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Global attractors for the extensible thermoelastic beam system

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ABSTRACT

This work is focused on the dissipative system

$$\begin{cases} \partial_{tt}u + \partial_{xxxx}u + \partial_{xx}\theta - \big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u = f, \\ \partial_t\theta - \partial_{xx}\theta - \partial_{xxt}u = g \end{cases}$$

describing the dynamics of an extensible thermoelastic beam, where the dissipation is entirely contributed by the second equation ruling the evolution of θ. Under natural boundary conditions, we prove the existence of bounded absorbing sets. When the external sources f and g are time-independent, the related semigroup of solutions is shown to possess the global attractor of optimal regularity for all parameters $\beta \in \mathbb{R}$. The same result holds true when the first equation is replaced by

$$\partial_{tt}u - \gamma\,\partial_{xxtt}u + \partial_{xxxx}u + \partial_{xx}\theta - \big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u = f$$

with $\gamma > 0$. In both cases, the solutions on the attractor are strong solutions.

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1. Introduction

For $t > 0$, we consider the evolution system

$$\begin{cases} \partial_{tt}u + \partial_{xxxx}u + \partial_{xx}\theta - \left(\beta + \displaystyle\int_0^1 \left|\partial_x u(x,\cdot)\right|^2 \mathrm{d}x\right)\partial_{xx}u = f, \\ \partial_t\theta - \partial_{xx}\theta - \partial_{xxt}u = g, \end{cases} \tag{1.1}$$

in the unknown variables $u = u(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ and $\theta = \theta(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$, having set $\mathbb{R}^+ = [0,\infty)$. The two equations are supplemented with the initial conditions

$$\begin{cases} u(x,0) = u_0(x), \\ \partial_t u(x,0) = u_1(x), \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{1.2}$$

for every $x \in [0,1]$, where u_0, u_1 and θ_0 are assigned data.

System (1.1) describes the vibrations of an extensible thermoelastic beam of unitary natural length, and is obtained by combining the pioneering ideas of Woinowsky-Krieger [33] with the theory of linear thermoelasticity [6]. Although a rigorous variational derivation of the model will be addressed in a forthcoming paper, it is worth noting that (1.1) is a mild quasilinear version of the nonlinear motion equations devised in [24, §3].

With regard to the physical meaning of the variables in play, u represents the vertical deflection of the beam from its configuration at rest, while the "temperature variation" θ actually arises from an approximation of the temperature variation with respect to a reference value, and it has the dimension of a temperature gradient (see [24]). The real function $f = f(x,t)$ is the lateral load distribution and $g = g(x,t)$ is the external heat supply, having the role of a control function. Finally, the parameter $\beta \in \mathbb{R}$ accounts for the axial force acting in the reference configuration: $\beta > 0$ when the beam is stretched, $\beta < 0$ when compressed. Concerning the boundary conditions, for all $t \geqslant 0$ we assume

$$\begin{cases} u(0,t) = u(1,t) = \partial_{xx}u(0,t) = \partial_{xx}u(1,t) = 0, \\ \theta(0,t) = \theta(1,t) = 0. \end{cases} \tag{1.3}$$

Namely, we take Dirichlet boundary conditions for the temperature variation θ and hinged boundary conditions for the vertical deflection u.

The focus of this paper is the study of the longterm properties of the dynamical system generated by problem (1.1)–(1.3) in the natural weak energy phase space. In particular, for the autonomous case, we prove the existence of the global attractor of optimal regularity, for all values of the real parameter β. The main difficulty arising in the asymptotic analysis comes from the very weak dissipation exhibited by the model, entirely contributed by the thermal component, whereas the mechanical component, by itself, does not cause any decrease of energy. Hence, the loss of mechanical energy is due only to the coupling, which propagates the thermal dissipation to the mechanical component with the effect of producing mechanical dissipation. From the mathematical side, in order to obtain stabilization properties it is necessary to introduce sharp energy functionals, which allow to exploit the thermal dissipation in its full strength. A similar situation has been faced in [1–3,10–12,20,25,31], dealing with linear and semilinear thermoelastic problems without mechanical dissipation. Along the same line, we also mention [26], which considers a quasilinear thermoelastic plate system.

After this work was finished, we learned of a paper by Bucci and Chueshov [5], which treats (actually, in a more general version) the same problem discussed here. In [5], borrowing some techniques from the recent article [12], the authors prove the existence of the global attractor of optimal regularity and finite fractal dimension for the semigroup generated by the autonomous version of (1.1)–(1.3). The existence of the (regular) attractor is also shown in presence of a rotational term in the first equation, whose dynamics becomes in turn of hyperbolic type. The proofs of [5] heavily rely on two basic facts: a key estimate, nowadays known in the literature as *stabilizability inequality* (cf. [10,12]), and the gradient system structure featured by the model. The regularity of the attractor is demonstrated only in a second moment, exploiting the peculiar form of the attractor itself: a section of all bounded complete orbits of the semigroup.

Nevertheless, we still believe that our paper might be of some interest, at least for the following reasons:

(i) We do not appeal to the gradient system structure, except for the characterization of the attractor as the unstable set of stationary solutions. Accordingly, the existence of absorbing sets is established via explicit energy estimates, providing a precise (uniform) control on the entering times of the trajectories. The method applies to the nonautonomous case as well, where the gradient system structure is lost.
(ii) Our proof of asymptotic compactness is rather direct and simpler than in [5]. Indeed, it merely boils down to the construction of a suitable decomposition of the semigroup. Incidentally, the required regularity is gained in just one single step, without making use of bootstrapping arguments.
(iii) In fact, we prove a stronger result: we find exponentially attracting sets of optimal regularity, obtaining at the same time the attractor and its regularity. Having such exponentially attracting sets, it is possible to show with little effort the existence of regular exponential attractors in the sense of [16], having finite fractal dimension.
(iv) In the rotational case, where the first equation contains the extra term $-\gamma\partial_{xxtt}u$ with $\gamma > 0$, we improve the regularity of the attractor devised in [5].

Plan of the paper. In Section 2, we consider an abstract generalization of (1.1)–(1.3), whose solutions are generated by a family of solution operators $S(t)$. Section 3 is devoted to the existence of an absorbing set for $S(t)$. In Section 4, we dwell on the autonomous case, where $S(t)$ is a semigroup, establishing the existence and the regularity of the global attractor. The proofs are carried out in the subsequent Section 5. In the last Section 6, we extend the results to a more general model, where an additional rotational inertia term is present.

2. The abstract problem

2.1. Notation

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a separable real Hilbert space, and let $A : H \to H$ be a strictly positive selfadjoint operator with domain $\mathcal{D}(A) \Subset H$. For $r \in \mathbb{R}$, we introduce the scale of Hilbert spaces generated by the powers of A

$$H^r = \mathcal{D}\big(A^{r/4}\big), \qquad \langle u, v\rangle_r = \big\langle A^{r/4}u, A^{r/4}v\big\rangle, \qquad \|u\|_r = \big\|A^{r/4}u\big\|.$$

We will always omit the index r when $r = 0$. The symbol $\langle\cdot,\cdot\rangle$ will also be used to denote the duality product between H^r and its dual space H^{-r}. In particular, we have the compact embeddings $H^{r+1} \Subset H^r$, along with the generalized Poincaré inequalities

$$\lambda_1\|u\|_r^4 \leqslant \|u\|_{r+1}^4, \quad \forall u \in H^{r+1},$$

where $\lambda_1 > 0$ is the first eigenvalue of A. Finally, we define the product Hilbert spaces

$$\mathcal{H}^r = H^{r+2} \times H^r \times H^r.$$

2.2. Formulation of the problem

For $\beta \in \mathbb{R}$, we consider the abstract Cauchy problem on $\mathcal{H}$ in the unknown variables $u = u(t)$ and $\theta = \theta(t)$

$$\begin{cases} \partial_{tt}u + Au - A^{1/2}\theta + \big(\beta + \|u\|_1^2\big)A^{1/2}u = f(t), \quad t > 0, \\ \partial_t\theta + A^{1/2}\theta + A^{1/2}\partial_t u = g(t), \quad t > 0, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \quad \theta(0) = \theta_0. \end{cases} \tag{2.1}$$

The following well-posedness result holds.

