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Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics

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Abstract

We propose the study of certain discretizations of geometric evolution equations as an approach to the study of the existence problem of some elliptic partial differential equations of a geometric nature as well as a means to obtain interesting dynamics on certain infinite-dimensional spaces. We illustrate the fruitfulness of this approach in the context of the Ricci flow, as well as another flow, in Kähler geometry. We introduce and study dynamical systems related to the Ricci operator on the space of Kähler metrics that arise as discretizations of these flows. We pose some problems regarding their dynamics. We point out a number of applications to well-studied objects in Kähler and conformal geometry such as constant scalar curvature metrics, Kähler–Ricci solitons, Nadel-type multiplier ideal sheaves, balanced metrics, the Moser–Trudinger–Onofri inequality, energy functionals and the geometry and structure of the space of Kähler metrics. E.g., we obtain a new sharp inequality strengthening the classical Moser–Trudinger–Onofri inequality on the two-sphere.

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1. Introduction

Our main purpose in this article is to propose the systematic use of certain discretizations of geometric evolution equations as an approach to the study of the existence problem of certain elliptic partial differential equations of a geometric nature as well as a means to obtain interesting dynamics on certain infinite-dimensional spaces. We illustrate the fruitfulness of this approach in the context of the Ricci flow, as well as another flow, in Kähler geometry. We describe how this approach gives a new method for the construction of canonical Kähler metrics. We also introduce a number of canonical dynamical systems on the space of Kähler metrics that we believe merit further study. Some of the results and constructions described here were announced previously [54].

Given an elliptic partial differential equation, several classical methods are available to approach the problem of existence of solutions. In essence, standard elliptic theory reduces the existence problem to the demonstration of certain a priori estimates for solutions. The main difficulty lies therefore in devising methods to obtain these estimates.

One common method, that goes back at least to Bernstein and Poincaré, is the continuity method. In this approach one continuously deforms the given elliptic operator to another (oftentimes in a linear fashion), for which the existence problem is known to have solutions. Ellipticity provides for existence of solutions for small perturbations of this easier problem. In order to prove existence for the whole deformation path one then seeks to establish a priori estimates, uniform along the deformation, for solutions of the family of elliptic problems.

Another approach, drawing some of its motivation from Physics, is the heat flow method, going back to Fourier. Here the idea is to study a deformation of the elliptic problem according to a parabolic heat equation whose equilibrium state is precisely a solution to the original elliptic equation. Much of the standard elliptic theory has a parabolic counterpart. First, one makes use of the latter in order to establish short-time existence. Long-time existence and convergence then hinge upon establishing a priori estimates, as before.

A third approach, going back, among others, to Euler and Cauchy, is the discretization method, that can be considered as a blend of the two above. Here the idea is to replace an evolution equation (or "flow") by a countable set of elliptic equations that arise by repeatedly solving a difference equation corresponding to discretizing the flow equation in the time variable. This approach provides common and elementary numerical algorithms, the Euler method and its variants, and is widely used in the "real world," for example to numerically integrate differential equations.

In this article we wish to explore this third approach in the context of certain geometric evolution equations. To the best of our knowledge, it seems that it has not been used before in a systematic manner in this context. We will concentrate here on the Ricci flow and other related flows in the context of Kähler geometry. We hope that this circle of ideas might find applications also in other geometric situations involving other flows (e.g., Yang–Mills flow).

We would like to emphasize that when a particular elliptic equation has a solution one morally expects all three methods to converge towards such a solution. Therefore one should not take as a surprise the fact that the discretization method converges in some of the cases we consider. The crux is thus not the convergence itself but rather the new point of view and insights that this method provides; both to the study of the original elliptic problem as well as to the understanding of the evolution equation, the continuity method and the relation between the two. In addition, one may obtain in this way non-trivial canonical discrete dynamical systems on infinite-dimensional spaces that may be of some interest in their own right.

Let us consider as a simple illustration the Laplace equation on a bounded smooth domain Ω in $\mathbb{R}^n$. The elliptic problem is then to find a function u satisfying

$$\begin{aligned} \Delta u &= 0, \quad \text{on } \Omega, \\ u &= \psi, \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Consider then the difference equations

$$\begin{aligned} u_k - u_{k-1} &= \Delta u_k, \quad \text{on } \Omega, \\ u_k &= \psi, \quad \text{on } \partial\Omega, \\ u_0 &= u, \end{aligned}$$

where u is any smooth function that agrees with the smooth function ψ on the boundary. Thus, one may write

$$u_k = (-\Delta + 1)^{-1} \circ \cdots \circ (-\Delta + 1)^{-1} u.$$

One may then readily show that the sequence $\{u_k\}_{k \geqslant 0}$ exists for each $k \in \mathbb{N}$ and converges exponentially fast in k to the unique solution of (1).

To give further intuition as to why this method works we consider the following finite-dimensional problem: Given a positive semi-definite matrix A and a vector v, find the projection of v onto the zero eigenspace of A. One possible solution is to consider the sequence of vectors defined iteratively by

$$\begin{aligned} v_k &= (A+I)^{-1} v_{k-1}, \\ v_0 &= v. \end{aligned}$$

Then $\lim_{k\to\infty} v_k$ exists and is the required projection. This algorithm is nothing but the discretization of the flow

$$\begin{aligned} \frac{dv(t)}{dt} &= -Av(t), \\ v(0) &= v. \end{aligned}$$

Discretizations corresponding to different time steps will produce equivalent dynamical systems

$$\begin{aligned} v_k &= (\tau A+I)^{-1} v_{k-1}, \\ v_0 &= v, \end{aligned}$$

whose convergence is faster the larger the time-step $\tau \in (0,\infty)$, with τ and the first nonzero eigenvalue of A controlling the exponential factor of the speed of convergence.

2. Constructing canonical metrics in Kähler geometry

In this article we wish to apply the method described in the previous section towards the study of canonical Kähler metrics and the space of Kähler metrics. In this section we very briefly describe the problem and some background. We refer to [3,10,31,63,71] for more background.

