Indag. Mathem., N.S., 6 (4), 419–432 December 18, 1995

Chebyshev-type quadrature for analytic weights on the circle and the interval

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Communicated by Prof. J. Korevaar at the meeting of January 30, 1995

ABSTRACT

We give a sharp asymptotic bound on the number of nodes needed for Chebyshev-type (= equal weight) quadrature of degree p for measures on $[-1,1]$ of the form $w(t)/(\pi\sqrt{1-t^2})dt$, where w is positive on $[-1,1]$ and analytic in a neighborhood of $[-1,1]$. This bound is derived from a corresponding bound for Chebyshev-type quadrature for analytic weights on the unit circle. In addition, we present some results on Chebyshev-type quadrature on certain algebraic curves.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let ν be a probability measure on a compact subset of the complex plane $\mathbb{C}$. A Chebyshev-type quadrature formula for ν is a numerical integration formula of the form

$$(1.1)\qquad \int \phi(\zeta)d\nu(\zeta) \approx \frac{1}{n}\sum_{j=1}^{n} \phi(\zeta_j)$$

where the nodes $\zeta_1, \zeta_2, \ldots, \zeta_n$ are points in $\mathbb{C}$. The (algebraic) degree of (1.1) is the maximal number p such that equality holds for all polynomials ϕ (in one complex variable) of degree $\leq p$. See [2], [3], [6] for surveys on Chebyshev-type quadrature.

Usually, we want to restrict the position of the nodes to some fixed compact

* Supported by the Netherlands Foundation for Mathematics SMC with financial aid from the Netherlands Organization for the Advancement of Scientific Research NWO. Work completed while at the University of South Florida, Tampa, Fl, USA.

set. For example, if the support of ν is a real interval we would like to have the nodes ζ_j in that interval. Accordingly we introduce the following definition.

Definition. Let ν be a compactly supported probability measure on $\mathbb{C}$ and let $K \subset \mathbb{C}$ be compact. We denote by $N_K(p;\nu)$ the minimal number n such that a Chebyshev-type quadrature formula of degree $\geq p$ exists with all its n nodes in K. If the measure ν is clear from the context we write simply $N_K(p)$.

We are interested in the behavior of $N_K(p;\nu)$ as $p \to \infty$.

To a large extent this problem is open. Only for some special measures ν and sets K the asymptotic behavior of $N_K(p;\nu)$ is known. Most of these results deal with the case when the support of ν is equal to the interval $I = [-1, 1]$ and $K = I$. The main example is the arcsin measure $\pi^{-1}(1-t^2)^{-1/2}dt$ for which the Gauss quadrature formula has equal weights and so $N_I(2p-1) = p$ for every p. Several other measures ν on $[-1, 1]$ are known for which $N_I(p;\nu) \leq p$ for every p, see [2] and the references given there. A general description of such measures has been given by Peherstorfer [11], [12].

In contrast, for the normalized Lebesgue measure $dt/2$ on $[-1, 1]$, Bernstein [1] proved that $N_I(p) \sim p^2$. This was later generalized by the author [8] to ultraspherical weights $C_\alpha(1-t^2)^\alpha dt$ with $\alpha > 0$ for which one has $N_I(p) \sim p^{2\alpha+2}$. The case $-\frac{1}{2} < \alpha < 0$ is still open and for $\alpha < -\frac{1}{2}$, see [10]. [Here and later $a_p \sim b_p$ means that a_p/b_p is bounded above and below by positive constants independent of p.]

In this paper we will present an estimate on $N_I(p;\nu)$ for measures ν of the form

$$(1.2) \qquad d\nu(t) = \frac{w(t)}{\pi\sqrt{1-t^2}}\, dt$$

with w positive on $[-1, 1]$ and analytic in a neighborhood of $[-1, 1]$. Our main result is the following.

Theorem 1. *Let ν be a probability measure on $[-1, 1]$ of the form (1.2) where w is a positive analytic function on $[-1, 1]$. Then*

$$(1.3) \qquad \limsup_{p\to\infty} \frac{N_I(p;\nu)}{p} \leq \max_{t\in I} \frac{1}{w(t)}.$$

Remark A. An inequality in the opposite direction may be derived from a result in Kahaner [5, Corollary 2] which we will adapt to our situation. Taking his ν equal to our $N_I(p)$, his m equal to our p, his W equal to 1 and his $p(x)$ equal to $w(x)/(\pi\sqrt{1-x^2})$, Kahaner's Corollary 2 asserts the following. If

$$(N_I(p) - p - 1)/[p/2] < P \quad \text{as } p \to \infty,$$

then

$$w(x) \geq 1/(P+2).$$

As a result,

$$(1.4)\qquad \liminf_{p\to\infty} \frac{N_I(p;\nu)}{p} \geq \max_{t\in I} \frac{1}{2w(t)}.$$

This inequality shows that, apart from a factor 2, inequality (1.3) is best possible. Actually, Kahaner states his result only for Chebyshev-type quadrature formulas with distinct nodes, but from his proofs it is easily seen that this restriction is unnecessary.

