}$$

LEMMA 2.1. *Let* $\mathbf{x} \in \mathbb{R}^m$. *Then*

$$\mathbf{x} \in \mathcal{P} \quad \Leftrightarrow \quad Z\mathbf{p} \lneqq 0,\ \mathbf{p} \geq 0 \text{ has no solution}, \tag{2.20}$$

where $Z = (\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^k)$ *is the matrix with columns*

$$\mathbf{z}^i = \mathbf{x} \circ (\mathbf{x}^i - \mathbf{x}), \qquad i = 1, \ldots, k. \tag{2.21}$$

PROOF.

$$\begin{aligned}
\mathbf{x} \in \mathcal{P} \quad &\Leftrightarrow \quad \exists\, D \in \mathcal{D}_m, \quad \langle D(\mathbf{y} - \mathbf{x}), D\mathbf{x} \rangle \geq 0, \quad \forall \mathbf{y} \in \mathcal{S} \quad [1, \text{p.41}], \\
&\Leftrightarrow \quad \exists\, D \in \mathcal{D}_m, \quad \mathbf{x}^T D^2 (\mathbf{y} - \mathbf{x}) \geq 0, \quad \forall \mathbf{y} \in \mathcal{S}, \\
&\Leftrightarrow \quad \exists\, D \in \mathcal{D}_m, \quad \mathbf{x}^T D^2 (\mathbf{x}^i - \mathbf{x}) \geq 0, \quad i = 1, \ldots, k, \\
&\Leftrightarrow \quad Z^T \mathbf{d} \geq \mathbf{0}, \quad \mathbf{d} > \mathbf{0} \text{ has a solution.}
\end{aligned}$$

By a theorem of alternatives [8, p. 29]

$$\mathbf{x} \in \mathcal{P} \quad \Leftrightarrow \quad Z\mathbf{p} \lneqq 0, \mathbf{p} \geq \mathbf{0} \text{ has no solution.}$$

■

THEOREM 2.2. $\mathcal{P} \subset \mathcal{A}$.

PROOF. For any $\mathbf{x} \in \mathcal{S} \setminus \mathcal{A}$, there is $\mathbf{y} \in \mathcal{S}$ such that $|\mathbf{y}| \lneqq |\mathbf{x}|$. Therefore

$$\|D\mathbf{y}\|_2 < \|D\mathbf{x}\|_2$$

for any $D \in \mathcal{D}_m$, which implies $\mathbf{x} \in \mathcal{S} \setminus \mathcal{P}$. ■

THEOREM 2.3. $\mathcal{A} \subset \operatorname{cl} \mathcal{P}$.

PROOF.

Case 1. $\mathbf{x} = (x_i) \in \mathcal{A}$, $x_i \neq 0, i = 1, \ldots, m$. We show that $\mathbf{x} \in \mathcal{P}$. If not, then by Lemma 2.1

$$Z\mathbf{p} \lneqq 0, \qquad \mathbf{p} \geq 0, \tag{2.22}$$

has a solution $\mathbf{p}$. Let

$$\mathbf{y} := \sum_{i=1}^{k} \lambda_i \mathbf{x}^i \in \mathcal{S}$$

with

$$\lambda_j := \frac{p_j}{\sum_{i=1}^{k} p_i}, \qquad j = 1, 2, \ldots, k.$$

Then (2.22) gives

$$\mathbf{x} \circ (\mathbf{y} - \mathbf{x}) \lneqq \mathbf{0}. \tag{2.23}$$

For sufficiently small $\lambda > 0$, the vector

$$\mathbf{z} := \lambda \mathbf{y} + (1 - \lambda)\mathbf{x} \in \mathcal{S}.$$

Then it follows from (2.23) that

$$|\mathbf{z}| \lneqq |\mathbf{x}|, \qquad \text{contradicting} \quad \mathbf{x} \in \mathcal{A}.$$

Case 2. $\mathbf{x} = (x_i) \in \mathcal{A}$, $I^c = \{i : x_i = 0\} \neq \emptyset$. Without loss of generality let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_I \\ \mathbf{0} \end{pmatrix}.$$