The search for a canonical metric representative of a fixed Kähler class has been at the heart of Kähler geometry since its birth. Indeed, in his article Kähler defined the eponymous manifold motivated by the fact that in this setting Einstein's equation simplifies considerably and reduces to a second order partial differential equation for a single function [39], namely, the local potential u which represents the Kähler form ω on the open domain U via $\omega|_U = \sqrt{-1}\partial\bar\partial u$ must satisfy

$$\det\left[\frac{\partial^2 u}{\partial z^i \partial \bar z^j}\right] = e^{-\mu u}, \quad \text{on } U,$$

where μ is the Einstein constant (cf. Schouten–van Dantzig [60] who independently discovered Kähler geometry). Two decades later, following Chern's work on characteristic classes, Calabi introduced the concept of the space of Kähler metrics in a fixed cohomology class and formulated the problem on a compact closed manifold as an equation for a global smooth function (Kähler potential) φ,

$$\omega_\varphi^n = \omega^n e^{f_\omega - \mu\varphi},$$

where f_ω satisfies $\sqrt{-1}\partial\bar\partial f_\omega = \operatorname{Ric}\omega - \mu\omega$. This showed that a necessary condition for the existence of solutions is that the first Chern class be definite or zero. Calabi proposed that this equation should always admit a unique solution in each Kähler class when $\mu = 0$. In addition he suggested the study of a more general notion, that of an extremal metric [12,13]. Since then much progress has been made towards understanding when such metrics exist. Regarding Kähler–Einstein metrics, the most general result in this direction is given by the work of Aubin in the negative Ricci curvature case [1] and by Yau in the case of nonpositive Ricci curvature which provided a solution to Calabi's conjecture [12,80]. Following this much work has gone into understanding the positive case, notably by Tian who provided a complete solution for complex surfaces, in addition to establishing an analytic characterization of Kähler–Einstein manifolds and a theory of stability [67,70]. For general extremal metrics however a general existence theory is not presently available although a conjectural picture, the so-called Yau–Tian–Donaldson conjecture, suggests that it should be intimately related with notions of stability in algebraic geometry [65].

The principal tool in the study of Kähler–Einstein metrics has been the continuity method, as suggested initially by Calabi [13], and later studied by Aubin and Yau. In the remaining case ($\mu > 0$) Bando and Mabuchi showed that the continuity method will converge to a Kähler–Einstein metric when one exists [7]. Another important tool has been the Ricci flow introduced by Hamilton [36] in the more general setting of Riemannian manifolds. Cao has shown that the continuity method proofs for the cases $\mu \leqslant 0$ may be phrased in terms of the convergence of the Ricci flow [14]. Later, much work has gone into understanding the Ricci flow on Fano manifolds and recently Perelman and Tian and Zhu proved that the analogous convergence result holds in this case [76].

The idea that there might be another way of approaching canonical metrics, in the form of a discrete infinite-dimensional iterative dynamical system, was suggested by Nadel [49]. More recently, Donaldson has proposed a program for the construction of constant scalar curvature Kähler metrics on projective manifolds using finite-dimensional iteration schemes and balanced metrics that are in essence computable [28].

The main motivation for our work came from trying to approach Nadel's basic problem: Find an (infinite-dimensional) iterative dynamical system on the space of Kähler metrics that converges to a Kähler–Einstein metric. In his note Nadel suggested one such dynamical system on Fano manifolds which, as we explain below, is not suited to the problem (see Section 10.6). Nevertheless his idea is related to the right answer, and it is our purpose in this article and its sequel [56] to describe our approach to Nadel's problem and some of its consequences.

Setup and notation

Let (M, J, g) be a connected compact closed Kähler manifold of complex dimension n and let $\Omega \in H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C})$ be a Kähler class with $d = \partial + \bar\partial$. Define the Laplacian $\Delta = -\bar\partial \circ \bar\partial^\star - \bar\partial^\star \circ \bar\partial$ with respect to a Riemannian metric g and assume that J is compatible with g and parallel with respect to its Levi-Civita connection. Let $g_{\mathrm{Herm}} = 1/\pi \cdot g_{i\bar j}(z)dz^i \otimes \overline{dz^j}$ be the associated Kähler metric, that is the induced Hermitian metric on $(T^{1,0}M, \mathrm{J})$, and let $\omega := \omega_g = \sqrt{-1}/2\pi \cdot g_{i\bar j}(z)dz^i \wedge d\bar z^j$ denote its corresponding Kähler form, a closed positive $(1,1)$-form on (M,J) such that $g_{\mathrm{Herm}} = \frac{1}{2}g - \frac{\sqrt{-1}}{2}\omega$. Similarly denote by g_ω the Riemannian metric induced from ω by $g_\omega(\cdot,\cdot) = \omega(\cdot, \mathrm{J}\,\cdot)$.

For any Kähler form we let $\mathrm{Ric}(\omega) = -\sqrt{-1}/2\pi \cdot \partial\bar{\partial} \log\det(g_{i\bar{j}})$ denote the Ricci form of ω. It is well-defined globally and represents the first Chern class $c_1 := c_1(T^{1,0}M, \mathrm{J}) \in H^2(M, \mathbb{Z})$. Alternatively it may be viewed as minus the curvature form of the canonical line bundle K_M, the top exterior product of the holomorphic cotangent bundle $T^{1,0\star}M$. One calls ω Kähler–Einstein if $\mathrm{Ric}\,\omega = a\omega$ for some real a. The trace of the Ricci form with respect to ω is called the scalar curvature and is denoted by $s(\omega)$. The average of the scalar curvature is denoted by s_0 and does not depend on the choice of $\omega \in \mathcal{H}_\Omega$. Nor does the volume $V = \Omega^n([M])$.

Let H_g denote the Hodge projection operator from the space of closed forms onto the kernel of Δ. Denote by $\mathcal{D}_\Omega$ the space of all closed (1, 1)-forms cohomologous to Ω, by $\mathcal{H}_\Omega$ the subspace of Kähler forms, and by $\mathcal{H}_\Omega^+$ the subspace of Kähler forms whose Ricci curvature is positive (nonempty if and only if $c_1 > 0$).

For a Kähler form ω with $[\omega] = \Omega$ we will consider the space of smooth strictly ω-plurisubharmonic functions (Kähler potentials)

$$\mathcal{H}_\omega = \left\{\varphi \in C^\infty(M)\colon\ \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\right\}.$$

Let $\mathrm{Aut}(M, \mathrm{J})$ denote the complex Lie group of automorphisms (biholomorphisms) of (M, J) and denote by $\mathrm{aut}(M, \mathrm{J})$ its Lie algebra of infinitesimal automorphisms composed of real vector fields X satisfying $\mathcal{L}_X\mathrm{J} = 0$. Let G be any compact real Lie subgroup of $\mathrm{Aut}(M, \mathrm{J})$, and let $\mathrm{Aut}(M, \mathrm{J})_0$ denote the identity component of $\mathrm{Aut}(M, \mathrm{J})$. We denote by $\mathcal{H}_\Omega(G) \subseteq \mathcal{H}_\Omega$ and $\mathcal{H}_\omega(G) \subseteq \mathcal{H}_\omega$ the corresponding subspaces of G-invariant elements.

3. The Ricci iteration

In this section we introduce the Ricci iteration and describe some of its elementary properties.

Hamilton's Ricci flow on a Kähler manifold of definite or zero first Chern class is defined as the set $\{\omega(t)\}_{t\in\mathbb{R}_+}$ satisfying the evolution equations

$$\begin{aligned} &\frac{\partial\omega(t)}{\partial t} = -\mathrm{Ric}\,\omega(t) + \mu\omega(t), \quad t \in \mathbb{R}_+, \\ &\omega(0) = \omega \in \mathcal{H}_\Omega, \end{aligned} \tag{2}$$

where Ω is a Kähler class satisfying $\mu\Omega = c_1$ for some $\mu \in \mathbb{R}$ (see, e.g., [22]). We will sometimes refer to Eqs. (2) themselves as the Ricci flow.