Proposition 2.1. *Assume that*

$$f \in L^1_{\mathrm{loc}}\big(\mathbb{R}^+, H\big), \qquad g \in L^1_{\mathrm{loc}}\big(\mathbb{R}^+, H\big) + L^2_{\mathrm{loc}}\big(\mathbb{R}^+, H^{-1}\big).$$

Then, for all initial data $(u_0, u_1, \theta_0) \in \mathcal{H}$, *problem* (2.1) *admits a unique* (*weak*) *solution*

$$\big(u(t), \partial_t u(t), \theta(t)\big) \in \mathcal{C}\big(\mathbb{R}^+, \mathcal{H}\big)$$

with

$$\big(u(0), \partial_t u(0), \theta(0)\big) = (u_0, u_1, \theta_0).$$

Moreover, calling $\bar{z}(t)$ *the difference of any two solutions corresponding to initial data having norm less than or equal to* $R \geqslant 0$, *there exists* $C = C(R) \geqslant 0$ *such that*

$$\big\|\bar{z}(t)\big\|_{\mathcal{H}} \leqslant C\mathrm{e}^{Ct}\big\|\bar{z}(0)\big\|_{\mathcal{H}}, \quad \forall t \geqslant 0.$$

We omit the proof, based on a standard Galerkin approximation procedure together with a slight generalization of the usual Gronwall lemma (cf. [30]). Proposition 2.1 translates into the existence of the *solution operators*

$$S(t) : \mathcal{H} \to \mathcal{H}, \quad t \geqslant 0,$$

acting as

$$z = (u_0, u_1, \theta_0) \mapsto S(t)z = \big(u(t), \partial_t u(t), \theta(t)\big),$$

and satisfying the joint continuity property

$$(t, z) \mapsto S(t)z \in \mathcal{C}\big(\mathbb{R}^+ \times \mathcal{H}, \mathcal{H}\big).$$

Remark 2.2. In the autonomous case, namely, when both f and g are time-independent, the family $S(t)$ fulfills the semigroup property

$$S(t + \tau) = S(t)S(\tau), \quad \forall t, \tau \geqslant 0.$$

Thus, $S(t)$ is a strongly continuous semigroup of operators on $\mathcal{H}$.

We define the *energy* at time $t \geqslant 0$ corresponding to the initial data $z = (u_0, u_1, \theta_0) \in \mathcal{H}$ as

$$\mathcal{E}(t) = \frac{1}{2}\big\|S(t)z\big\|_{\mathcal{H}}^2 + \frac{1}{4}\big(\beta + \big\|u(t)\big\|_1^2\big)^2.$$

Multiplying the first equation of (2.1) by $\partial_t u$ and the second one by θ, we find the *energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \|\theta\|_1^2 = \langle \partial_t u, f\rangle + \langle \theta, g\rangle. \tag{2.2}$$

Indeed,

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\big(\beta + \|u\|_1^2\big)^2 &= \frac{1}{2}\big(\beta + \|u\|_1^2\big)\frac{\mathrm{d}}{\mathrm{d}t}\big(\beta + \|u\|_1^2\big)\\ &= \frac{1}{2}\big(\beta + \|u\|_1^2\big)\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_1^2 = \big(\beta + \big\|u(t)\big\|_1^2\big)\big\langle A^{1/2}u, \partial_t u\big\rangle.\end{aligned}$$

As a consequence, for every $T > 0$, there exists a positive increasing function $\mathcal{Q}_T$ such that

$$\mathcal{E}(t) \leqslant \mathcal{Q}_T\big(\mathcal{E}(0)\big), \quad \forall t \in [0, T]. \tag{2.3}$$

2.3. The concrete problem

The abstract system (2.1) serves as a model to describe quite general situations, including thermoelastic plates. In particular, problem (1.1)–(1.3) is a concrete realization of (2.1), obtained by putting $H = L^2(0,1)$ and $A = \partial_{xxxx}$ with domain

$$\mathcal{D}(A) = \big\{u \in H^4(0,1){:}\ u(0) = u(1) = \partial_{xx}u(0) = \partial_{xx}u(1) = 0\big\}.$$

In which case,

$$A^{1/2} = -\partial_{xx}, \qquad \mathcal{D}\big(A^{1/2}\big) = H^2(0,1) \cap H_0^1(0,1).$$

Although we supposed in (1.3) that both ends of the beam are hinged, different boundary conditions for u are physically significant as well, such as

$$u(0,t) = u(1,t) = \partial_x u(0,t) = \partial_x u(1,t) = 0, \tag{2.4}$$

when both ends of the beam are clamped, or

$$u(0,t) = u(1,t) = \partial_x u(0,t) = \partial_{xx} u(1,t) = 0, \tag{2.5}$$

when one end is clamped and the other one is hinged. On the contrary, the so-called cantilever boundary condition (one end clamped and the other one free) does not comply with the extensibility assumption, since no geometric constraints compel the beam length to change. However, the analysis carried out in this work depends on the specific structure of the hinged boundary conditions, whereas, assuming the boundary conditions (2.4) or (2.5), major modifications on the needed tools are required.

3. The absorbing set

In this section, we prove the existence of an absorbing set for the family $S(t)$. This is a bounded set $\mathfrak{B} \subset \mathcal{H}$ with the following property: for every $R \geqslant 0$, there is an *entering time* $t_R \geqslant 0$ such that

$$\bigcup_{t \geqslant t_R} S(t)z \subset \mathfrak{B},$$

whenever $\|z\|_{\mathcal{H}} \leqslant R$. Actually, we establish a more general result.

Theorem 3.1. *Let $f \in L^\infty(\mathbb{R}^+, H)$, and let $\partial_t f$ and g be translation bounded functions in $L^2_{\mathrm{loc}}(\mathbb{R}^+, H^{-1})$, that is,*

$$\sup_{t \geqslant 0} \int_t^{t+1} \left\{ \left\| \partial_t f(\tau) \right\|_{-1}^2 + \left\| g(\tau) \right\|_{-1}^2 \right\} \mathrm{d}\tau < \infty. \tag{3.1}$$

Then, there exists $R_0 > 0$ with the following property: in correspondence of every $R \geqslant 0$, there is $t_0 = t_0(R) \geqslant 0$ such that

$$\mathcal{E}(t) \leqslant R_0, \quad \forall t \geqslant t_0,$$

whenever $\mathcal{E}(0) \leqslant R$. Both R_0 and t_0 can be explicitly computed.

The absorbing set, besides giving a first rough estimate of the dissipativity of the system, is the preliminary step to prove the existence of much more interesting objects describing the asymptotic dynamics, such as global or exponential attractors (see, for instance, [4,7–9,22,29,32]). Unfortunately, in certain situations where the dissipation is very weak, a direct proof of the existence of the absorbing set via *explicit* energy estimates might be very hard to find. On the other hand, for a quite general class of autonomous problems (the so-called gradient systems), it is possible to use an alternative approach and overcome this obstacle, appealing to the existence of a Lyapunov functional (see [13,22,23]). In which case, if the semigroup possesses suitable smoothing properties, one obtains right away the global attractor, and the absorbing set is then recovered as a byproduct. Though, the procedure provides no quantitative information on the entering time t_R, which is somehow unsatisfactory, especially in view of numerical simulations. This technique has been successfully adopted in the recent paper [21], concerned with the longterm analysis of an integrodifferential equation with low dissipation, modelling the transversal motion of an extensible viscoelastic beam.

As mentioned in the introduction, the problem considered in the present work is also weakly dissipative. But if we assume f and g independent of time, there is a way to define a Lyapunov functional (actually, for an equivalent problem), which would allow to exploit the method described above. In any case, in order to exhibit an actual bound on t_R, and also to deal with time-dependent external forces, a direct proof of Theorem 3.1 would be much more desirable. However, due to presence of the coupling term, when performing the standard (and unavoidable) estimates, some "pure" energy terms having a power strictly greater than one pop up with the wrong sign. Such terms cannot be handled by means of standard Gronwall-type lemmas. Nonetheless, we are still able to establish the result, leaning on the following novel Gronwall-type lemma with parameter devised in [18].