Of course, (1.3) and (1.4) imply that $N_I(p;\nu) \sim p$ for every measure ν of the form (1.2). For the special case $w(t) = 1 - at$ with $-1 < a < 1$ the relation $N_I(p) \sim p$ was established in [9] using different methods.

Remark B. Weights of the form (1.2) were also considered by Peherstorfer. In [12, Theorem 4(a)] he proved that $N_I(p;\nu) \leq p + 1$ for every p, if w satisfies

$$w(\cos(\theta)) = \operatorname{Re} G(e^{i\theta})$$

where $G(z)$ is analytic and has positive real part in the disk $|z| < 3 + 2\sqrt{2}$.

This result applies for example if w is analytic and has a positive real part in an ellipse E_ρ,

$$E_\rho := \{t \in \mathbb{C} \,|\, |t + \sqrt{t^2 - 1}| < \rho\},$$

with $\rho > 5(3 + 2\sqrt{2})$. The constant $5(3 + 2\sqrt{2})$ is certainly not best possible.

To see this, note that $W(z) = w((z + z^{-1})/2)$ is analytic and has positive real part in the annulus $\rho^{-1} < |z| < \rho$. Then the coefficients a_j in the Laurent expansion $W(z) = \sum_{j=-\infty}^{\infty} a_j z^j$ satisfy the estimates

$$|a_j|(\rho^j + \rho^{-j}) \leq 2, \quad j = 1, 2, \ldots,$$

cf. [4, p. 102]. We also have $a_0 = 1$ and $a_{-j} = a_j$. Now for the function

$$G(z) = 1 + 2\sum_{j=1}^{\infty} a_j z^j,$$

is is clear that $w(\cos\theta) = \operatorname{Re} G(e^{i\theta})$ and for $|z| = r < 3 + 2\sqrt{2}$, we have

$$\operatorname{Re} G(z) \geq 1 - 2\sum_{j=1}^{\infty} |a_j| r^j > 0.$$

Thus Theorem 4(a) of [12] can be applied.

I am grateful to Franz Peherstorfer for pointing out to me that a result of this type should hold.

To obtain Theorem 1 we first consider measures on the unit circle C in the complex plane, $C := \{z \in \mathbb{C} \mid |z| = 1\}$. We denote by D the unit disk, $D := \{z \in \mathbb{C} \mid |z| \leq 1\}$, and by λ the normalized Lebesgue measure on C (total mass one). Let ν be a probability measure on the unit circle of the form $d\nu = W d\lambda$ where W is positive on C and analytic in a neighborhood of C. For such a measure, we will prove the following two estimates.

Theorem 2. *Let ν be a probability measure on the unit circle of the form $d\nu = W d\lambda$ where W is positive on C and analytic in a neighborhood of C. Then we have* (a)

$$(1.5)\qquad \limsup_{p\to\infty} \frac{N_D(p;\nu)}{p} \le \max_{|z|=1} \frac{1}{W(z)}$$

and (b)

$$(1.6)\qquad \limsup_{p\to\infty} \frac{N_C(p;\nu)}{p} \le \max_{|z|=1} \frac{2}{W(z)}.$$

Remark C. For $\nu = \lambda$ we have $N_D(p) = 1$ (one node at the origin) and $N_C(p) = p + 1$ (nodes at the vertices of a regular $(p+1)$-gon). This example shows that it is not always true that $N_D(p) \sim p$ for measures satisfying the conditions of Theorem 2. Thus (1.5) is not always best possible.

Remark D. The estimate (1.6) is best possible, except for the factor 2, since we have the following estimate from below

$$(1.7)\qquad \liminf_{p\to\infty} \frac{N_C(p;\nu)}{p} \ge \max_{|z|=1} \frac{1}{W(z)}.$$

This can be proved using the same ideas used in [5] to prove (1.4).

We finish this introduction by stating briefly the contents of the following sections. We start in Section 2 by proving three lemmas that will be used in the proofs of Theorem 2. In Section 3 we prove part (a) of Theorem 2 and in Section 4 we prove part (b). In Section 5 Theorem 1 is deduced from Theorem 2. Finally, we present in Section 6 an application of Theorem 2 to Chebyshev-type quadrature on certain algebraic curves, including ellipses.