We introduce the following dynamical system that is our main object of study in this article. It is a discrete version of this flow.

Definition 3.1. Let Ω be a Kähler class satisfying $\mu\Omega = c_1$ for some $\mu \in \mathbb{R}$. Given a Kähler form $\omega \in \mathcal{H}_\Omega$ and a number $\tau > 0$ define the time τ Ricci iteration to be the sequence of forms $\{\omega_{k\tau}\}_{k\geqslant 0} \subseteq \mathcal{H}_\Omega$, satisfying the equations

$$\begin{aligned} &\omega_{k\tau} = \omega_{(k-1)\tau} + \tau\mu\omega_{k\tau} - \tau\,\mathrm{Ric}\,\omega_{k\tau}, \quad k \in \mathbb{N}, \\ &\omega_0 = \omega, \end{aligned} \tag{3}$$

for each $k \in \mathbb{N}$ for which a solution exists in $\mathcal{H}_\Omega$.

We pose the following elementary conjecture concerning the limiting behavior of the Ricci iteration in the presence of fixed points.

Conjecture 3.2. *Let (M, J) be a compact Kähler manifold admitting a Kähler–Einstein metric. Let Ω be a Kähler class such that $\mu\Omega = c_1$ with $\mu \in \mathbb{R}$. Then for any $\omega \in \mathcal{H}_\Omega$ and for any $\tau > 0$, the time τ Ricci iteration exists for all $k \in \mathbb{N}$ and converges in the sense of Cheeger–Gromov to a Kähler–Einstein metric.*

Regarding this conjecture we prove in this article the following result.

Theorem 3.3. *Let (M, J) be a compact Kähler manifold admitting a unique Kähler–Einstein metric. Let Ω be a Kähler class such that $\mu\Omega = c_1$ with $\mu \in \mathbb{R}$. Then for any $\omega \in \mathcal{H}_\Omega$ and for any $\tau > 1/\mu$ (when $\mu > 0$) or $\tau > 0$ (when $\mu \leqslant 0$), the time τ Ricci iteration exists for all $k \in \mathbb{N}$ and converges to a Kähler–Einstein metric in the $C^l(M)$-topology for any $l \in \mathbb{N}$. When $\alpha_G(M) > 1$ (see (31)) for some compact group $G \subset \mathrm{Aut}(M, \mathrm{J})$ and $\omega \in \mathcal{H}_\Omega(G)$ the same also holds for $\tau = 1/\mu$ (when $\mu > 0$).*

It is worth noting that while many ingredients of the proof rely on similar techniques to those used in the continuity and flow methods, some of the arguments in proving certain a priori estimates are novel and are special to our discrete dynamics situation.

Let the Ricci potential $f : \varphi \in \mathcal{H}_\omega \to f_{\omega_\varphi} \in C^\infty(M)$ be the vector field on $\mathcal{H}_\omega$ satisfying

$$\sqrt{-1}\partial\bar{\partial} f_{\omega_\varphi} = \mathrm{Ric}\,\omega_\varphi - \mu\omega_\varphi, \qquad \frac{1}{V}\int_M e^{f_{\omega_\varphi}}\omega_\varphi^n = 1. \tag{4}$$

For each k write

$$\omega_{k\tau} = \omega_{\psi_{k\tau}}, \quad \text{with } \psi_{k\tau} = \sum_{l=1}^{k} \varphi_{l\tau}.$$

The iteration (3) on $\mathcal{H}_\Omega$ can be written as the following system of complex Monge–Ampère equations on $\mathcal{H}_\omega$,

$$\omega_{\psi_{k\tau}}^n = \omega^n e^{f_\omega + \frac{1}{\tau}\varphi_{k\tau} - \mu\psi_{k\tau}} = \omega_{\psi_{(k-1)\tau}}^n e^{(\frac{1}{\tau}-\mu)\varphi_{k\tau} - \frac{1}{\tau}\varphi_{(k-1)\tau}}, \quad k \in \mathbb{N} \tag{5}$$

(implicit in this equation is also a normalization for $\varphi_{k\tau}$ that eliminates the ambiguity in passing from an equation on $\mathcal{H}_\Omega$ to one on $\mathcal{H}_\omega$). The term $\frac{1}{\tau}\varphi_{k\tau}$ replaces the term "$\dot{\psi}$" that would appear in the Ricci flow.

We now mention some basic features of the iteration.

The most elementary one is that at each step one gains regularity (two derivatives). This is a discrete version of the infinite smoothing property of heat equations.

Another distinctive feature of the iteration is that it can be used to turn the solution of each type of Monge–Ampère equation into the next simplest one. Indeed, to find a Kähler–Einstein metric of negative scalar curvature $-n$ one needs to solve the equation

$$\omega_\varphi^n = \omega^n e^{f_\omega + \varphi}. \tag{6}$$

The corresponding time one Ricci iteration requires solving at each step the equation

$$\omega_{\varphi}^{n} = \omega^{n} e^{f_{\omega}+2\varphi}. \tag{7}$$

Similarly, the Calabi–Yau equation

$$\omega_{\varphi}^{n} = \omega^{n} e^{f_{\omega}}, \tag{8}$$

is traded for a sequence of equations of the previous type (6), and finally, the most difficult equation,

$$\omega_{\varphi}^{n} = \omega^{n} e^{f_{\omega}-\varphi}, \tag{9}$$

for a Kähler–Einstein metric of positive scalar curvature n, is turned into a sequence of Calabi–Yau equations (8) via the time one iteration, or to a sequence of equations of the type (6) for smaller time steps. For large time steps it remains an equation of the same type.

We now discuss the link the iteration creates between classical continuity method paths and the heat equation. In several places in the literature on canonical metrics it is mentioned that the continuity method and the heat flow method are morally equivalent (see, e.g., [38,63]). The following discussion comes to make this statement somewhat more explicit.

Indeed, note that one may also consider the time step of the iteration as a dynamical parameter and study the continuity method path it defines. More precisely, the time τ Ricci iteration is given by a sequence $\{\omega_{k\tau}\}_{k\geqslant 1}$ of Kähler forms in $\mathcal{H}_{\Omega}$ satisfying

$$\omega_{k\tau} = \omega_{(k-1)\tau} + \tau\mu\omega_{k\tau} - \tau \operatorname{Ric}\omega_{k\tau}, \qquad \omega_0 = \omega, \tag{10}$$

for each $k \in \mathbb{N}$ for which a solution exists. Now set $k = 1$ and consider the path $\{\omega_\tau := \omega_{\varphi_\tau}\}_{\tau\geqslant 0}$ in $\mathcal{H}_{\Omega}$ (for each τ for which it exists). In $\mathcal{H}_{\omega}$ we obtain the path

$$\omega_{\varphi_\tau}^{n} = \omega^{n} e^{f_{\omega}+(\frac{1}{\tau}-\mu)\varphi_\tau}, \quad \tau \in (0,\infty). \tag{11}$$

Let us compare this path to others that appeared previously in the literature.