We define

$$\mathcal{S}_I := \left\{\mathbf{y}_I : \begin{pmatrix}\mathbf{y}_I\\ \mathbf{0}\end{pmatrix} \in \mathcal{S}\right\}, \tag{2.24}$$

$$\mathcal{A}_I := \{\mathbf{x}_I \in \mathcal{S}_I : \nexists\, \mathbf{y}_I \in \mathcal{S}_I \text{ such that } |\mathbf{y}_I| \lneqq |\mathbf{x}_I|\}. \tag{2.25}$$

Then $\mathcal{S}_I$ is a polytope and $\mathbf{x}_I \in \mathcal{A}_I$. By case 1, there is a positive diagonal matrix D_I such that

$$\mathbf{x}_I = \arg \min_{\mathbf{y}_I \in \mathcal{S}_I} \|D_I \mathbf{y}_I\|_2. \tag{2.26}$$

Let

$$D_n = \begin{pmatrix} \frac{1}{n} D_I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} \in \mathcal{D}_m,$$

and let $\mathbf{x}_n := \mathbf{x}_{D_n}$. Then by the definition (2.17)

$$\left\| \begin{pmatrix} \frac{1}{n} D_I (\mathbf{x}_n)_I \\ (\mathbf{x}_n)_{I^c} \end{pmatrix} \right\|_2 \le \left\| \begin{pmatrix} \frac{1}{n} D_I \mathbf{x}_I \\ \mathbf{0} \end{pmatrix} \right\|_2. \tag{2.27}$$

Since $\mathcal{S}$ is bounded, the sequence $\{\mathbf{x}_n\}$ has a convergent subsequence. Without loss of generality, let $\mathbf{x}_n \to \overline{\mathbf{x}} \in \operatorname{cl} \mathcal{P}$. Then it follows from (2.27) that

$$\overline{\mathbf{x}}_{J^c} = \mathbf{0}$$

and

$$\|D_I \overline{\mathbf{x}}_I\|_2 \le \|D_I \mathbf{x}_I\|_2.$$

By the uniqueness of $\mathbf{x}_I$ in (2.26), we have $\mathbf{x} = \overline{\mathbf{x}} \in \operatorname{cl} \mathcal{P}$. ■

THEOREM 2.4. $\operatorname{cl} \mathcal{X}^{\{2\}} = \operatorname{cl} \mathcal{X}^{\{F\}}$.

PROOF. $\operatorname{cl} \mathcal{X}^{\{2\}} \subset \operatorname{cl} \mathcal{X}^{\{F\}}$ is obviously true. We prove $\operatorname{cl} \mathcal{X}^{\{F\}} \subset \operatorname{cl} \mathcal{X}^{\{2\}}$ by showing $\mathcal{X}^{\{F\}} \subset \operatorname{cl} \mathcal{X}^{\{2\}}$: Let $\mathcal{S}$ be the polytope defined by

$$\mathcal{S} := \left\{\mathbf{r}(\mathbf{x}) = A\mathbf{x} - \mathbf{b} : \mathbf{x} \in \mathcal{C}\right\}.$$

Define $\mathcal{P}$, $\mathcal{A}$ as before, and let $\mathbf{x} \notin \operatorname{cl} \mathcal{X}^{\{2\}}$. Then $\mathbf{r}(\mathbf{x}) \notin \operatorname{cl} \mathcal{P}$. By Theorem 2.3, $\mathbf{r}(\mathbf{x}) \notin \mathcal{A}$. Therefore there is $\mathbf{y} \in \mathcal{C}$ such that

$$|A\mathbf{y} - \mathbf{b}| \lneqq |A\mathbf{x} - \mathbf{b}|,$$

which implies

$$f\big(|A\mathbf{y} - \mathbf{b}|\big) < f\big(|A\mathbf{x} - \mathbf{b}|\big)$$

for any $f \in \mathcal{F}_m$. Therefore $\mathbf{x} \notin \mathcal{X}^{\{F\}}$, proving that $\mathcal{X}^{\{F\}} \subset \mathrm{cl}\,\mathcal{X}^{\{2\}}$. ∎

REMARK 2.3. Theorem 2.4 shows that all linear approximation problems, minimizing a strictly isotone function of the residual, can be solved using scaled least-squares problems. Compare with Remark 2.1.