2. PRELIMINARIES

With any probability measure ν with support in a disk $|z| \le R$, we associate the function

$$(2.1)\qquad f_\nu(z) = \exp\left(\int \log(z-\zeta)\, d\nu(\zeta)\right), \quad |z| > R,$$

where we take $\log(z-\zeta)$ equal to an arbitrary value of $\log z$ plus the principal value of $\log(1-\zeta/z)$. Clearly, f_ν is an analytic function for $|z| > R$ and $f_\nu(z)$ behaves like z for $|z| \to \infty$. It is easy to see that

$$(2.2)\qquad f_\nu(z) = z \exp\left(-\sum_{k=1}^{\infty} \frac{c_k}{k} z^{-k}\right), \quad |z| > R,$$

where the c_k are the moments of ν,

$$(2.3)\qquad c_k = \int \zeta^k\, d\nu(\zeta), \quad k \ge 0.$$

Our construction of Chebyshev-type quadrature formulas is based on the following lemma. This result is not new, but we have included the proof for convenience. Closely related results can be found in [9], [10], [11], [12], [14].

Lemma 3. *Let ν be a compactly supported probability measure. Let $p, n \in \mathbb{N}$ and suppose $F_n(z) = \prod_{j=1}^{n}(z-\zeta_j)$ is a polynomial of degree n such that*

$$(2.4) \qquad f_\nu(z)^n = F_n(z) + O(z^{n-p-1}), \quad (z \to \infty).$$

Then

$$\int \phi(\zeta)d\nu(\zeta) = \frac{1}{n}\sum_{j=1}^{n}\phi(\zeta_j)$$

for every polynomial ϕ of degree $\leq p$.

Proof. Dividing (2.4) by z^n and taking logarithms we get by (2.2)

$$-n\sum_{k=1}^{\infty}\frac{c_k}{k}z^{-k} = \log\prod_{j=1}^{n}\left(1-\frac{\zeta_j}{z}\right) + O(z^{-p-1})$$
$$= -\sum_{k=1}^{\infty}\sum_{j=1}^{n}\frac{\zeta_j^k}{k}z^{-k} + O(z^{-p-1}).$$

Comparing the coefficients of z^{-k} and using (2.3) we find

$$\int \zeta^k d\nu(\zeta) = \frac{1}{n}\sum_{j=1}^{n}\zeta_j^k, \quad k = 1,\ldots,p.$$

Since ν is a probability measure we also have equality for $k = 0$ and the lemma follows. □

For measures ν as in Theorem 2 the function f_ν has an analytic continuation to the exterior of a smaller disk. The precise statement is in the following lemma. For $r > 0$, we denote by C_r the circle $|z| = r$ and by λ_r the normalized Lebesgue measure on C_r.

Lemma 4. *Let ν be a probability measure on the unit circle such that $d\nu = Wd\lambda$ where W is positive on C and analytic in a neighborhood of C. Then there is $r_0 < 1$ and for every $r \in (r_0, 1)$ a probability measure ν_r on C_r of the form $d\nu_r = W_r d\lambda_r$ such that*

$$(2.5) \qquad f_{\nu_r}(z) = f_\nu(z), \quad |z| > 1,$$

[that is, f_{ν_r} is an analytic continuation of f_ν to $|z| > r$], and

$$(2.6) \qquad \lim_{r \nearrow 1}\sup_{|z|=1}|W_r(rz) - W(z)| = 0.$$

Proof. Suppose W has the Laurent expansion

$$(2.7) \qquad W(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \quad a_{-j} = \bar{a}_j.$$

Then the negative Laurent coefficients are equal to the moments of ν. Since W has an analytic continuation to a neighborhood of C the series (2.7) converges in an annulus $R^{-1} < |z| < R$ for some $R > 1$. Set for $r \in (R^{-1}, 1)$,

$$(2.8) \qquad W_r(z) := \sum_{j=-\infty}^{0} a_j z^j + \sum_{j=1}^{\infty} a_j r^{-2j} z^j, \quad R^{-1} < |z| < Rr^2.$$

The function W_r is real on the circle C_r and it is easy to see that (2.6) holds. Since W is positive on C, this implies that there is $r_0 \in [R^{-1}, 1)$ such that W_r is strictly positive on C_r for every $r \in (r_0, 1)$.
Let $r \in (r_0, 1)$ and write $d\nu_r = W_r d\lambda_r$. Because of (2.8) we have

$$\int \zeta^k d\nu_r(\zeta) = a_{-k}, \quad k \geq 0.$$

Therefore ν_r is a probability measure on C_r and the moments of ν_r agree with the moments of ν. This implies (2.5), cf. (2.2), (2.3). □

As a final preliminary to the proofs we need a result on the dependence of f_ν on the measure ν. Recall that a sequence of measures $(\nu_n)_n$ whose supports are in a fixed compact set K converges to a measure ν on K in the weak-star topology if

$$\lim_{n\to\infty} \int \phi d\nu_n = \int \phi d\nu$$

for every continuous function ϕ on K.