In the case $\mu = 1$, when restricted to the segment $\tau \geqslant 1$ this is just a reparametrization of Aubin's path [2] given by

$$\omega_{\varphi_s}^{n} = \omega^{n} e^{f_{\omega}-s\varphi_s}, \quad s \in [0,1], \tag{12}$$

via $s = 1 - \frac{1}{\tau}$. Here the solution for (12) at $s = 0$ is given by the Calabi–Yau Theorem. Namely, one typically first solves the family of equations introduced by Calabi [13, (11)], [80]

$$\omega_{\varphi_s}^{n} = \omega^{n} e^{(s+1)f_{\omega}+c_s}, \quad s \in [-1,0], \tag{13}$$

and then continues to work with the path (12).

For the path (11) we still need to invoke the Calabi–Yau Theorem to show closedness at $\tau \to 1^{-}$. However, this path can be viewed as a continuity version of the Ricci flow and has various monotonicity properties that (13) does not. When studying the Ricci iteration for the case $\Omega = c_1$ this will be useful (note also that this path may be used in place of (13) to prove the Calabi–Yau Theorem).

Remark 3.4. We note that Calabi's path (13) can in fact be interpreted as a continuity path arising from a flow, however not the Ricci flow, see (49) below.

In light of this relation to the continuity method, the Ricci iteration is seen to interpolate between the continuity method ($\tau = \infty$) and the Ricci flow ($\tau = 0$). Cao and Perelman proved that when a Kähler–Einstein metric exists, the flow will converge to it in the sense of Cheeger–Gromov. Aubin, Bando–Mabuchi and Yau proved the analogous result for the continuity method. These results are the main motivation for Conjecture 3.2.

Next, in the case $\mu = -1$, the continuity path (11) that arises from the Ricci iteration equation is the same as the continuity path considered by Tian and Yau in their study of Kähler–Einstein metrics of negative Ricci curvature on some non-compact manifolds [72, p. 586]. The innovative idea of Tian and Yau was to consider a continuity parameter "starting from infinity" observing that along this path one has a uniform lower bound for the Ricci curvature.

Now an open problem concerning the flow equation (2) with $\mu > 0$ is whether one has a uniform lower bound for the Ricci curvature depending only on the initial data in the absence of a Kähler–Einstein metric. Along the time τ iteration one does have such a bound, depending on τ, namely, $\operatorname{Ric}\omega_{k\tau} > \frac{\tau\mu-1}{\tau}\omega_{k\tau}$. One possible approach to this problem might be to show that the Ricci flow stays asymptotically close to the time τ Ricci iteration for some range of time steps τ.

Remark 3.5. We remark that when $\mu = 1$ another path has been considered previously by Demailly and Kollár [23, (6.2.3)], given by $\omega_{\varphi_t}^n = \omega^n e^{tf_\omega - t\varphi_t}$, $t \in [0, 1]$. As written, this path also does not require to start from a solution to a Calabi–Yau equation. Yet in order to get openness for it one assumes $\operatorname{Ric}\omega > 0$ and this involves solving a Calabi–Yau equation (indeed there is no way to produce a Kähler–Einstein metric without entering $\mathcal{H}_{c_1}^+$, and the Calabi–Yau Theorem amounts to $\mathcal{H}_{c_1}^+ \neq \emptyset$). This path can also be explained in terms of a discretization; see (50).

Remark 3.6. Another relation between a continuity path, defined by Tian and Zhu, and a discretized flow, this time a modified Ricci flow, will be discussed in Section 9.

4. Some energy functionals on the space of Kähler metrics

In this section we will obtain a monotonicity result along the Ricci iteration for a family of energy functionals. This result has independent interest, and it seems interesting to compare it with corresponding studies for the Ricci flow (see Remark 4.4).

We briefly recall the pertinent definitions and properties of these energy functionals. For more details on this subject we refer to a previous article [58] and the references therein.

We call a real-valued function A defined on a subset $\operatorname{Dom}(A)$ of $\mathcal{D}_\Omega \times \mathcal{D}_\Omega$ an energy functional if it is zero on the diagonal restricted to $\operatorname{Dom}(A)$. By a Donaldson-type functional, or exact energy functional, we will mean an energy functional that satisfies the cocycle condition $A(\omega_1, \omega_2) + A(\omega_2, \omega_3) = A(\omega_1, \omega_3)$ with each of the pairs appearing in the formula belonging to $\operatorname{Dom}(A)$ [26,44,71]. We will occasionally refer to both of these simply as functionals and exact functionals, respectively.

The functionals I, J, introduced by Aubin [2], are defined for each pair $(\omega, \omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi) \in \mathcal{D}_\Omega \times \mathcal{D}_\Omega$ by

$$I(\omega,\omega_\varphi)=V^{-1}\int_M\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{l=0}^{n-1}\omega^{n-1-l}\wedge\omega_\varphi^l=V^{-1}\int_M\varphi\big(\omega^n-\omega_\varphi^n\big),\tag{14}$$

$$J(\omega,\omega_\varphi)=\frac{V^{-1}}{n+1}\int_M\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{l=0}^{n-1}(n-l)\omega^{n-l-1}\wedge\omega_\varphi^l.\tag{15}$$

We say that an exact functional A is bounded from below on $U\subseteq\mathcal{H}_\omega$ if for every ω such that $(\omega,\omega_\varphi)\in\mathrm{Dom}(A)$ and $\omega_\varphi\in U$ holds $A(\omega,\omega_\varphi)\geqslant C_\omega$ with C_ω independent of ω_φ. We say it is proper (in the sense of Tian) on a set $U\subseteq\mathcal{H}_\Omega(G)$ if for each $\omega\in\mathcal{H}_\Omega(G)$ there exists a smooth function $\nu_\omega:\mathbb{R}\to\mathbb{R}$ satisfying $\lim_{s\to\infty}\nu_\omega(s)=\infty$ such that $A(\omega,\omega_\varphi)\geqslant\nu_\omega((I-J)(\omega,\omega_\varphi))$ for every $\omega_\varphi\in U$ [69, Definition 5.1], [71]. This is well-defined, in other words depends only on $[\omega]$. Properness of a functional implies it has a lower bound.

Previously [58] we introduced the following collection of energy functionals for each $k\in\{0,\dots,n\}$,

$$\begin{aligned}I_k(\omega,\omega_\varphi)&=\frac{1}{V}\int_M\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{l=0}^{k-1}\frac{k-l}{k+1}\omega^{n-1-l}\wedge\omega_\varphi^l\\&=\frac{V^{-1}}{k+1}\int_M\varphi\left(k\omega^n-\sum_{l=1}^{k}\omega^{n-l}\wedge\omega_\varphi^l\right).\end{aligned}\tag{16}$$

Note that $I_n=J$, $I_{n-1}=((n+1)J-I)/n$, and $I_0=0$. These functionals can be referred to as the "staircase functionals" (see [57, §§4.4.1] for an explanation of this choice of name).