Lemma 3.2. *Let $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be an absolutely continuous function satisfying, for some $K \geqslant 0$, $Q \geqslant 0$, $\varepsilon_0 > 0$ and every $\varepsilon \in (0, \varepsilon_0]$, the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + \varepsilon \Lambda(t) \leqslant K\varepsilon^2 \left[\Lambda(t)\right]^{3/2} + \varepsilon^{-2/3}\varphi(t),$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *fulfills the relation* $\sup_{t \geqslant 0} \int_t^{t+1} \varphi(\tau)\,\mathrm{d}\tau \leqslant Q$. *Then, there exist* $R_\star > 0$ *and* $\kappa > 0$ *such that, for every* $R \geqslant 0$,

$$\Lambda(t) \leqslant R_\star, \quad \forall t \geqslant R^{1/\kappa}(1+\kappa Q)^{-1},$$

whenever $\Lambda(0) \leqslant R$.

Remark 3.3. Both $R_\star$ and κ can be calculated in terms of K, Q and ε_0 (cf. [18]).

We are now ready to proceed to the proof of the theorem.

Proof of Theorem 3.1. Here and in the sequel, we will tacitly use several times the Young and the Hölder inequalities, besides the usual Sobolev embeddings. The generic positive constant C appearing in this proof may depend on β and $\|f\|_{L^\infty(\mathbb{R}^+,H)}$.

On account of (2.2), the functional

$$\mathcal{L}(t) = \mathcal{E}(t) - \langle u(t), f(t)\rangle$$

satisfies the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L} + \|\theta\|_1^2 = -\langle u, \partial_t f\rangle + \langle \theta, g\rangle.$$

Observing that

$$\|u\|_1 \leqslant C\big|\beta + \|u\|_1^2\big|^{1/2} + C|\beta|^{1/2} \leqslant C\mathcal{E}^{1/4} + C, \tag{3.2}$$

we have the control

$$-\langle u, \partial_t f\rangle \leqslant C\varepsilon^{2/3}\mathcal{E}^{1/2} + \varepsilon^{-2/3}\|\partial_t f\|_{-1}^2 \leqslant C\varepsilon^2\mathcal{E}^{3/2} + \varepsilon^{-2/3}\|\partial_t f\|_{-1}^2 + C,$$

for all $\varepsilon \in (0,1]$. Moreover,

$$\langle \theta, g\rangle \leqslant \frac{1}{2}\|\theta\|_1^2 + C\|g\|_{-1}^2.$$

Thus, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L} + \frac{1}{2}\|\theta\|_1^2 \leqslant C\varepsilon^2\mathcal{E}^{3/2} + \varepsilon^{-2/3}\|\partial_t f\|_{-1}^2 + C\|g\|_{-1}^2 + C. \tag{3.3}$$

Next, we consider the auxiliary functionals

$$\Phi(t) = \langle \partial_t u(t), u(t)\rangle, \qquad \Psi(t) = \langle \partial_t u(t), \theta(t)\rangle_{-1}.$$

Concerning Φ, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \|u\|_2^2 + \big(\beta + \|u\|_1^2\big)^2 - \beta\big(\beta + \|u\|_1^2\big) = \|\partial_t u\|^2 + \langle u, \theta\rangle_1 + \langle u, f\rangle.$$

Noting that

$$\frac{1}{2}\big(\beta + \|u\|_1^2\big)^2 - \beta\big(\beta + \|u\|_1^2\big) = \frac{1}{2}\|u\|_1^4 - \frac{1}{2}\beta^2 \geqslant -C,$$

and

$$\langle u, \theta\rangle_1 + \langle u, f\rangle \leqslant \frac{1}{4}\|u\|_2^2 + C\|\theta\|_1^2 + C,$$

we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{3}{4}\|u\|_2^2 + \frac{1}{2}\big(\beta + \|u\|_1^2\big)^2 \leqslant \|\partial_t u\|^2 + C\|\theta\|_1^2 + C. \tag{3.4}$$

Turning to Ψ, we have the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi + \|\partial_t u\|^2 = \|\theta\|^2 - \langle \partial_t u, \theta\rangle - \langle u, \theta\rangle_1 + \langle \partial_t u, g\rangle_{-1} + \langle \theta, f\rangle_{-1} + \mathcal{J},$$

having set

$$\mathcal{J} = -\big(\beta + \|u\|_1^2\big)\langle u, \theta\rangle.$$

We easily see that

$$\begin{aligned}&\|\theta\|^2 - \langle \partial_t u, \theta\rangle - \langle u, \theta\rangle_1 + \langle g, \partial_t u\rangle_{-1} + \langle f, \theta\rangle_{-1}\\ &\quad\leqslant \frac{1}{8}\|u\|_2^2 + \frac{1}{4}\|\partial_t u\|^2 + C\|\theta\|_1^2 + C\|g\|_{-1}^2 + C,\end{aligned}$$

whereas, in light of (3.2), the remaining term $\mathcal{J}$ is controlled as

$$\mathcal{J} \leqslant C\|\theta\|\big(\|u\|_1^3 + 1\big) \leqslant C\|\theta\|\mathcal{E}^{3/4} + C\|\theta\| \leqslant C\|\theta\|\mathcal{E}^{3/4} + C\|\theta\|_1^2 + C.$$

In conclusion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi + \frac{3}{4}\|\partial_t u\|^2 \leqslant \frac{1}{8}\|u\|_2^2 + C\|\theta\|_1^2 + C\|\theta\|\mathcal{E}^{3/4} + C\|g\|_{-1}^2 + C. \tag{3.5}$$

Collecting (3.4)–(3.5), we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\Phi + 2\Psi\} + \mathcal{E} \leqslant C\|\theta\|_1^2 + C\|\theta\|\mathcal{E}^{3/4} + C\|g\|_{-1}^2 + C. \tag{3.6}$$

Finally, for $\varepsilon \in (0, 1]$, we write

$$\Lambda(t) = \mathcal{L}(t) + 2\varepsilon\big\{\Phi(t) + 2\Psi(t)\big\} + C,$$

where the above C is large enough and ε is small enough such that

$$\frac{1}{2}\mathcal{E} \leqslant \Lambda \leqslant 2\mathcal{E} + C. \tag{3.7}$$

Then, calling

$$\varphi(t) = C + C\big\|\partial_t f(t)\big\|_{-1}^2 + C\big\|g(t)\big\|_{-1}^2,$$

the inequalities (3.3), (3.6) and (3.7) entail

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Lambda+\varepsilon\Lambda+\frac{1}{2}(1-C\varepsilon)\|\theta\|_1^2 &\leqslant C\varepsilon^2\Lambda^{3/2}+C\varepsilon\|\theta\|\Lambda^{3/4}+\varepsilon^{-2/3}\varphi\\ &\leqslant C\varepsilon^2\Lambda^{3/2}+\varepsilon^{-2/3}\varphi+\frac{1}{4}\|\theta\|_1^2.\end{aligned}$$

It is apparent that there exists $\varepsilon_0>0$ small such that, for every $\varepsilon\in(0,\varepsilon_0]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda+\varepsilon\Lambda\leqslant C\varepsilon^2\Lambda^{3/2}+\varepsilon^{-2/3}\varphi.$$

By virtue of (3.1), we are in a position to apply Lemma 3.2. Using once more (3.7), the proof is finished. $\square$

4. The global attractor

In the remaining of the paper, we assume

$$f,g\in H\quad\text{independent of time.}$$

In which case, $S(t)$ is a strongly continuous semigroup on $\mathcal{H}$. The main result, proved in the next section, reads as follows.

Theorem 4.1. *The absorbing set $\mathfrak{B}$ of the semigroup $S(t)$ is exponentially attracted by a closed ball $\mathfrak{C}$ of $\mathcal{H}^2$, i.e.*

$$\operatorname{dist}_{\mathcal{H}}\big(S(t)\mathfrak{B},\mathfrak{C}\big)\leqslant M\mathrm{e}^{-\varkappa t},$$

for some constants $M,\varkappa>0$, which can be explicitly calculated.