Lemma 5. *Let $r > 0$ and let ν and ν_n, $n = 1, 2, \ldots$ be probability measures on the circle C_r such that $(\nu_n)_n$ converges to ν in the weak-star sense. Then*

$$\lim_{n\to\infty} \frac{f_\nu(z)}{f_{\nu_n}(z)} = 1, \quad |z| > r,$$

and the convergence is uniform on $|z| \geq \rho$ for every $\rho > r$.

Proof. From (2.1) it is clear that $f_{\nu_n}(z)$ converges to $f_\nu(z)$ for every $|z| > r$. To prove the uniform convergence, let $\rho > r$ and note that the functions $f_{\nu_n}(z)/z$ are analytic on $|z| > r$, including ∞, and that they are uniformly bounded for $|z| \geq \rho$. By Montel's theorem, we find that $(f_{\nu_n}(z)/z)_n$ is a normal family and so the pointwise convergence implies that $f_{\nu_n}(z)/z$ converges to $f_\nu(z)/z$ uniformly for $|z| \geq \rho$. This proves the lemma. □

3. NODES INSIDE THE UNIT DISK

Proof of Theorem 2(a). Let r_0, ν_r and W_r be as in Lemma 4. We write f instead of f_ν. By Lemma 4 it has an analytic continuation to $|z| > r_0$ which we also denote by f.

Choose $\alpha > 0$ such that $\alpha < \min_{|z|=1} W(z)$. By (2.6) we can take $r \in (r_0, 1)$ such that $W_r(z) > \alpha$ for every z on the circle C_r. Then $\nu_r - \alpha\lambda_r$ is a positive measure on C_r with total mass $1 - \alpha$. We introduce a sequence $(Q_m)_m$ of polynomials such that

- Q_m is a monic polynomial of degree m,
- the zeros of Q_m are on $|z| = r$, and
- the asymptotic zero distribution of the Q_m is equal to $(\nu_r - \alpha\lambda_r)/(1-\alpha)$.

This last requirement means the following. With the polynomial Q_m is associated its normalized zero distribution

$$\mu_m := \frac{1}{m}\sum_{j=1}^{m} \delta_{\zeta_j}$$

where $\zeta_1, \ldots, \zeta_m$ are the zeros of Q_m counted according to multiplicity. Then it is required that in weak-star sense

$$\lim_{m\to\infty} \mu_m = (\nu_r - \alpha\lambda_r)/(1-\alpha).$$

Then it also follows that

$$\text{(3.1)} \qquad \lim_{m\to\infty} [(1-\alpha)\mu_m + \alpha\lambda_r] = \nu_r.$$

Since $f_{\mu_m}(z) = Q_m(z)^{1/m}$ and $f_{\lambda_r}(z) = z$ we obtain from Lemma 5

$$\lim_{m\to\infty} \frac{f(z)}{z^{\alpha} Q_m(z)^{(1-\alpha)/m}} = 1$$

uniformly for $|z| \geq \rho$ for every $\rho > r$. [Recall that $f_{\nu_r} = f$ by Lemma 4.]

Somewhat more generally, we can consider sequences $(m_p)_p$ and $(\alpha_p)_p$ such that

$$m_p \in \mathbb{N}, \qquad \lim_{p\to\infty} m_p = \infty, \qquad 0 \leq \alpha_p \leq 1, \qquad \lim_{p\to\infty} \alpha_p = \alpha.$$

Then it follows from (3.1) that in weak-star sense

$$\lim_{p\to\infty} [(1-\alpha_p)\mu_{m_p} + \alpha_p\lambda_r] = \nu_r,$$

and again from Lemma 5 we get

$$\text{(3.2)} \qquad \lim_{p\to\infty} \frac{f(z)}{z^{\alpha_p} Q_{m_p}(z)^{(1-\alpha_p)/m_p}} = 1$$

uniformly for $|z| \geq \rho$ for every $\rho > r$.
We fix a number $\rho \in (r, 1)$, and for $p \in \mathbb{N}$ we take

$$n = n_p = [p/\alpha], \qquad m_p = n_p - p, \qquad \alpha_p = p/n_p.$$

Here $[x]$ denotes the largest integer $\leq x$. Since $0 < \alpha < 1$, we have $m_p \to \infty$ and $\alpha_p \to \alpha$. Hence (3.2) holds, i.e.,

$$\text{(3.3)} \qquad \lim_{p\to\infty} \frac{f(z)}{z^{p/n} Q_{n-p}(z)^{1/n}} = 1$$

uniformly for $|z| \geq \rho$. Now we take the polynomial P_p of degree p such that

$$\text{(3.4)} \qquad f(z)^n = P_p(z)Q_{n-p}(z) + \mathcal{O}(z^{n-p-1}) \quad (z\to\infty).$$