When the Kähler class is proportional to the first Chern class, that is $\mu\Omega=c_1$ for some $\mu\in\mathbb{R}$, define the Ding functional

$$F_\mu(\omega,\omega_\varphi)=\begin{cases}-\dfrac{1}{n+1}\dfrac{1}{V}\displaystyle\int_M\varphi\sum_{l=0}^{n}\omega^{n-l}\wedge\omega_\varphi^l-\frac{1}{\mu}\log\frac{1}{V}\int_M e^{f_\omega-\mu\varphi}\omega^n, & \text{for } \mu\neq0,\\ -\dfrac{1}{n+1}\dfrac{1}{V}\displaystyle\int_M\varphi\sum_{l=0}^{n}\omega^{n-l}\wedge\omega_\varphi^l+\frac{1}{V}\int_M\varphi e^{f_\omega}\omega^n, & \text{for } \mu=0.\end{cases}\tag{17}$$

The critical points of these functionals are Kähler–Einstein metrics [24]. However, there is an important difference between the two cases in (17). While for the first the functional is exact, for $\mu=0$ this is not true. This is because the second term of F_0 is not exact on the space $\mathcal{H}_\omega$, while the first is. This rather peculiar phenomenon is reflected also by a property of the generalized Ding functional (see the end of Section 10.3).

Let $\mu_k:=\frac{c_1^{k+1}\cup[\omega]^{n-k-1}([M])}{[\omega]^n([M])}$. When $\mu\Omega=c_1$ we have $\mu_k=\mu^{k+1}$. Define the Chen–Tian functionals

$$E_k(\omega,\omega_\varphi)=\frac{1}{k+1}\frac{1}{V}\int_{M\times[0,1]}\big[\mu_k\omega_\varphi^{k+1}-(\mathrm{Ric}\,\omega_\varphi-\Delta_{\omega_\varphi}\dot\varphi)^{k+1}\big]\wedge(\omega_\varphi+\dot\varphi)^{n-k}\wedge dt.\tag{18}$$

Note our convention contains the factor $1/(k+1)$. These are well-defined independently of the choice of path and are exact [21] (for a different proof see [57, §§4.4.5]). Moreover, in the case $\mu \neq 0$, the functionals E_k are related to the K-energy E_0 introduced by Mabuchi [44] via the following relation:

Proposition 4.1. *Let $\mu \neq 0$. Let $k \in \{0, \ldots, n\}$. For every $(\omega, \omega_\varphi) \in \mathcal{H}_{c_1/\mu} \times \mathcal{H}_{c_1/\mu}$,*

$$E_k(\omega, \omega_\varphi) = \mu^k E_0(\omega, \omega_\varphi) + \mu^{k+1} I_k\big(\omega_\varphi, \mu^{-1} \operatorname{Ric} \omega_\varphi\big) - \mu^{k+1} I_k\big(\omega, \mu^{-1} \operatorname{Ric} \omega\big). \tag{19}$$

Proof. This relation has been previously demonstrated for the case $\mu = 1$ [58, Proposition 2.6] (recall that the case $\mu = 1$, $k = n$ is a result of Bando and Mabuchi). In general the same proof goes through by keeping track of the constant μ. □

Finally, recall the definitions of the following subsets of the space of Kähler forms on a Fano manifold [58]:

$$\mathcal{A}_k(\omega) = \big\{\omega_\varphi \in \mathcal{H}_{c_1}\colon\ E_k(\omega, \omega_\varphi) \geqslant 0\big\}, \tag{20}$$

$$\mathcal{B}_k = \big\{\omega_\varphi \in \mathcal{H}_{c_1}\colon\ I_k(\omega_\varphi, \operatorname{Ric} \omega_\varphi) \geqslant 0\big\}. \tag{21}$$

When a Kähler–Einstein form ω exists we denote $\mathcal{A}_k := \mathcal{A}_k(\omega)$. This is well-defined and does not depend on the choice of the Kähler–Einstein form. We recall that both the sets $\mathcal{A}_k$ and $\mathcal{B}_k$ strictly contain the set $\mathcal{H}^+_{c_1}$, of Kähler forms of positive Ricci curvature (see Theorem 10.6 below for a more precise statement).

The following monotonicity result will be useful later. For E_0, F_μ and E_1 with $1/\mu = \tau = 1$ this was proven before [5,25,64].

Proposition 4.2.

(i) *The functional E_0 is monotonically decreasing along the time τ iteration ($\tau > 0$) whenever the initial point is not Kähler–Einstein.*
(ii) *When $1/\mu = \tau = 1$ the same is true for F_1, E_1, and, when the initial metric lies in $\mathcal{B}_k \supset \mathcal{H}^+_{c_1}$, also for E_k, $k \geqslant 2$.*

Proof. By exactness, it suffices to show monotonicity at each step of the iteration.

(i) For concreteness we derive the result only for $\tau = 1$ but explicitly compute the energy decrease along the iteration (in general see Lemma 9.3, stated for $\mu = 1$, that works for all μ). Consider the equation $\omega^n_{\varphi_1} = \omega^n e^{f_\omega - (\mu - 1)\varphi_1}$. One has [18], [69, p. 254], [70, (5.14)]

$$E_0(\omega, \omega_{\varphi_1}) = \frac{1}{V} \int_M \log \frac{\omega^n_{\varphi_1}}{\omega^n} \omega^n_{\varphi_1} - \mu (I - J)(\omega, \omega_{\varphi_1}) + \frac{1}{V} \int_M f_\omega \big(\omega^n - \omega^n_{\varphi_1}\big).$$

First, let $\mu = 1$. One has,

$$E_0(\omega, \omega_{\varphi_1}) = -(I - J)(\omega, \omega_{\varphi_1}) + \frac{1}{V} \int_M f_\omega \omega^n. \tag{22}$$

This is nonpositive by the definition of f_ω and Jensen's inequality.

When $\mu = -1$ one has

$$\begin{aligned} E_0(\omega, \omega_{\varphi_1}) &= \frac{1}{V}\int_M 2\varphi_1 \omega_{\varphi_1}^n + (I-J)(\omega, \omega_{\varphi_1}) + \frac{1}{V}\int_M f_\omega \omega^n \\ &= \frac{1}{V}\int_M 2\varphi_1 \omega_{\varphi_1}^n + \frac{1}{V}\int_M \varphi_1\big(\omega^n - \omega_{\varphi_1}^n\big) - J(\omega, \omega_{\varphi_1}) + \frac{1}{V}\int_M f_\omega \omega^n \\ &= -(I+J)(\omega, \omega_{\varphi_1}) + \frac{1}{V}\int_M (f_\omega + 2\varphi_1)\omega^n. \end{aligned}$$

Each term is nonpositive, once again by using the normalization inherent in (7).