Here, $\operatorname{dist}_{\mathcal{H}}$ denotes the usual Hausdorff semidistance in $\mathcal{H}$, given by

$$\operatorname{dist}_{\mathcal{H}}(B_1,B_2)=\sup_{z_1\in B_1}\inf_{z_2\in B_2}\|z_1-z_2\|_{\mathcal{H}}.$$

On account of the compact embedding $\mathcal{H}^2\Subset\mathcal{H}$, by standard arguments of the theory of dynamical systems, we infer the existence of the global attractor (cf. [4,22,32]). This is, by definition, the unique compact set $\mathfrak{A}\subset\mathcal{H}$ at the same time fully invariant and attracting for the semigroup; namely, $S(t)\mathfrak{A}=\mathfrak{A}$ for all $t\geqslant 0$, and

$$\lim_{t\to\infty}\operatorname{dist}_{\mathcal{H}}\big(S(t)B,\mathfrak{A}\big)=0,$$

for every bounded set $B\subset\mathcal{H}$.

Corollary 4.2. *The semigroup $S(t)$ acting on $\mathcal{H}$ possesses the (connected) global attractor $\mathfrak{A}\subset\mathfrak{C}$. In particular, $\mathfrak{A}$ is bounded in $\mathcal{H}^2$.*

Remark 4.3. Within our hypotheses, the regularity of $\mathfrak{A}$ is optimal. On the other hand, one can prove that $\mathfrak{A}$ is as regular as f and g permit. For instance, if $f,g\in H^n$ for every $n\in\mathbb{N}$, then each component of $\mathfrak{A}$ belongs to H^n for every $n\in\mathbb{N}$.

The proof of the theorem will be carried out by exploiting a particular decomposition of $S(t)$ devised in [21]. Besides, it is not hard to demonstrate (e.g. following [16]) the existence of regular exponential attractors for $S(t)$ having finite fractal dimension in $\mathcal{H}$. Since the global attractor is the *minimal* closed attracting set, a straightforward consequence is

Corollary 4.4. *The fractal dimension of $\mathfrak{A}$ in $\mathcal{H}$ is finite.*

Remark 4.5. In fact, having proved the existence of the absorbing set $\mathfrak{B}$, we could also consider the nonautonomous case (when f and g depend on time), establishing a more general result on the existence of the global attractor for a process of operators, provided that f and g fulfill suitable translation compactness properties (see [7,8] for more details). However, in that case, the decomposition from [21] fails to work, and other techniques should be employed in order to establish asymptotic compactness, such as the α-contraction method [22] (see also [15], where the method is applied to a similar, albeit autonomous, problem).

We now dwell on the structure of the global attractor. To this aim, we introduce the set

$$\mathcal{S} = \{z \in \mathcal{H}\colon\ S(t)z = z,\ \forall t \geqslant 0\}$$

of stationary points of $S(t)$, which consists of all vectors of the form $(u, 0, \theta_g)$, where $\theta_g = A^{-1/2}g \in H^2$ and $u \in H^4$ is a solution to the elliptic problem

$$Au + \big(\beta + \|u\|_1^2\big)A^{1/2}u = f + g.$$

The set $\mathcal{S}$ turns out to be nonempty and bounded in $\mathcal{H}^2$. Then, calling $z(t)$ a complete trajectory of $S(t)$ when

$$z(t+\tau) = S(t)z(\tau), \quad \forall t \geqslant 0,\ \forall \tau \in \mathbb{R},$$

we obtain the following characterization of $\mathfrak{A}$.

Proposition 4.6. *The global attractor $\mathfrak{A}$ coincides with the unstable set of $\mathcal{S}$; namely,*

$$\mathfrak{A} = \Big\{z(0)\colon\ z(t) \text{ is a complete trajectory of } S(t) \text{ and } \lim_{t\to\infty}\big\|z(-t) - \mathcal{S}\big\|_{\mathcal{H}} = 0\Big\}.$$

Corollary 4.7. *If $\mathcal{S}$ is finite, then*

$$\mathfrak{A} = \Big\{z(0)\colon\ \lim_{t\to\infty}\big\|z(-t) - z_1\big\|_{\mathcal{H}} = \lim_{t\to\infty}\big\|z(t) - z_2\big\|_{\mathcal{H}} = 0\Big\},$$

for some $z_1, z_2 \in \mathcal{S}$. If $\mathcal{S}$ consists of a single element $z_s \in \mathcal{H}^2$, then $\mathfrak{A} = \{z_s\}$.

As shown in [14], the set $\mathcal{S}$ is always finite when all the eigenvalues λ_n of A (recall that $\lambda_n \uparrow \infty$) satisfying the relation

$$\beta < -\sqrt{\lambda_n}$$

are simple (this is the case in the concrete problem (1.1)–(1.3)), while it possesses a single element z_s if $\beta \geqslant -\sqrt{\lambda_1}$. In particular, we have

Corollary 4.8. *If $\beta > -\sqrt{\lambda_1}$ and $f + g = 0$, then $\mathfrak{A} = \{(0, 0, \theta_g)\}$, with θ_g as above, and*

$$\operatorname{dist}_{\mathcal{H}}\big(S(t)B, \mathfrak{A}\big) = \sup_{z\in B}\big\|S(t)z - (0, 0, \theta_g)\big\|_{\mathcal{H}} \leqslant \mathcal{Q}\big(\|B\|_{\mathcal{H}}\big)e^{-\varkappa t},$$

for some $\varkappa > 0$ and some positive increasing function $\mathcal{Q}$, explicitly computable.

We conclude the section discussing the injectivity of $S(t)$ on $\mathfrak{A}$.

Proposition 4.9. *The map $S(t)_{|\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ fulfills the backward uniqueness property; namely, the equality $S(t)z_1 = S(t)z_2$, for some $t > 0$ and $z_1, z_2 \in \mathfrak{A}$, implies that $z_1 = z_2$.*

As a consequence, the map $S(t)_{|\mathfrak{A}}$ is a bijection on $\mathfrak{A}$, and so it can be extended to negative times by the formula

$$S(-t)_{|\mathfrak{A}} = \left[S(t)_{|\mathfrak{A}}\right]^{-1}.$$

In this way, $S(t)_{|\mathfrak{A}}$, $t \in \mathbb{R}$, is a strongly continuous (in the topology of $\mathcal{H}$) group of operators on $\mathfrak{A}$.

5. Proofs of the results

5.1. An equivalent problem

We define

$$\theta_g = A^{-1/2}g, \qquad z_g = (0, 0, \theta_g) \in \mathcal{H}^2.$$

Writing as usual $S(t)z = (u(t), \partial_t u(t), \theta(t))$, for some given $z = (u_0, u_1, \theta_0) \in \mathcal{H}$, we introduce the function

$$\omega(t) = \theta(t) - \theta_g.$$

It is apparent that $(u(t), \partial_t u(t), \omega(t))$ solves

$$\begin{cases} \partial_{tt}u + Au - A^{1/2}\omega + \left(\beta + \|u\|_1^2\right)A^{1/2}u = h, \\ \partial_t\omega + A^{1/2}\omega + A^{1/2}\partial_t u = 0, \end{cases} \tag{5.1}$$

where

$$h = f + g \in H,$$

with initial conditions

$$\left(u(0), \partial_t u(0), \omega(0)\right) = z - z_g.$$

According to Proposition 2.1, system (5.1) generates a strongly continuous semigroup $S_0(t)$ on $\mathcal{H}$, which clearly fulfills the relation

$$S(t)(z_g + \zeta) = z_g + S_0(t)\zeta, \quad \forall \zeta \in \mathcal{H}. \tag{5.2}$$

Thus, from Theorem 3.1, we learn that $S_0(t)$ possesses the absorbing set

$$\mathfrak{B}_0 = -z_g + \mathfrak{B},$$

and, using (2.3), we infer the uniform bound

$$\sup_{t \geqslant 0}\sup_{\zeta \in \mathfrak{B}_0}\left\|S_0(t)\zeta\right\|_{\mathcal{H}} \leqslant C. \tag{5.3}$$

Remark 5.1. Here and till the end of the section, the generic constant C depends only on β, $\|h\|$ and the size of the absorbing set $\mathfrak{B}_0$.