This implies that P_p is the polynomial part of $f(z)^n/Q_{n-p}(z)$ and therefore P_p is determined uniquely by (3.4). We want to prove that for p large enough, P_p has all its zeros in the unit disk. From (3.4) we deduce the integral representation

$$P_p(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)^n}{Q_{n-p}(\zeta)} \frac{d\zeta}{\zeta - z}, \quad |z| < R,$$

which is valid for every $R > 1$. By the residue theorem, we have for $|z| \geq 1$,

$$P_p(z) = \frac{f(z)^n}{Q_{n-p}(z)} + \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)^n}{Q_{n-p}(\zeta)} \frac{d\zeta}{\zeta - z}. \tag{3.5}$$

Choose $\delta > 0$ such that $(1+\delta)\rho^\alpha < 1 - \delta$. Then for p, and thus $n = n_p$, large enough we have

$$(1+\delta)^n \rho^{\alpha n} \frac{\rho}{1-\rho} < (1-\delta)^n \tag{3.6}$$

and also by (3.3)

$$\sup_{|z|\geq\rho} \left| \frac{f(z)}{z^{p/n} Q_{n-p}(z)^{1/n}} - 1 \right| < \delta. \tag{3.7}$$

Hence, if $|\zeta| = \rho$,

$$\left| \frac{f(\zeta)^n}{Q_{n-p}(\zeta)} \right| \leq (1+\delta)^n \rho^p \leq (1+\delta)^n \rho^{\alpha n},$$

so that for every $|z| \geq 1$,

$$\left| \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)^n}{Q_{n-p}(\zeta)} \frac{d\zeta}{\zeta - z} \right| \leq (1+\delta)^n \rho^{\alpha n} \frac{\rho}{1-\rho}. \tag{3.8}$$

From (3.7) it also follows that for $|z| \geq 1$,

$$\left| \frac{f(z)^n}{Q_{n-p}(z)} \right| \geq (1-\delta)^n. \tag{3.9}$$

Using (3.5), (3.6), (3.8) and (3.9) we find for p large enough and $|z| \geq 1$,

$$|P_p(z)| \geq (1-\delta)^n - (1+\delta)^n \rho^{\alpha n} \frac{\rho}{1-\rho} > 0.$$

This implies that for large p, the polynomial P_p has no zeros in $|z| \geq 1$ and therefore all zeros are in the unit disk. Also Q_{n-p} has its zeros in the unit disk (they are on the circle $|z| = r$). Therefore, by (3.4) and Lemma 3, there exists a Chebyshev-type quadrature formula for ν of degree $\geq p$ with n nodes in the unit disk. Thus for large p, we have $N_D(p) \leq n \leq p/\alpha$, so that $\limsup_{p\to\infty} N_D(p)/p \leq 1/\alpha$. This holds for every $\alpha < \min_{|z|=1} W(z)$ and (1.5) follows. □

For the proof of part (b) of Theorem 2 we follow closely the proof of part (a) as given in the previous section. However, we have to use an additional trick to make sure that the nodes are exactly on the circle and therefore we have to set up things just a little differently. This will cause the extra factor 2 that appears in the estimate (1.6).

Proof of Theorem 2(b). We choose α, r and the sequence $(Q_m)_m$ as in the proof of Theorem 2(a). We fix $\rho \in (r, 1)$ and for $p \in \mathbb{N}$ we now take

$$n = n_p = 2[p/\alpha], \qquad m_p = n - 2p, \qquad \alpha_p = 2p/n. \tag{4.1}$$

Clearly $m_p \to \infty$ and $\alpha_p \to \alpha$, so that (3.2) holds, that is,

$$\lim_{p\to\infty} \frac{f(z)}{z^{2p/n} Q_{n-2p}(z)^{1/n}} = 1 \tag{4.2}$$

uniformly for $|z| \geq \rho$.
Now we take the polynomial P_p such that

$$f(z)^n = z^p P_p(z) Q_{n-2p}(z) + \mathcal{O}(z^{n-p-1}), \quad (z \to \infty). \tag{4.3}$$

Then we have for $|z| \geq 1$, cf. (3.5),

$$P_p(z) = \frac{f(z)^n}{z^p Q_{n-2p}(z)} + \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)^n}{\zeta^p Q_{n-2p}(\zeta)} \frac{d\zeta}{\zeta - z}. \tag{4.4}$$

Let $\delta > 0$ be such that $(1+\delta)\rho^{\alpha/2} < 1 - \delta$. Then it follows from (4.2) that we have for p large enough,

$$\left| \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)^n}{\zeta^p Q_{n-2p}(\zeta)} \frac{d\zeta}{\zeta - z} \right| \leq (1+\delta)^n \rho^{\alpha n/2} \frac{\rho}{1-\rho}, \quad |z| \geq 1, \tag{4.5}$$

and

$$\left| \frac{f(z)^n}{z^p Q_{n-2p}(z)} \right| \geq (1-\delta)^n |z|^p \geq (1-\delta)^n, \quad |z| \geq 1. \tag{4.6}$$