When $\mu = 0$ one has

$$\begin{aligned} E_0(\omega, \omega_{\varphi_1}) &= \frac{1}{V}\int_M \varphi_1 \omega_{\varphi_1}^n + \frac{1}{V}\int_M f_\omega \omega^n \\ &= -I(\omega, \omega_{\varphi_1}) + \frac{1}{V}\int_M (f_\omega + \varphi_1)\omega^n, \end{aligned}$$

and we may argue as before.

(ii) Using (14)–(15) and (17) one has

$$\begin{aligned} F_1(\omega, \omega_{\varphi_1}) &= -(I-J)(\omega, \omega_{\varphi_1}) - \frac{1}{V}\int_M \varphi_1 \omega_{\varphi_1}^n - \log \frac{1}{V}\int_M e^{f_\omega - \varphi_1}\omega^n && (23) \\ &= -(I-J)(\omega, \omega_{\varphi_1}) - \frac{1}{V}\int_M \varphi_1 \omega_{\varphi_1}^n - \log \frac{1}{V}\int_M e^{-\varphi_1}\omega_{\varphi_1}^n. && (24) \end{aligned}$$

By Jensen's inequality the last two terms combined are nonpositive, and so we conclude by the positivity of $I - J$. (Alternatively, the iteration stays on the submanifold of $\mathcal{H}_\omega$ defined by the equation $\frac{1}{V}\int_M e^{f_\omega - \varphi}\omega^n = 1$ and hence the third term is identically zero, while the second one is negative, again by a special case of Jensen's inequality: $1 - \frac{1}{V}\int_M \varphi_1 \omega_1^n \leqslant \frac{1}{V}\int_M e^{-\varphi_1}\omega_1^n = \frac{1}{V}\int_M \omega_2^n = 1$.)

Next, when $1/\mu = \tau = 1$ Proposition 4.1 gives

$$E_k(\omega, \omega_{\varphi_1}) = E_0(\omega, \omega_{\varphi_1}) + I_k(\omega_{\varphi_1}, \operatorname{Ric}\omega_{\varphi_1}) - I_k(\omega, \operatorname{Ric}\omega). \tag{25}$$

Next, recall the inequality $I_k \leqslant J$ [58, (9)]. Since in our case $\operatorname{Ric}\omega_{\varphi_1} = \omega$ we deduce from (25) and (22) that

$$E_k(\omega, \omega_{\varphi_1}) \leqslant -(I-J)(\omega, \omega_{\varphi_1}) + J(\omega_{\varphi_1}, \omega) - I_k(\omega, \operatorname{Ric}\omega).$$

Since $J(\omega,\omega_{\varphi_1})+J(\omega_{\varphi_1},\omega)=I(\omega,\omega_{\varphi_1})$, one has $E_k(\omega,\omega_{\varphi_1})\leqslant 0$ if $I_k(\omega,\operatorname{Ric}\omega)\geqslant 0$, or in other words, if $\omega\in\mathcal{B}_k$. Finally, note that the subspace $\mathcal{B}_k$ is preserved under the iteration since, in fact, after the first step the iteration will stay in $\mathcal{H}^+_{c_1}$. □

Remark 4.3. An alternative derivation of the second part of (ii) could be to choose a particular path connecting ω and ω_1 and note that each of the contributions has a preferred sign. For example, choosing Calabi's continuity path (13) produces three terms of which two are evidently nonpositive. However then one still needs to manipulate the third term which comes up, $\frac{V^{-1}}{k+1}\int_M\sum_{i=1}^k\binom{i+1}{k+1}f_\omega(\sqrt{-1}\partial\bar\partial f_\omega)^i\wedge\omega^{n-i}$, and argue that it equals precisely $-I_k(\omega,\operatorname{Ric}\omega)$ and then use the results of [58] as above. However to derive (i) for all time steps one needs to use instead the path (11) along which E_0 is monotonic, as alluded to after (13) above.

Remark 4.4. Here it is interesting to compare with the Ricci flow. One knows that F_1, E_0 are monotonically decreasing along the flow and that as long as $\operatorname{Ric}\omega>-\omega$ the same is true for E_1 [21, §§3.3, Proposition 4.9]. However an analogous result is not known along the Ricci flow for E_k, $k\geqslant 2$. In the case that a Kähler–Einstein metric exists one knows that the flow will converge. One also knows that when restricted to the space $\mathcal{B}_k$ the functional E_k attains a minimum precisely on the space of Kähler–Einstein metrics, however that outside this space it is not true that these functionals are bounded from below on $\mathcal{H}_{c_1}$ [58, §5] (see also Section 10.2 below). Thus all that is apparent at the present moment is that once the flow stays in $\mathcal{B}_k$, the functional E_k will eventually decrease, however even then we do not know whether this will happen monotonically.

5. The Ricci iteration for negative and zero first Chern class

In this section we prove the existence and convergence of the Ricci iteration in the case that either $c_1<0$ and $\Omega=-c_1$, or that $c_1=0$ and Ω is an arbitrary Kähler class.

We start with a result that is a simple consequence of the theory of elliptic complex Monge–Ampère equations. This result is the existence part of Theorem 3.3 in the cases under consideration.

Lemma 5.1. *Let (M,J) be a compact Kähler manifold whose first Chern class is negative or zero. When $c_1=0$ denote by Ω a Kähler class; otherwise let $\Omega=\mu c_1$ denote a Kähler class with $\mu<0$. Then for any $\omega\in\mathcal{H}_\Omega$, the time τ Ricci iteration exists for all $k\in\mathbb{N}$ and all $\tau\in(0,\infty)$.*

Proof. It is enough to show existence for one step of the iteration in order to show the iteration exists for each $k\in\mathbb{N}$ (by repeating the argument at each step).

Fix $\tau\in(0,\infty)$. The existence of ω_1 amounts to solving the equation

$$\omega_1=\omega_0+\tau\mu\omega_1-\tau\operatorname{Ric}\omega_1.$$

Let $\omega_{\varphi_1}=\omega_1$ with $\varphi_1\in\mathcal{H}_\omega$. This can be written as a complex Monge–Ampère equation:

$$\omega^n_{\varphi_1}=\omega^n e^{f_\omega+(\frac{1}{\tau}-\mu)\varphi_1},\qquad \int_M\omega^n e^{f_\omega+(\frac{1}{\tau}-\mu)\varphi_1}=V. \tag{26}$$

Under the assumption $\mu < 1/\tau$, and hence in particular if $\mu \leqslant 0$, the maximum principle gives an a priori L^∞ estimate on φ_1. Then the work of Aubin and Yau [1,80] immediately applies to give higher-order estimates. We conclude that a unique solution $\omega_{\varphi_1} \in \mathcal{H}_\Omega$ exists. □

We now turn to the proof of the convergence statement of Theorem 3.3 in the cases under consideration.