Theorem 4.1 (and so Corollary 4.2) is an immediate consequence of

Theorem 5.2. *There exist explicitly computable constants $R, M, \varkappa > 0$ such that, writing*

$$\mathfrak{C}_0 = \{\zeta \in \mathcal{H}^2 \colon \|\zeta\|_{\mathcal{H}^2} \leqslant R\},$$

the exponential attraction property

$$\operatorname{dist}_{\mathcal{H}}\big(S_0(t)\mathfrak{B}_0, \mathfrak{C}_0\big) \leqslant M e^{-\varkappa t}$$

holds. Accordingly, the semigroup $S_0(t)$ acting on $\mathcal{H}$ possesses the (connected) global attractor $\mathfrak{A}_0 \subset \mathfrak{C}_0$.

Indeed, in light of (5.2), Theorem 4.1 and Corollary 4.2 are recovered at once by setting

$$\mathfrak{C} = z_g + \mathfrak{C}_0, \qquad \mathfrak{A} = z_g + \mathfrak{A}_0.$$

We postpone the proof of Theorem 5.2, which requires several steps. In what follows, for $\zeta = (u_0, u_1, \omega_0) \in \mathcal{H}$, we denote

$$S_0(t)\zeta = \big(u(t), \partial_t u(t), \omega(t)\big),$$

whose corresponding energy is given by

$$\mathcal{E}_0(t) = \frac{1}{2}\|S_0(t)\zeta\|_{\mathcal{H}}^2 + \frac{1}{4}\big(\beta + \|u(t)\|_1^2\big)^2.$$

Therefore, the functional

$$\mathcal{L}_0(t) = \mathcal{E}_0(t) - \langle h, u(t)\rangle$$

satisfies the differential equality

$$\frac{d}{dt}\mathcal{L}_0 + \|\omega\|_1^2 = 0. \tag{5.4}$$

It is then easy to see that $\mathcal{L}_0$ is a Lyapunov functional for $S_0(t)$, and by means of standard arguments (cf. [4,22,32]), we conclude that

$$\mathfrak{A}_0 = \Big\{\zeta(0)\colon \ \zeta(t) \text{ is a complete trajectory of } S_0(t) \text{ and } \lim_{t\to\infty}\|\zeta(-t) - \mathcal{S}_0\|_{\mathcal{H}} = 0\Big\},$$

where

$$\mathcal{S}_0 = -z_g + \mathcal{S}$$

is the (nonempty) set of stationary points of $S_0(t)$. Besides, if $\mathcal{S}_0$ is finite,

$$\mathfrak{A}_0 = \Big\{\zeta(0)\colon \ \lim_{t\to\infty}\|\zeta(-t) - \zeta_1\|_{\mathcal{H}} = \lim_{t\to\infty}\|\zeta(t) - \zeta_2\|_{\mathcal{H}} = 0\Big\},$$

for some $\zeta_1, \zeta_2 \in \mathcal{S}_0$. In particular, when $\mathcal{S}_0$ consists of a single element, recalling that the Lyapunov functional is decreasing along the trajectories, there exists only one (constant) complete trajectory of $S_0(t)$, so implying that $\mathfrak{A}_0$ is a singleton. On account of (5.2), this gives the proofs of Proposition 4.6 and Corollary 4.7. In the same fashion, Proposition 4.9 follows from the analogous statement for $S_0(t)$, detailed in the next proposition.

Proposition 5.3. *The map $S_0(t)_{|\mathfrak{A}_0} : \mathfrak{A}_0 \to \mathfrak{A}_0$ fulfills the backward uniqueness property.*

Proof. We follow a classical method devised by Ghidaglia [19] (see also [32, §III.6]), along with a factorization scheme from [27] (see also [11] and [28, §3D]). For $\zeta_1, \zeta_2 \in \mathfrak{A}_0$, denote

$$S_0(t)\zeta_\iota = \big(u_\iota(t), v_\iota(t), \omega_\iota(t)\big), \quad \iota = 1, 2,$$

and

$$\zeta(t) = \big(u(t), v(t), \omega(t)\big) = S_0(t)\zeta_1 - S_0(t)\zeta_2.$$

Assume that $\zeta(T) = 0$ for some $T > 0$. We draw the conclusion if we show that

$$\zeta(0) = \zeta_1 - \zeta_2 = 0. \tag{5.5}$$

On account of (5.1), the column vector

$$\xi(t) = \big(A^{1/2}u(t), v(t), \omega(t)\big)^\top$$

satisfies the differential equation

$$\partial_t \xi + A^{1/2} \cdot \mathbb{B}\xi = \mathcal{G},$$

where we set

$$\mathbb{B} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} 0 \\ (\|u_2\|_1^2 - \|u_1\|_1^2)A^{1/2}u_2 - (\beta + \|u_1\|_1^2)A^{1/2}u \\ 0 \end{pmatrix}.$$

The matrix $\mathbb{B}$ possesses three distinct eigenvalues of strictly positive real parts, precisely: $a \sim 0.57$ and $b \pm ic$, with $b \sim 0.22$ and $c \sim 1.31$. Hence, there exists a (complex) invertible (3×3)-matrix $\mathbb{U}$ such that

$$\mathbb{U}^{-1}\mathbb{B}\mathbb{U} = \mathbb{D},$$

where $\mathbb{D}$ is the diagonal matrix whose entries are the eigenvalues of $\mathbb{B}$. Accordingly, setting $\mathcal{G}_\star = \mathbb{U}^{-1}\mathcal{G}$, the (complex) function $\xi_\star(t) = \mathbb{U}^{-1}\xi(t)$ fulfills

$$\partial_t \xi_\star + A^{1/2} \cdot \mathbb{D}\xi_\star = \mathcal{G}_\star.$$

Besides, $\xi_\star(T) = 0$. At this point, for $r = 0, 1$, we consider the complex Hilbert spaces

$$\mathcal{W}^r = H^r_{\mathbb{C}} \times H^r_{\mathbb{C}} \times H^r_{\mathbb{C}},$$

$H^r_{\mathbb{C}}$ being the complexification of H^r. It is convenient to endow $\mathcal{W}^1$ with the equivalent norm

$$\|\xi_\star\|^2_{\mathcal{W}^1} = a\|w_\star\|_1^2 + b\|v_\star\|_1^2 + b\|\omega_\star\|_1^2, \quad \xi_\star = (w_\star, v_\star, \omega_\star).$$

With this choice,

$$\langle A^{1/2}\cdot\mathbb{D}\xi_\star,\xi_\star\rangle_{\mathcal{W}}=\|\xi_\star\|^2_{\mathcal{W}^1}.$$

It is also apparent that

$$\|\mathcal{G}\|_{\mathcal{W}}\leqslant k\|A^{1/2}u\|=k\|\xi\|_{\mathcal{W}},$$

for some $k>0$ independent of $\zeta_1,\zeta_2\in\mathfrak{A}_0$. Thus, up to redefining $k>0$,

$$\|\mathcal{G}_\star\|_{\mathcal{W}}\leqslant k\|\xi_\star\|_{\mathcal{W}}.$$

Next, we define the function

$$\Gamma(t)=\frac{\|\xi_\star(t)\|^2_{\mathcal{W}^1}}{\|\xi_\star(t)\|^2_{\mathcal{W}}}.$$

Taking the time-derivative of Γ, and exploiting the equality

$$\langle A^{1/2}\cdot\mathbb{D},\xi_\star-\Gamma\xi_\star,\Gamma\xi_\star\rangle_{\mathcal{W}}=0,$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma&=\frac{-2\|A^{1/2}\cdot\mathbb{D}\xi_\star-\Gamma\xi_\star\|^2_{\mathcal{W}}}{\|\xi_\star\|^2_{\mathcal{W}}}+\frac{2\Re\langle A^{1/2}\cdot\mathbb{D}\xi_\star-\Gamma\xi_\star,\mathcal{G}_\star\rangle_{\mathcal{W}}}{\|\xi_\star\|^2_{\mathcal{W}}}\\&\leqslant\frac{-2\|A^{1/2}\cdot\mathbb{D}\xi_\star-\Gamma\xi_\star\|^2_{\mathcal{W}}}{\|\xi_\star\|^2_{\mathcal{W}}}+\frac{\|A^{1/2}\cdot\mathbb{D}\xi_\star-\Gamma\xi_\star\|^2_{\mathcal{W}}}{\|\xi_\star\|^2_{\mathcal{W}}}+\frac{\|\mathcal{G}_\star\|^2_{\mathcal{W}}}{\|\xi_\star\|^2_{\mathcal{W}}}\leqslant k^2,\end{aligned}$$

and an integration in time provides the estimate

$$\Gamma(t)\leqslant\Gamma(0)+k^2t.$$

If (5.5) is false, by continuity, $\zeta(t)\neq 0$ in a neighborhood of zero. In turn, $\xi_\star(t)\neq 0$ in the same neighborhood. Recalling that $\xi_\star(T)=0$, there exists $T_0\in(0,T]$ such that $\xi_\star(t)\neq 0$ on $[0,T_0)$ and $\xi_\star(T_0)=0$. Taking the time-derivative of $\log\|\xi_\star\|^{-1}_{\mathcal{W}}$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|\xi_\star\|^{-1}_{\mathcal{W}}=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\log\|\xi_\star\|^2_{\mathcal{W}}=\Gamma-\frac{\Re\langle\xi_\star,\mathcal{G}_\star\rangle_{\mathcal{W}}}{\|\xi_\star\|^2_{\mathcal{W}}}\leqslant\Gamma+\frac{\|\mathcal{G}_\star\|_{\mathcal{W}}}{\|\xi_\star\|_{\mathcal{W}}}\leqslant\Gamma+k.$$