COROLLARY 2.1. $\mathrm{cl}\,\mathcal{X}^{\{1\}} = \mathrm{cl}\,\mathcal{X}^{\{2\}}$.

PROOF. Follows from Remark 2.2 and Theorem 2.4. ∎

3. *A* IS OF FULL ROW RANK

Throughout this section let $A \in \mathbb{R}_m^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The convex hull of the basic solutions (1.6) is

$$\mathcal{C} := \mathrm{conv}\{\widehat{A_{*J}^{-1}\mathbf{b}} : J \in \mathcal{J}\}, \tag{3.1}$$

where $\widehat{A_{*J}^{-1}\mathbf{b}}$ has $A_{*J}^{-1}\mathbf{b}$ in position J, zeros elsewhere. For any $1 \le p \le \infty$ and $N \in \mathcal{D}_n$, consider the problem

$$\min_{\mathbf{x}}\{\|N^{-1}\mathbf{x}\|_p : A\mathbf{x} = \mathbf{b}\} \tag{3.2}$$

and its solutions, called *scaled minimum-l_p-norm solutions*, which are unique for $1 < p < \infty$.

If $N = I$, these solutions are simply called *minimum-l_p-norm solutions*.

For $p = 2$ and any $D \in \mathcal{D}_n$, the *scaled minimum-l_2-norm solution* of

$$\min_{\mathbf{x}}\{\|D^{-1/2}\mathbf{x}\|_2 : A\mathbf{x} = \mathbf{b}\}, \tag{3.3}$$

is easily computed (see, e.g., [3]):

$$\mathbf{x} = DA^T(ADA^T)^{-1}\mathbf{b}. \tag{3.4}$$

Let the set of scaled minimum-l_p-norm solutions be

$$\mathcal{X}_{\{p\}} := \bigcup_{N \in \mathcal{D}_n} \left\{\mathbf{x} : \mathbf{x} \in \arg\min_{\mathbf{x}}\{\|N^{-1}\mathbf{x}\|_p : A\mathbf{x} = \mathbf{b}\}\right\}. \tag{3.5}$$

Then (3.4) gives

$$\mathcal{X}_{\{2\}} = \{DA^T(ADA^T)^{-1}\mathbf{b} : D \in \mathcal{D}_n\}. \tag{3.6}$$

LEMMA 3.1. *Let* $A \in \mathbb{R}_m^{m\times n}$. *Then* $\mathcal{X}_{\{2\}} \subset \mathcal{C}$.

PROOF. Let $\mathbf{x}$ be the solution of (3.3), $\mathbf{y} := D^{-1/2}\mathbf{x}$, $B := AD^{1/2}$. Then $\mathbf{y}$ is the minimum-l_2-norm solution of $B\mathbf{y} = \mathbf{b}$ and, by [2], a convex combination of basic solutions,

$$\mathbf{y} = \sum_{J\in\mathcal{J}} \gamma_J \widehat{B_{*J}^{-1}\mathbf{b}}.$$

Therefore

$$\begin{aligned}\mathbf{x} &= D^{1/2}\mathbf{y},\\ &= \sum_{J\in\mathcal{J}} \gamma_J \widehat{A_{*J}^{-1}\mathbf{b}} \in \mathcal{C}.\end{aligned}$$

■

The following theorem is analogous to Theorem 2.1.

THEOREM 3.1. *Let* $A \in \mathbb{R}_m^{m\times n}$, $1 < p < \infty$. *Then* $\mathcal{X}_{\{p\}} = \mathcal{X}_{\{2\}}$.

PROOF. Analogous to the proof of Theorem 2.1. ■

THEOREM 3.2. *Let* $A \in \mathbb{R}_m^{m\times n}$. *Then there is a solution* $\mathbf{x}^*$ *of*

$$\min_{\mathbf{x}}\{\|\mathbf{x}\|_1 : A\mathbf{x} = \mathbf{b}\} \tag{3.7}$$

which is a basic solution of $A\mathbf{x} = \mathbf{b}$, *i.e.*, $\mathbf{x}^* = \widehat{A_{*J}^{-1}\mathbf{b}}$ *for some* $J \in \mathcal{J}$.