Proof of Theorem 3.3 *($\mu \leqslant 0$).* Assume first that $c_1 < 0$ and let $\Omega = -c_1$. We have the following system of Monge–Ampère equations:

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega + \psi_{k\tau} + \frac{1}{\tau}\varphi_{k\tau}}, \quad k \in \mathbb{N}. \tag{27}$$

We first prove an a priori uniform bound, independent of k in an inductive manner. The first equation reads $\omega^n_{\varphi_\tau} = \omega^n e^{f_\omega + (1+\frac{1}{\tau})\varphi_\tau}$. At the maximum of φ_τ we have $\omega_{\varphi_\tau} \leqslant \omega$ and thus $(1+\frac{1}{\tau})\sup\varphi_\tau \leqslant -\inf f_\omega$. A similar argument at the minimum of φ_τ gives $-(1+\frac{1}{\tau})\inf\varphi_\tau \leqslant \sup f_\omega$. The second equation reads $\omega^n_{\varphi_\tau+\varphi_{2\tau}} = \omega^n_{\varphi_\tau} e^{-\frac{1}{\tau}\varphi_\tau + (1+\frac{1}{\tau})\varphi_{2\tau}}$. The maximum/minimum principle now gives $(1+\frac{1}{\tau})\sup\varphi_{2\tau} \leqslant \frac{1}{\tau}\sup\varphi_\tau$ and $-(1+\frac{1}{\tau})\inf\varphi_{2\tau} \leqslant -\frac{1}{\tau}\inf\varphi_\tau$ or $\sup\varphi_{2\tau} \leqslant -\frac{\tau}{(1+\tau)^2}\inf f_\omega$ and $-\inf\varphi_{2\tau} \leqslant \frac{\tau}{(1+\tau)^2}\sup f_\omega$. We then have $\sup\psi_{k\tau} \leqslant -\inf f_\omega$, $-\inf\psi_{k\tau} \leqslant \sup f_\omega$. This uniform bound implies the existence of an a priori $C^{2,\alpha}$ bound on $\psi_{k\tau}$, independently of k. Such a claim would follow directly from Aubin and Yau's arguments [1,80] if the term $\varphi_{k\tau}$ did not appear in the right-hand side of (27). To justify the claim in our context where such a term does appear we will argue differently in order to obtain a uniform estimate for $\Delta_\omega\psi_{k\tau}$ (that is in fact equivalent, via the usual Aubin–Yau argument (see, e.g., [63]), to a uniform estimate on $\Delta_\omega\varphi_{k\tau}$). Such an estimate then implies a uniform $C^{2,\alpha}$ estimate on $\psi_{k\tau}$ using standard elliptic regularity techniques for the nondegenerate Monge–Ampère equation [11, §5]. To obtain the Laplacian estimate we follow Bando–Kobayashi's derivation of a Laplacian estimate in a different context, that adapts to our setting [6, p. 179]. Let $f:(Q,\alpha_1) \to (R,\alpha_2)$ be a map between two complete Kähler manifolds. The Chern–Lu inequality in the context of Yau's Schwarz Lemma gives [42,43,79]

$$\Delta_{\alpha_1}\log|\partial f|^2 \geqslant \frac{\operatorname{Ric}\alpha_1(\partial f, \bar\partial f)}{|\partial f|^2} - \frac{\operatorname{Bisect}_{\alpha_2}(\partial f, \bar\partial f, \partial f, \bar\partial f)}{|\partial f|^2}.$$

Let now f be the identity map from $(M, \omega_{\psi_{k\tau}})$ to (M, ω). In our setting we have

$$\operatorname{Ric}\omega_{\psi_{k\tau}} = -(1+1/\tau)\omega_{\psi_{k\tau}} + 1/\tau\,\omega_{\psi_{(k-1)\tau}} > -(1+1/\tau)\omega_{\psi_{k\tau}},$$

that is the Ricci curvature is uniformly bounded from below along the iteration. Since $|\partial f|^2 = \operatorname{tr}_{\omega_{\psi_{k\tau}}}\omega = n - \Delta_{\omega_{\psi_{k\tau}}}\psi_{k\tau}$ it follows that

$$\Delta_{\omega_{\psi_{k\tau}}}\log(n - \Delta_{\omega_{\psi_{k\tau}}}\psi_{k\tau}) \geqslant -C(1 + n - \Delta_{\omega_{\psi_{k\tau}}}\psi_{k\tau}),$$

for some uniform constant $C > 0$, and hence

$$\Delta_{\omega_{\psi_{k\tau}}}\big(\log(n - \Delta_{\omega_{\psi_{k\tau}}}\psi_{k\tau}) - (C+1)\psi_{k\tau}\big) \geqslant -n - C(1+n) + (n - \Delta_{\omega_{\psi_{k\tau}}}\psi_{k\tau}). \tag{28}$$

We may now apply the maximum principle to obtain a uniform upper bound for $\operatorname{tr}_{\omega_{\psi_{k\tau}}}\omega$, using the fact that we already have a uniform L^∞ estimate for $\psi_{k\tau}$. Now observe the uniform L^∞ estimate on $\psi_{k\tau}$ together with (27) implies that the volume forms $\omega^n_{\psi_{k\tau}}$ converge in L^∞ to a fixed uniformly positive and bounded volume form. Namely, $\omega^n_{\psi_{k\tau}}/\omega^n = \det_\omega \omega_{\psi_{k\tau}}$ is uniformly bounded. This then implies that also $\operatorname{tr}_\omega \omega_{\psi_{k\tau}}$ is uniformly bounded, namely we have a uniform bound for $n + \Delta_\omega \psi_{k\tau}$ as required.

As a result, by elliptic regularity theory, a subsequence converges to a smooth solution which we denote by ψ_∞. In fact the convergence is exponentially fast and there is no need to take a subsequence: $\|\psi_{k\tau} - \psi_{(k-1)\tau}\|_{C^{2,\alpha}} \leqslant C\tau(1+\tau)^{-k}$.

Now, by Proposition 4.2, we notice that unless ω_0 is itself Kähler–Einstein, the functional E_0 is strictly decreasing along the iteration. In particular, since ω_∞ is a fixed point of the iteration it must be Kähler–Einstein.