Integrating on $(0,t)$, with $t<T_0$, we conclude that

$$\log\|\xi_\star(t)\|^{-1}_{\mathcal{W}}\leqslant\log\|\xi_\star(0)\|^{-1}_{\mathcal{W}}+T_0\Gamma(0)+kT_0+\frac{1}{2}k^2T_0^2.$$

This produces a uniform bound on $\log\|\xi_\star(t)\|^{-1}_{\mathcal{W}}$ over $[0,T_0)$, in contradiction with the fact that $\|\xi_\star(T_0)\|_{\mathcal{W}}=0$. $\square$

5.2. Proof of Theorem 5.2

We need first to prove a suitable dissipation integral for the norm of $\partial_t u$.

Lemma 5.4. *For every $\nu > 0$ small, there is $C_\nu > 0$ such that*

$$\int_s^t \left\|\partial_t u(\tau)\right\|^2 \mathrm{d}\tau \leqslant \nu(t-s) + C_\nu, \quad \forall t > s \geqslant 0,$$

whenever $\zeta \in \mathfrak{B}_0$.

Proof. Let $\zeta \in \mathfrak{B}_0$. An integration of (5.4), together with (5.3), yield the uniform bound

$$\int_0^\infty \left\|\omega(t)\right\|_1^2 \mathrm{d}t \leqslant C \tag{5.6}$$

uniformly with respect to $\zeta \in \mathfrak{B}_0$. We now consider the auxiliary functionals, analogous to those in the proof of Theorem 3.1,

$$\Phi_0(t) = \left\langle \partial_t u(t), u(t)\right\rangle, \qquad \Psi_0(t) = \left\langle \partial_t u(t), \omega(t)\right\rangle_{-1},$$

which satisfy the equalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0 + \|u\|_2^2 + \|u\|_1^4 + \beta\|u\|_1^2 = \|\partial_t u\|^2 + \langle u, \omega\rangle_1 + \langle u, h\rangle,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_0 + \|\partial_t u\|^2 = \|\omega\|^2 - \langle \partial_t u, \omega\rangle - \langle u, \omega\rangle_1 + \langle \omega, h\rangle_{-1} - \left(\beta + \|u\|_1^2\right)\langle u, \omega\rangle.$$

Then, we easily get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0 + \frac{1}{2}\|u\|_2^2 \leqslant \|\partial_t u\|^2 + C + C\|\omega\|_1^2,$$

and, for all positive $\varepsilon \leqslant 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_0 + \frac{1}{2}\|\partial_t u\|^2 \leqslant \frac{\varepsilon}{2}\|u\|_2^2 + C\varepsilon + \frac{C}{\varepsilon}\|\omega\|_1^2.$$

Therefore, for every $\varepsilon \leqslant 1/4$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\varepsilon\Phi_0 + \Psi_0\} + \frac{1}{4}\|\partial_t u\|^2 \leqslant C\varepsilon + \frac{C}{\varepsilon}\|\omega\|_1^2.$$

Integrating the last inequality on (s, t), and using (5.3) and (5.6), we conclude that

$$\int_s^t \left\|\partial_t u(\tau)\right\|^2 \mathrm{d}\tau \leqslant C\varepsilon(t-s) + \frac{C}{\varepsilon}.$$

Setting $\nu = C\varepsilon$ and $C_\nu = C/\varepsilon$ the proof is completed. $\square$

We will also make use of the following Gronwall-type lemma (see, e.g. [17]).

Lemma 5.5. *Let $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be an absolutely continuous function satisfying, for some $\nu > 0$, the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + 2\nu\Lambda(t) \leqslant \psi(t)\Lambda(t),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is any locally summable functions such that

$$\int_s^t \psi(\tau)\,\mathrm{d}\tau \leqslant \nu(t-s) + c, \quad \forall t > s \geqslant 0,$$

with $c \geqslant 0$. Then,

$$\Lambda(t) \leqslant \mathrm{e}^c \Lambda(0)\mathrm{e}^{-\nu t}.$$

At this point, exploiting the interpolation inequality

$$\|u\|_1^2 \leqslant \|u\|\|u\|_2,$$

we choose $\alpha > 0$ large enough such that

$$\frac{1}{4}\|u\|_2^2 \leqslant \frac{1}{2}\|u\|_2^2 + \beta\|u\|_1^2 + \alpha\|u\|^2 \leqslant \mu\|u\|_2^2, \tag{5.7}$$

for some $\mu = \mu(\alpha, \beta) \geqslant 1$. Then, following [21], we decompose the solution $S_0(t)\zeta$, with $\zeta \in \mathfrak{B}_0$, into the sum

$$S_0(t)\zeta = L(t)\zeta + K(t)\zeta,$$

where

$$L(t)\zeta = \big(v(t), \partial_t v(t), \eta(t)\big) \quad \text{and} \quad K(t)\zeta = \big(w(t), \partial_t w(t), \rho(t)\big)$$

are the (unique) solutions to the Cauchy problems

$$\begin{cases} \partial_{tt} v + Av - A^{1/2}\eta + \big(\beta + \|u\|_1^2\big)A^{1/2}v + \alpha v = 0, \\ \partial_t \eta + A^{1/2}\eta + A^{1/2}\partial_t v = 0, \\ \big(v(0), \partial_t v(0), \eta(0)\big) = \zeta, \end{cases} \tag{5.8}$$

and

$$\begin{cases} \partial_{tt} w + Aw - A^{1/2}\rho + \big(\beta + \|u\|_1^2\big)A^{1/2}w - \alpha v = h, \\ \partial_t \rho + A^{1/2}\rho + A^{1/2}\partial_t w = 0, \\ \big(w(0), \partial_t w(0), \rho(0)\big) = 0. \end{cases} \tag{5.9}$$

We begin to prove the exponential decay of $L(t)\zeta$.

Lemma 5.6. *There is $\varkappa > 0$ such that*

$$\sup_{\zeta \in \mathfrak{B}_0} \big\|L(t)\zeta\big\|_{\mathcal{H}} \leqslant C\mathrm{e}^{-\varkappa t}.$$

Proof. Denoting for simplicity

$$E_0(t) = \left\| L(t)\zeta \right\|_{\mathcal{H}}^2,$$

we define, for $\varepsilon > 0$, the functional

$$\Lambda_0(t) = \Theta_0(t) + \varepsilon \Upsilon_0(t),$$

where

$$\Theta_0(t) = E_0(t) + \beta \left\| v(t) \right\|_1^2 + \alpha \left\| v(t) \right\|^2 + \left\| u(t) \right\|_1^2 \left\| v(t) \right\|_1^2,$$
$$\Upsilon_0(t) = \left\langle \partial_t v(t), v(t) \right\rangle + 2\left\langle \partial_t v(t), \eta(t) \right\rangle_{-1}.$$