Combining (4.4), (4.5) and (4.6) we find for p large enough,

$$|P_p(z)| \geq (1-\delta)^n - (1+\delta)^n \rho^{\alpha n/2} \frac{\rho}{1-\rho}, \quad |z| \geq 1.$$

By the choice of δ it follows that $|P_p(z)| > 0$ for $|z| \geq 1$ and p large enough. Thus for large p, say $p \geq p_0$, the polynomial P_p has all its zeros inside the open unit disk.

Finally, we take $p \geq p_0$ and we define the polynomial F_n of degree $n = n_p = 2[p/\alpha]$ by

$$F_n(z) := z^p P_p(z) Q_{n-2p}(z) + P_p^*(z) Q_{n-2p}^*(z). \tag{4.7}$$

Here the * denotes the reversed polynomial, that is, $P_p^*(z) = z^p \overline{P_p(1/\bar{z})}$ and similarly for $Q_{n-2p}^*(z)$. Because P_p and Q_{n-2p} have their zeros in the open unit disk, the polynomial F_n has all its zeros on the unit circle and all these zeros are simple, see Schur [13, p. 230]. Furthermore by (4.3) and (4.7),

$$(4.8) \qquad f(z)^n = F_n(z) + O(z^{n-p}), \quad (z \to \infty).$$

Then it follows from (4.8) and Lemma 3 that the zeros of F_n are the nodes of a Chebyshev-type quadrature formula for ν of degree $\geq p-1$. So for large p, we have $N_C(p-1) \leq n \leq 2p/\alpha$, which implies

$$\limsup_{p\to\infty} \frac{N_C(p;\nu)}{p} \leq \frac{2}{\alpha}.$$

This holds for every $\alpha < \min_{|z|=1} W(z)$, so that (1.6) follows. □

Remark E. Let ν be a probability measure on the circle C and suppose the Chebyshev-type quadrature formula

$$(4.9) \qquad \int \phi(\zeta)d\nu(\zeta) = \frac{1}{n}\sum_{j=1}^{n} \phi(\zeta_j)$$

is exact for all polynomials ϕ of degree $\leq p$ with nodes $\zeta_j \in C$. Using $\phi(\zeta) = \zeta^k$ and taking complex conjugates in (4.9), we obtain

$$\int \zeta^{-k} d\nu(\zeta) = \frac{1}{n}\sum_{j=1}^{n} \zeta_j^{-k}, \quad k = 1,\ldots,p.$$

Note that this works because all nodes are on the unit circle. Thus it follows that (4.9) holds for every Laurent polynomial ϕ of degree $\leq p$.

Remark F. Theorem 2(b) also gives results on Chebyshev-type quadrature for trigonometric polynomials on $[0, 2\pi]$.

The context is the following. Let μ be a probability measure on $[0, 2\pi]$. The trigonometric degree of a Chebyshev-type quadrature formula

$$\int_0^{2\pi} T(t)d\mu(t) \approx \frac{1}{n}\sum_{j=1}^{n} T(t_j),$$

is the maximal p such that equality holds for all trigonometric polynomials T of degree $\leq p$. We denote by $N^T_{[0,2\pi]}(p;\mu)$ the minimal number n such that a Chebyshev-type formula for μ exists with n nodes in $[0, 2\pi]$ and trigonometric degree $\geq p$.

By means of the mapping $\zeta = e^{it}$ we obtain from μ a measure ν on the unit circle. Using Remark E one can prove easily that the following relation holds for every p,

$$(4.10) \qquad N^T_{[0,2\pi]}(p;\mu) = N_C(p;\nu).$$

Then we have the following corollary to Theorem 2(b), which will be used in Section 6.

Corollary 6. *Let μ be a probability measure on $[0, 2\pi]$ of the form $d\mu(t) = w(t)dt$*

where w is a positive 2π-periodic function on $\mathbb{R}$ and analytic in a neighborhood of $\mathbb{R}$. Then

$$\limsup_{p\to\infty} \frac{N^T_{[0,2\pi]}(p;\mu)}{p} \le \max_{t\in[0,2\pi]} \frac{1}{\pi w(t)}. \tag{4.11}$$

Proof. The measure ν on the unit circle corresponding to μ is given by $d\nu = 2\pi W d\lambda$ where $W(e^{it}) = w(t)$. So (4.11) follows from (4.10) and Theorem 2(b). □

5. WEIGHTS ON THE INTERVAL

In this section we prove Theorem 1. Let ν be a probability measure on the interval $[-1,1]$ as in the statement of Theorem 1. We associate with ν the function

$$W(z) := w((z+z^{-1})/2), \quad z \in C.$$

Then $W d\lambda$ is a probability measure on the circle which is symmetric with respect to the real axis. The following lemma shows how to obtain a Chebyshev-type quadrature formula for ν from a symmetric Chebyshev-type quadrature formula for $W d\lambda$.