PROOF. Let $\mathbf{y}$ be any solution of (3.7), and let $\mathbf{c} = \operatorname{sign}\mathbf{y}$. Consider the linear programming problem

$$\text{(LP)} \qquad \begin{aligned} \min \quad & c^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b}, \\ & x_i \geq 0 \quad \text{if} \quad c_i = 1, \\ & x_i = 0 \quad \text{if} \quad c_i = 0, \\ & x_i \leq 0 \quad \text{if} \quad c_i = -1. \end{aligned}$$

Clearly $\mathbf{y}$ is an optimal solution of (LP), and any solution of (LP) is a solution of (3.7). By the theory of linear programming, there is a solution of (LP) which is a basic solution of $A\mathbf{x} = \mathbf{b}$. ■

The following theorem is analogous to Lemma 1.1.

THEOREM 3.3. *Let $A \in \mathbb{R}_m^{m\times n}$ and let f be isotone. Then the problem*

$$\min_{\mathbf{x}}\{f(|\mathbf{x}|): A\mathbf{x} = \mathbf{b}\}, \tag{3.8}$$

has a solution in $\mathcal{C}$. If f is strictly isotone then every solution of (3.8) *lies in $\mathcal{C}$.*

PROOF. Let $\mathbf{x}^* = (x_i^*)$ be any solution of (3.8), and define three index sets for the signs of x_i^*,

$$\pi := \{i : x_i^* > 0\}, \qquad \zeta := \{i : x_i^* = 0\}, \qquad \nu := \{i : x_i^* < 0\}$$

Consider the polyhedral set

$$\mathcal{Y} := \{\mathbf{y} : A\mathbf{y} = \mathbf{b},\ \mathbf{y}_\pi \geq 0,\ \mathbf{y}_\zeta = 0,\ \mathbf{y}_\nu \leq 0\}.$$

Since $\mathbf{x}^* \in \mathcal{Y}$, there exist extreme points $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(r)}$ and extreme directions $\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(t)}$ of $\mathcal{Y}$ such that

$$\mathbf{x}^* = \sum_{i=1}^{r} \lambda_i \mathbf{y}^{(i)} + \sum_{j=1}^{t} \mu_j \mathbf{d}^{(j)},$$

where

$$\sum_{i=1}^{r} \lambda_i = 1, \quad \lambda_i \geq 0, \qquad \mu_j \geq 0.$$

Moreover, the extreme points of $\mathcal{Y}$ are given by $\mathbf{y}^{(i)} = \widehat{A_{*J}^{-1}\mathbf{b}}$ for some $J \in \mathcal{J}$, and the extreme directions belong to the cone

$$\mathcal{D} := \{\mathbf{d} : A\mathbf{d} = 0,\ \mathbf{d}_\pi \geq 0,\ \mathbf{d}_\zeta = 0,\ d_\mu \leq 0\}.$$

Let

$$\mathbf{x}^* = \mathbf{s} + \mathbf{d},$$

where

$$\mathbf{s} = \sum_{i=1}^{r} \lambda_i \mathbf{y}^{(i)}, \qquad \mathbf{d} = \sum_{j=1}^{t} \mu_j \mathbf{d}^{(j)}.$$

Then

$$|\mathbf{x}^*| = |\mathbf{s}| + |\mathbf{d}| \tag{3.9}$$

and

$$f(|\mathbf{x}^*|) \geq f(|\mathbf{s}|).$$

By the optimality of $\mathbf{x}^*$,

$$f(|\mathbf{x}^*|) = f(|\mathbf{s}|), \tag{3.10}$$

showing $\mathbf{s} \in C$ is a solution of (3.8).

Next, suppose that f is strictly isotone. Then (3.10) implies $|\mathbf{x}^*| = |\mathbf{s}|$.

$$\therefore \quad \mathbf{d} = 0, \quad \text{by (3.9)}; \qquad \therefore \quad \mathbf{x}^* = \mathbf{s} \in C.$$

■

The following result, analogous to Theorem 2.4 and Corollary 2.1, is stated without proof.