It is clear from (5.3) and (5.7) that, for all ε small enough,

$$\frac{1}{2} E_0 \leqslant \Lambda_0 \leqslant C E_0. \tag{5.10}$$

Due to (5.3), (5.8) and (5.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \Theta_0 + 2\|\eta\|_1^2 = 2\left\langle A^{1/2} u, \partial_t u \right\rangle \|v\|_1^2 \leqslant C\|\partial_t u\| \|v\|_1^2 \leqslant C\|\partial_t u\| \Lambda_0,$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \Upsilon_0 + \left(\|v\|_2^2 + \beta\|v\|_1^2 + \alpha\|v\|^2\right) + \|u\|_1^2\|v\|_1^2 + \|\partial_t v\|^2 \\
&\quad = 2\|\eta\|^2 - 2\langle \partial_t v, \eta\rangle - \langle v, \eta\rangle_1 - 2\left(\beta + \|u\|_1^2\right)\langle v, \eta\rangle - 2\alpha\langle v, \eta\rangle_{-1} \\
&\quad \leqslant \frac{1}{2}\|\partial_t v\|^2 + \frac{1}{2}\|v\|_2^2 + C\|\eta\|_1^2.
\end{aligned}$$

Thus, using (5.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \Upsilon_0 + \frac{1}{4}\|v\|_2^2 + \frac{1}{2}\|\partial_t v\|^2 \leqslant C\|\eta\|_1^2.$$

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \Lambda_0 + \frac{\varepsilon}{4}\|v\|_2^2 + \frac{\varepsilon}{2}\|\partial_t v\|^2 + (2 - C\varepsilon)\|\eta\|_1^2 \leqslant C\|\partial_t u\| \Lambda_0 \leqslant \frac{\varepsilon}{16}\Lambda_0 + C\|\partial_t u\|^2 \Lambda_0.$$

Appealing again to (5.10), it is apparent that, for all $\varepsilon > 0$ small enough, we have the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \Lambda_0 + \frac{\varepsilon}{16}\Lambda_0 \leqslant C\|\partial_t u\|^2 \Lambda_0.$$

The claim follows from Lemmas 5.4 and 5.5, using again (5.10). □

Corollary 5.7. *If $\beta > -\sqrt{\lambda_1}$ and $h = 0$, then*

$$\left\| S_0(t)\zeta \right\|_{\mathcal{H}} \leqslant \mathcal{Q}\left(\|\zeta\|_{\mathcal{H}}\right) \mathrm{e}^{-\varkappa t},$$

for some $\varkappa > 0$ and some positive increasing function $\mathcal{Q}$, explicitly computable.

Proof. We preliminarily observe that it suffices to prove the result for $\zeta \in \mathfrak{B}_0$. Moreover, choosing a constant σ such that

$$-\beta\lambda_1^{-1/2} < \sigma < 1,$$

we find the controls

$$m\|u\|_2^2 \leqslant \sigma\|u\|_2^2 + \beta\|u\|_1^2 \leqslant \left(1 + |\beta|\lambda_1^{-1/2}\right)\|u\|_2^2, \tag{5.11}$$

with

$$m = \begin{cases} \sigma & \text{if } \beta \geqslant 0, \\ \sigma + \beta\lambda_1^{-1/2} & \text{if } \beta < 0. \end{cases}$$

Then, we just recast the proof of Lemma 5.6, with (5.11) in place of (5.7). □

Remark 5.8. Exploiting (5.2), from Corollary 5.7 we readily get the proof of Corollary 4.8.

The next lemma shows the uniform boundedness of $K(t)\mathfrak{B}_0$ in the more regular space $\mathcal{H}^2$, compactly embedded into $\mathcal{H}$.

Lemma 5.9. *The estimate*

$$\sup_{\zeta \in \mathfrak{B}_0} \left\|K(t)\zeta\right\|_{\mathcal{H}^2} \leqslant C$$

holds for every $t \geqslant 0$.

Proof. We first observe that, from (5.3) and Lemma 5.6, we have

$$\|w\|_3^2 \leqslant \|w\|_2\|w\|_4 \leqslant \left(\|u\|_2 + \|v\|_2\right)\|w\|_4 \leqslant C\|w\|_4. \tag{5.12}$$

Setting

$$E_1(t) = \left\|K(t)\zeta\right\|_{\mathcal{H}^2}^2,$$

we define, for $\varepsilon > 0$, the functional

$$\Lambda_1(t) = \Theta_1(t) + \varepsilon\Upsilon_1(t),$$

where

$$\Theta_1(t) = E_1(t) + \left(\beta + \left\|u(t)\right\|_1^2\right)\left\|w(t)\right\|_3^2 - 2\langle Aw(t), h\rangle,$$
$$\Upsilon_1(t) = \langle\partial_t w(t), w(t)\rangle_2 + 2\langle\partial_t w(t), \rho(t)\rangle_1.$$

Note that, from (5.3) and (5.12),

$$\frac{1}{2}E_1 - C \leqslant \Lambda_1 \leqslant CE_1 + C, \tag{5.13}$$

for all ε small enough. Exploiting Lemma 5.6, (5.3) and (5.12), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_1 + 2\|\rho\|_3^2 = 2\alpha\langle v, \partial_t w\rangle_2 + 2\bigl\langle A^{1/2}u, \partial_t u\bigr\rangle\|w\|_3^2 \leqslant \frac{\varepsilon}{4}\|w\|_4^2 + \frac{\varepsilon}{4}\|\partial_t w\|_2^2 + \frac{C}{\varepsilon},$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_1 + \|w\|_4^2 + \|\partial_t w\|_2^2 + \|u\|_1^2\|w\|_3^2 \\
&\quad = -\beta\|w\|_3^2 + 2\|\rho\|_2^2 - \langle w, \rho\rangle_3 - 2\bigl(\beta + \|u\|_1^2\bigr)\langle w, \rho\rangle_2 - 2\langle \partial_t w, \rho\rangle_2 \\
&\qquad + \alpha\langle v, w\rangle_2 + 2\alpha\langle v, \rho\rangle_1 + \bigl\langle Aw + 2A^{1/2}\rho, h\bigr\rangle \\
&\quad \leqslant \frac{1}{4}\|w\|_4^2 + \frac{1}{4}\|\partial_t w\|_2^2 + C\|\rho\|_3^2 + C,
\end{aligned}$$

which entails

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_1 + \frac{3}{4}\|w\|_4^2 + \frac{3}{4}\|\partial_t w\|_2^2 \leqslant C\|\rho\|_3^2 + C.$$

Collecting the above estimates, we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \frac{\varepsilon}{2}\|w\|_4^2 + \frac{\varepsilon}{2}\|\partial_t w\|_2^2 + (2 - C\varepsilon)\|\rho\|_3^2 \leqslant \frac{C}{\varepsilon}.$$

On account of (5.13), we can now fix ε small enough such that the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \nu\Lambda_1 \leqslant C$$

holds for some $\nu > 0$. Applying the Gronwall lemma, and using again (5.13), we are done. $\square$

Lemmas 5.6 and 5.9 yield the exponential attraction property stated in Theorem 5.2.

6. A more general model

In this final section, we discuss a more general abstract problem, obtained by adding the term $\gamma A^{1/2}\partial_{tt}u$ to the first equation of (2.1), where $\gamma \geqslant 0$ is the so-called rotational parameter.

Given $\gamma \geqslant 0$, we define the strictly positive selfadjoint operator on H

$$M_\gamma = 1 + \gamma A^{1/2},$$

with domain $\mathcal{D}(M_\gamma) = H^2$ (when $\gamma > 0$). Since the operator M_γ commutes with A and all its powers, we introduce the spaces

$$H_\gamma^r = \mathcal{D}\bigl(A^{(r-1)/4}M_\gamma^{1/2}\bigr), \quad r \in \mathbb{R},$$

with inner products and norms

$$\langle u, v\rangle_{r,\gamma} = \bigl\langle A^{(r-1)/4}M_\gamma^{1/2}u, A^{(r-1)/4}M_\gamma^{1/2}v\bigr\rangle, \qquad \|u\|_{r,\gamma} = \bigl\|A^{(r-1)/4}M_\gamma^{1/2}u\bigr\|.$$

Finally, we set

$$\mathcal{V}_\gamma^r = H^{r+2} \times H_\gamma^{r+1} \times H^r.$$

Again, we agree to omit the index r when $r = 0$.