Lemma 7. *Let $p, m \in \mathbb{N}$ and let $\zeta_1, \zeta_2, \ldots, \zeta_{2m}$ be points on the unit circle such that $\zeta_{2m+1-j} = \bar{\zeta}_j, j = 1, \ldots, m$. Suppose that for every polynomial ϕ of degree $\le p$ we have*

$$\int \phi(\zeta) W(\zeta) d\lambda(\zeta) = \frac{1}{2m} \sum_{j=1}^{2m} \phi(\zeta_j). \tag{5.1}$$

Then for every polynomial ϕ of degree $\le p$

$$\int_{-1}^{1} \phi(t) \frac{w(t)}{\pi\sqrt{1-t^2}} dt = \frac{1}{m} \sum_{j=1}^{m} \phi(t_j), \tag{5.2}$$

where $t_j = \operatorname{Re} \zeta_j, j = 1, \ldots, m$.

Proof. Let ϕ be a polynomial of degree $\le p$. It is easy to verify that

$$\int_{-1}^{1} \phi(t) \frac{w(t)}{\pi\sqrt{1-t^2}} dt = \int \phi((\zeta+\zeta^{-1})/2) W(\zeta) d\lambda(\zeta). \tag{5.3}$$

Since $\phi((\zeta+\zeta^{-1})/2)$ is a Laurent polynomial of degree $\le p$, we obtain from (5.1) and Remark E

$$\begin{cases} \int \phi((\zeta+\zeta^{-1})/2) W(\zeta) d\lambda(\zeta) = \dfrac{1}{2m} \displaystyle\sum_{j=1}^{2m} \phi((\zeta_j+\zeta_j^{-1})/2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad = \dfrac{1}{2m} \displaystyle\sum_{j=1}^{2m} \phi(t_j). \end{cases} \tag{5.4}$$

Since $\zeta_{2m+1-j} = \bar{\zeta}_j$ we find $t_{2m+1-j} = t_j$. Then (5.2) follows from (5.3) and (5.4). $\square$

Proof of Theorem 1. Let $W(z) := w((z + z^{-1})/2)$ and take $\alpha > 0$ such that

$$\alpha < \min_{t \in I} w(t) = \min_{z \in C} W(z).$$

Then $W d\lambda$ satisfies the conditions of Theorem 2 and so by part (b) we have $N_C(p; W d\lambda) \leq 2p/\alpha$ for every large p. Therefore for large p, there are $n \leq 2p/\alpha$ and nodes $\zeta_1, \ldots, \zeta_n$ on the unit circle such that

$$\int \phi(\zeta) W(\zeta) d\lambda(\zeta) = \frac{1}{n} \sum_{j=1}^{n} \phi(\zeta_j) \tag{5.5}$$

for every polynomial ϕ of degree $\leq p$.

By (4.1) we may assume n is even, say $n = 2m$. Since W is symmetric with respect to the real axis an inspection of the proof of Theorem 2(b) shows that we may assume that the nodes ζ_j are non-real and satisfy $\zeta_{2m+1-j} = \bar{\zeta}_j$, $j = 1, \ldots, m$. Indeed, in the proof of Theorem 2(b) we may assume that Q_{n-2p} is chosen to be a real polynomial. Since f (as defined in the proof of Theorem 2) has real coefficients in the Laurent expansion around ∞, it follows from (4.3) and (4.7) that P_p and F_n are real polynomials as well. Moreover

$$F_n(1) = 2P_p(1) Q_{n-2p}(1) \neq 0,$$
$$F_n(-1) = 2(-1)^p P_p(-1) Q_{n-2p}(-1) \neq 0.$$

Thus the zeros of F_n are non-real and they come in conjugate pairs.

Using Lemma 7 and (5.5) we then find that for large p there is $m \leq p/\alpha$ such that

$$\int_{-1}^{1} \phi(t) \frac{w(t)}{\pi\sqrt{1-t^2}} dt = \frac{1}{m} \sum_{j=1}^{m} \phi(t_j)$$

for every polynomial of degree $\leq p$, where $t_j = \operatorname{Re} \zeta_j \in [-1, 1]$, $j = 1, \ldots, m$. Hence for large p, we have $N_I(p) \leq p/\alpha$ and this implies

$$\limsup_{p \to \infty} \frac{N_I(p; \nu)}{p} \leq \frac{1}{\alpha}.$$

Since $\alpha < \min_{t \in I} w(t)$ can be chosen arbitrarily the estimate (1.3) follows. $\square$