THEOREM 3.4. $\text{cl}\,\mathcal{X}_{(1)} = \text{cl}\,\mathcal{X}_{(2)} = \text{cl}\,\mathcal{X}_{\{F\}}$, *where*

$$\mathcal{X}_{\{F\}} := \bigcup_{f \in \mathcal{F}_n} \{\mathbf{x} : \mathbf{x} \in \arg\min_{\mathbf{x}} \{f(|\mathbf{x}|) : A\mathbf{x} = \mathbf{b}\}\} \tag{3.11}$$

4. THE GENERAL CASE

Throughout this section let $A \in \mathbb{R}_r^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The basic solutions (1.7) are denoted

$$\mathbf{x}_{IJ} := \widehat{A_{IJ}^{-1}\mathbf{b}_I}, \qquad (I, J) \in \mathcal{N}, \tag{4.1}$$

and their convex hull

$$C := \text{conv}\{\mathbf{x}_{IJ} : (I, J) \in \mathcal{N}\}. \tag{4.2}$$

Let f_1, f_2 be isotone functions. Consider the problem

$$\min \{f_2(|\mathbf{x}|) : \mathbf{x} \in \arg\min_{\mathbf{x}} f_1(|A\mathbf{x} - \mathbf{b}|)\}. \tag{4.3}$$

For any full-rank factorization $A = CR$, the above problem can be solved in stages:

$$\min_{\mathbf{y}} f_1(|C\mathbf{y} - \mathbf{b}|), \tag{4.4}$$

$$\min_{\mathbf{x}} \left\{f_2(|\mathbf{x}|) : R\mathbf{x} = \mathbf{y},\ \mathbf{y} \in \arg\ \min_{\mathbf{y}} f_1(|C\mathbf{y} - \mathbf{b}|)\right\}. \tag{4.5}$$

Combining Lemma 1.1 and Theorem 3.3, we have

THEOREM 4.1. *Let f_2 be isotone, and let f_1 be strictly isotone. Then there is a solution of* (4.3) *which is in C. If in addition f_2 is strictly isotone, every solution of* (4.3) *lies in C.*

PROOF. Let $A = CR$ be any full-rank factorization of A. Then clearly

$$\mathcal{I}(A) = \mathcal{I}(C), \qquad \mathcal{J}(A) = \mathcal{J}(R), \tag{4.6}$$

and $A_{IJ} = C_{I*}R_{*J}\,\forall(I,J) \in \mathcal{N}$. By Lemma 1.1 every solution $\mathbf{y}$ of (4.4) is a convex combination

$$\mathbf{y} = \sum_{I\in\mathcal{I}} \mu_I C_{I*}^{-1}\mathbf{b}_I. \tag{4.7}$$

It follows from Theorem 3.3 that a solution of (4.5) is a convex combination

$$\begin{aligned}
\mathbf{x} &= \sum_{J\in\mathcal{J}} \nu_J \widehat{R_{*J}^{-1}}\mathbf{y}, \\
&= \sum_{J\in\mathcal{J}} \nu_J \sum_{I\in\mathcal{I}} \mu_I \widehat{R_{*J}^{-1} C_{I*}^{-1}}\mathbf{b}_I, \qquad \text{by (4.7)}, \\
&= \sum_{(I,J)\in\mathcal{N}} \lambda_{IJ}\mathbf{x}_{IJ},
\end{aligned} \tag{4.8}$$

where

$$\lambda_{IJ} := \mu_I \nu_J, \qquad (I,J) \in \mathcal{N}, \tag{4.9}$$

are also convex weights. The second part follows by applying the second part of Theorem 3.3. ∎

An immediate corollary of Theorem 4.1 is

COROLLARY 4.1. *Let* $1 \le p_1 < \infty$. *Then the problem*

$$\min\left\{\|\mathbf{x}\|_{p_2} : \mathbf{x} \in \arg\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_{p_1}\right\} \tag{4.10}$$

has a solution in $\mathcal{C}$. *Moreover, if* $1 \le p_2 < \infty$ *then every solution of* (4.10) *lies in* $\mathcal{C}$.

The next example shows that Corollary 4.1 does not hold for $p_1 = \infty$.

EXAMPLE 4.1. Let

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then the solution set of

$$\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_\infty \tag{4.11}$$

is $0 \leq \mathbf{x} \leq 2$. For all p, the minimum-l_p-norm best l_∞-approximate solution is $\mathbf{x} = 0$ and does not belong to C, which here is the singleton $\{1\}$.

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Received 5 November 1992; final manuscript accepted 19 April 1993