Remark 6.1. Note that

$$\|u\|_{r,\gamma}^2 = \|u\|_{r-1}^2 + \gamma\|u\|_r^2 \leqslant \left(\frac{1}{\sqrt{\lambda_1}} + \gamma\right)\|u\|_r^2. \tag{6.1}$$

Hence, when $\gamma > 0$, the space H_γ^r is just H^r endowed with an equivalent norm, whereas $H_0^r = H^{r-1}$ and $\mathcal{V}_0^r = \mathcal{H}^r$.

We consider the Cauchy problem on $\mathcal{V}_\gamma$

$$\begin{cases} M_\gamma \partial_{tt} u + Au - A^{1/2}\theta + \left(\beta + \|u\|_1^2\right)A^{1/2}u = f(t), \quad t > 0, \\ \partial_t\theta + A^{1/2}\theta + A^{1/2}\partial_t u = g(t), \quad t > 0, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \quad \theta(0) = \theta_0, \end{cases} \tag{6.2}$$

of which (2.1) is just the particular instance corresponding to $\gamma = 0$. In concrete models, the additional term $\gamma A^{1/2}\partial_{tt}u$ accounts for the presence of rotational inertia. With f and g as in Proposition 2.1, this system generates a family of solution operators $S^\gamma(t)$ on $\mathcal{V}_\gamma$, satisfying the joint continuity property

$$(t, z) \mapsto S^\gamma(t)z \in \mathcal{C}\left(\mathbb{R}^+ \times \mathcal{V}_\gamma, \mathcal{V}_\gamma\right).$$

The energy at time t corresponding to the initial data $z \in \mathcal{V}_\gamma$ now reads

$$\mathcal{E}^\gamma(t) = \frac{1}{2}\left\|S^\gamma(t)z\right\|_{\mathcal{V}_\gamma}^2 + \frac{1}{4}\left(\beta + \left\|u(t)\right\|_1^2\right)^2,$$

and the energy identity (2.2) is still true replacing $\mathcal{E}$ with $\mathcal{E}^\gamma$. As a matter of fact, all the results stated in the previous sections extend to the present case.

Theorem 6.2. *Theorems* 3.1, 4.1, *Corollaries* 4.2, 4.4, 4.7, 4.8 *and Proposition* 4.6 *continue to hold with* $S^\gamma(t)$ *and* $\mathcal{V}_\gamma^r$ *in place of* $S(t)$ *and* $\mathcal{H}^r$*, respectively.*

Sketch of the proof. Repeat exactly the same demonstrations, simply replacing (clearly, besides $S(t)$ with $S^\gamma(t)$ and $\mathcal{H}^r$ with $\mathcal{V}_\gamma^r$) each occurrence of $\partial_t u$ [or $\partial_t v$, $\partial_t w$] with $M_\gamma\partial_t u$ [or $M_\gamma\partial_t v$, $M_\gamma\partial_t w$] in the definitions of the auxiliary functionals Φ, Ψ, Φ_0, Ψ_0, Υ_0, Υ_1. The integral estimate of Lemma 5.4 improves to

$$\int_s^t \left\|\partial_t u(\tau)\right\|_{1,\gamma}^2 \,\mathrm{d}\tau \leqslant \nu(t-s) + C_\nu, \quad \forall t > s \geqslant 0,$$

although, as we will see in a while, the original estimate would suffice. For example, let us examine more closely the modifications needed in the new proofs of Lemmas 5.6 and 5.9. We keep the same notation, just recalling that now the terms $\partial_{tt}v$ and $\partial_{tt}w$ in (5.8)–(5.9) are replaced by $M_\gamma\partial_{tt}v$ and $M_\gamma\partial_{tt}w$, respectively. The estimate on $\frac{\mathrm{d}}{\mathrm{d}t}\Theta_0$ remains unchanged. Concerning $\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_0$, the term $\|\partial_t v\|^2$ in the left-hand side turns into $\|\partial_t v\|_{1,\gamma}^2$, and in the right-hand side we have $-2\langle M_\gamma\partial_t v, \eta\rangle$ instead of $-2\langle\partial_t v, \eta\rangle$. But thanks to (6.1),

$$-2\langle M_\gamma\partial_t v, \eta\rangle \leqslant 2\|\partial_t v\|_{1,\gamma}\|\eta\|_{1,\gamma} \leqslant \frac{1}{2}\|\partial_t v\|_{1,\gamma}^2 + C\|\eta\|_1^2,$$

for some $C > 0$ independent of γ, provided that, say, $\gamma \leqslant 1$. So, we end up with the same differential inequality for Λ_0, which yields the desired claim in light of the dissipation integral for $\|\partial_t u\|$. Coming to Lemma 5.9, we readily have (cf. (6.1))

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_1 + 2\|\rho\|_3^2 \leqslant \frac{\varepsilon}{4}\|w\|_4^2 + \frac{\varepsilon}{4}\|\partial_t w\|_2^2 + \frac{C}{\varepsilon} \leqslant \frac{\varepsilon}{4}\|w\|_4^2 + \frac{\varepsilon}{4}\|\partial_t w\|_{3,\gamma}^2 + \frac{C}{\varepsilon},$$

whereas in the estimate of $\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_1$ the term $\|\partial_t w\|_2^2$ in the left-hand side becomes $\|\partial_t w\|_{3,\gamma}^2$, and in the right-hand side $-2\langle M_\gamma \partial_t w, \rho\rangle_2$ substitutes $-2\langle \partial_t w, \rho\rangle_2$. A further use of (6.1) entails the control

$$-2\langle M_\gamma \partial_t w, \rho\rangle_2 \leqslant 2\|\partial_t w\|_{3,\gamma}\|\rho\|_{3,\gamma} \leqslant \frac{1}{4}\|\partial_t w\|_{3,\gamma}^2 + C\|\rho\|_3^2.$$

Once again, we are led to the same differential inequality for Λ_1. $\square$

We point out that all the estimates appearing in the proofs are *uniform* with respect to the rotational parameter γ, provided that γ is assumed to be bounded from above. Indeed, the dependence on γ enters only through the definition of the norm. For instance, we can subsume the new versions of Theorems 3.1 and 4.1 into the following result.

Theorem 6.3. *There exist $\varkappa > 0$ and an increasing positive function $\mathcal{Q}$, both independent of $\gamma \geqslant 0$, and a closed ball $\mathfrak{C}^\gamma$ of the space $\mathcal{V}_\gamma^2$ such that the exponential attraction property*

$$\mathrm{dist}_{\mathcal{V}_\gamma}\big(S^\gamma(t)B, \mathfrak{C}^\gamma\big) \leqslant \mathcal{Q}\big(\|B\|_{\mathcal{V}_\gamma}\big)\mathrm{e}^{-\varkappa t}$$

holds for all bounded sets $B \subset \mathcal{V}_\gamma$.

Thus, the global attractor $\mathfrak{A}^\gamma$ of the semigroup $S^\gamma(t)$ is a (uniformly with respect to γ) bounded subset of $\mathcal{V}_\gamma^2$.

Remark 6.4. In particular, for any *fixed* $\gamma > 0$, we deduce the boundedness (this time, not uniform) of $\mathfrak{A}^\gamma$ in $H^4 \times H^3 \times H^2$. This improves the conclusions of [5], where the boundedness of $\mathfrak{A}^\gamma$ for $\gamma > 0$ is obtained only in the *intermediate* space $H^3 \times H^2 \times H^2$.

A straightforward albeit relevant consequence of the $\mathcal{V}_\gamma^2$-regularity of the attractor is emphasized in the next corollary.

Corollary 6.5. *For every $\gamma \geqslant 0$, the solutions to* (6.2) *with initial data on the attractor are strong solutions, i.e. the equations are fulfilled almost everywhere.*

Finally, recasting a standard argument from [22], it is easy to prove

Proposition 6.6. *The family $\{\mathfrak{A}^\gamma\}$ is upper semicontinuous at $\gamma = 0$; namely,*

$$\lim_{\gamma\to 0} \mathrm{dist}_{\mathcal{H}}\big(\mathfrak{A}^\gamma, \mathfrak{A}\big) = 0.$$

On the contrary, the analogue of Proposition 4.9 does not seem to follow by a direct adaptation of the former argument. However, the backward uniqueness property on the attractor holds for $\gamma > 0$ as well, and it can be proved as in [12].

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