6. CHEBYSHEV-TYPE QUADRATURE ON ALGEBRAIC CURVES

As a further application of Theorem 2 we address the problem of finding Chebyshev-type quadrature on certain algebraic curves in $\mathbb{R}^2$. In this case we consider quadrature formulas of the form

$$\frac{1}{L} \int_{\Gamma} \phi(x, y) ds \approx \frac{1}{n} \sum_{j=1}^{n} \phi(x_j, y_j). \tag{6.1}$$

Here Γ is a smooth closed curve in $\mathbb{R}^2$, ds denotes arc length on Γ, L is the length of Γ and the nodes (x_j, y_j) are on Γ. The degree of (6.1) is the maximal number p such that equality holds for every polynomial ϕ (in two variables) of total degree $\leq p$. We denote by $N_\Gamma^{(2)}(p)$ the minimal number n such that there exists a Chebyshev-type quadrature formula of degree $\geq p$ whose nodes lie on Γ.

The superscript $^{(2)}$ is used since we are dealing with polynomials in two variables.

Theorem 8. *Let Γ be the closed curve in $\mathbb{R}^2$ parametrized by $(P(t), Q(t))$, $t \in [0, 2\pi]$, where P and Q are real trigonometric polynomials. We assume $(P'(t), Q'(t)) \neq (0,0)$ for every t. Then*

$$\text{(6.2)} \qquad \limsup_{p\to\infty} \frac{N_\Gamma^{(2)}(p)}{p} \leq \frac{L}{\pi} \frac{\max(\deg P, \deg Q)}{\min_{t\in[0,2\pi]}(P'(t)^2 + Q'(t)^2)^{1/2}}.$$

Proof. Write

$$w(t) := (P'(t)^2 + Q'(t)^2)^{1/2}/L$$

and $d\mu(t) = w(t)dt$, $t \in [0, 2\pi]$. Then μ satisfies the hypotheses of Corollary 6 and so

$$\text{(6.3)} \qquad \limsup_{p\to\infty} \frac{N^T_{[0,2\pi]}(p;\mu)}{p} \leq \max_{t\in[0,2\pi]} \frac{1}{\pi w(t)}.$$

Choose $\alpha > 0$ such that $\alpha < \min_{t\in[0,2\pi]} w(t)$ and let D denote the maximum of $\deg P$ and $\deg Q$. Because of (6.3) there is p_0 such that for every $p \geq p_0$, we have

$$N^T_{[0,2\pi]}(pD;\mu) \leq \frac{pD}{\pi\alpha}.$$

Now let $p \geq p_0$. Then there exist a positive number $n \leq (pD)/(\pi\alpha)$ and nodes $t_1, \ldots, t_n \in [0, 2\pi]$ such that

$$\text{(6.4)} \qquad \int_0^{2\pi} T(t)w(t)dt = \frac{1}{n}\sum_{j=1}^n T(t_j)$$

for every trigonometric polynomial T of degree $\leq pD$.

Let ϕ be a polynomial of two variables of degree $\leq p$. Since

$$\frac{1}{L}\int_\Gamma \phi(x,y)ds = \int_0^{2\pi} \phi(P(t), Q(t))w(t)dt$$

and $\phi(P(t), Q(t))$ is a trigonometric polynomial of degree $\leq pD$, we find by (6.4) that

$$\frac{1}{L}\int_\Gamma \phi(x,y)ds = \frac{1}{n}\sum_{j=1}^n \phi(P(t_j), Q(t_j)).$$

Hence $N_\Gamma^{(2)}(p) \leq n \leq (pD)/(\pi\alpha)$ for every $p \geq p_0$. This implies

$$\limsup_{p\to\infty} \frac{N_\Gamma^{(2)}(p)}{p} \le \frac{D}{\pi\alpha}.$$

Since $\alpha < \min w(t)$ can be chosen arbitrarily, we obtain (6.2). □

Remark G. As an example, we take for Γ the ellipse with parametrization $(a\cos t, b\sin t)$ with $a > b > 0$, say. Then (6.2) gives

$$\limsup_{p\to\infty} \frac{N_\Gamma^{(2)}(p)}{p} \le \frac{4}{\pi}\frac{a}{b} E(\sqrt{1-b^2/a^2}) \tag{6.5}$$

where $E(k)$ is the complete elliptic integral of second kind. This is essentially best possible since we have the following estimate from below which can be proved using (1.7):

$$\liminf_{p\to\infty} \frac{N_\Gamma^{(2)}(p)}{p} \ge \frac{2}{\pi}\frac{a}{b} E(\sqrt{1-b^2/a^2}).$$

The finiteness of the lim sup in (6.5) was conjectured in [7, Remark 3.2].

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