We now consider the case $\mu = 0$, for which we have the following system of equations,

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega + \frac{1}{\tau}\varphi_{k\tau}}, \quad k \in \mathbb{N}. \tag{29}$$

We may rewrite this as $\omega^n_{\psi_{k\tau}} = \omega^n_{\psi_{(k-1)\tau}} e^{-\frac{1}{\tau}\varphi_{(k-1)\tau} + \frac{1}{\tau}\varphi_{k\tau}}$ from which we have $\sup \varphi_{k\tau} \leqslant \sup \varphi_{(k-1)\tau} \leqslant \cdots \leqslant -\tau \inf f_\omega$. Therefore we have

$$\left\| e^{f_\omega + \frac{1}{\tau}\varphi_{k\tau}} \right\|_{L^\infty(M)} \leqslant e^{\operatorname{osc} f_\omega}, \quad \forall k \in \mathbb{N}.$$

Now, by Yau's work it follows that there exists an a priori L^∞ bound on $\psi_{k\tau}$, independently of k. We may now invoke the same arguments as before since once again the Ricci curvature is uniformly bounded from below (this time by $-1/\tau$), and again the uniform L^∞ estimate together with (29) implies that the volume ratios $\omega^n_{\psi_{k\tau}}/\omega^n$ are uniformly bounded. It follows that a uniform $C^{2,\alpha}$ estimate holds.

Combined with the monotonicity result it follows, as before, that a subsequence converges to a Kähler potential of a Kähler–Einstein metric. Moreover, since the Kähler–Einstein metric is unique (in each fixed Kähler class) [13] any converging subsequence will necessarily converge to the same limit point. This then implies that our original sequence converges to this limit. □

6. The Ricci iteration for positive first Chern class

We turn to the study of the iteration on Fano manifolds that, as noted in the Introduction, is our main motivation for introducing the Ricci iteration. Most of the applications described in Section 10 are for this class of manifolds.

We first introduce an operator that arises very naturally although it does not seem to have been defined previously in the literature. It exists and is well-defined by the Calabi–Yau Theorem [80].

Definition 6.1. Define the inverse Ricci operator $\operatorname{Ric}^{-1}: \mathcal{D}_{c_1} \to \mathcal{H}_{c_1}$ by letting $\operatorname{Ric}^{-1}\omega := \omega_\varphi$ with ω_φ the unique Kähler form in $\mathcal{H}_{c_1}$ satisfying $\operatorname{Ric}\omega_\varphi = \omega$. Similary denote higher order iterates of this operator by Ric^{-l} for each $l \in \mathbb{N}$. Let $\operatorname{Ric}^0 := \operatorname{Id}$ denote the identity operator.

There exists a generalization of this operator to any Kähler manifold (Definition 8.1). For another direction in which this operator may be generalized see Definition 9.2.

We then see that the dynamical system corresponding to the time one Ricci iteration on a Fano manifold with $\mu = 1$ is nothing but the evolution of iterates of the inverse Ricci operator,

$$\omega_l = \operatorname{Ric}^{-l} \omega_0.$$

The following result concerns the "allowed" time steps in the iteration for any Fano manifold and is well-known. Note that unlike in the previous, unobstructed, cases, the allowed range for the time step is restricted unless an analytic "semi-stability" condition holds.

Define

$$\tau_G(M) = \sup\bigl\{t\colon \text{(11) has a solution for each } \tau \in (0,t) \text{ and } \omega \in \mathcal{H}_{c_1}(G)\bigr\}. \tag{30}$$

By definition this is a holomorphic invariant. Recall also the definition of Tian's invariants [66,68]

$$\alpha_G(M) = \sup\left\{a\colon \sup_{\varphi \in \mathcal{H}_\omega(G)} \frac{1}{V} \int_M e^{-a(\varphi - \sup\varphi)} \omega^n < \infty\right\}, \tag{31}$$

$$\beta_G(M) = \sup\bigl\{b\colon \operatorname{Ric}\omega \geqslant b\omega,\ \omega \in \mathcal{H}_\Omega(G)\bigr\}, \tag{32}$$

where in (31) ω is any element of $\mathcal{H}_{c_1}(G)$.

Lemma 6.2. *Let (M, J) be a Fano manifold and let G be a compact subgroup of* $\operatorname{Aut}(M, \mathrm{J})$.

(i) *For any $\omega \in \mathcal{H}_{c_1}(G)$, the time τ Ricci iteration exists for all $k \in \mathbb{N}$ and all $\tau \in [0, \tau_G(M))$. One has $\frac{1}{1-\beta_G(M)} \geqslant \tau_G(M) \geqslant |\frac{1}{\max\{1-(n+1)\alpha_G(M)/n, 0\}}| > 1$.*
(ii) *Assume that E_0 is bounded from below on $\mathcal{H}^+_{c_1}(G)$. Then $\tau_G(M) = \infty$.*

Proof. (i) By the Calabi–Yau Theorem $\tau_G(M) \geqslant 1$. According to Tian [66] the path (12) exists for each $s \in [0, (n+1)\alpha_G(M)/n) \cap [0, 1]$ whenever $\omega \in \mathcal{H}_{c_1}(G)$. Note that $\tau = 1/(1-s)$ and that $\alpha_G(M) > 0$.

(ii) This is equivalent to a result of Bando and Mabuchi [7, Theorem 5.7]. □

Combined with Theorem 10.7 we therefore obtain the existence part of Theorem 3.3 for $\mu > 0$.

Corollary 6.3. *Let (M, J) be a Kähler–Einstein Fano manifold. Then for any $\omega \in \mathcal{H}_{c_1}$, the time τ Ricci iteration exists for all $k \in \mathbb{N}$ and all $\tau \in (0, \infty)$.*

Proof of Theorem 3.3 *($\mu > 0$).* We now turn to the proof of the remaining part of Theorem 3.3 with the exception of the statement concerning G-invariant initial data that will be proved in Section 10.4. We assume for simplicity, as in the statement of the theorem, that $\operatorname{aut}(M, \mathrm{J}) = \{0\}$. We set $\mu = 1$. The computations for other values of $\mu > 0$ will then follow by rescaling. We consider the following system of equations on $\mathcal{H}_\omega$ corresponding to the Ricci iteration on $\mathcal{H}_{c_1}$:

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega - \psi_{k\tau} + \frac{1}{\tau}\varphi_{k\tau}}, \quad k \in \mathbb{N}. \tag{33}$$

Let $G_{k\tau}$ be a Green function for $-\Delta_{k\tau} = -\Delta_{\bar\partial,\omega_{\varphi_{k\tau}}}$ satisfying $\int_M G_{k\tau}(\cdot, y)\omega^n_{\psi_{k\tau}}(y) = 0$. Set $A_{k\tau} = -\inf_{M\times M} G_{k\tau}$. Since $-n < \Delta_0\psi_{k\tau}$ and $n > \Delta_{k\tau}\psi_{k\tau}$ the Green formula gives

$$\psi_{k\tau}(x) - \frac{1}{V}\int_M \psi_{k\tau}\omega_0^n = -\frac{1}{V}\int_M G_0(x, y)\Delta_0\psi_{k\tau}(y)\omega_0^n(y) \leqslant nA_0,$$

$$\psi_{k\tau}(x) - \frac{1}{V}\int_M \psi_{k\tau}\omega^n_{\psi_{k\tau}} = -\frac{1}{V}\int_M G_{k\tau}(x, y)\Delta_{k\tau}\psi_{k\tau}(y)\omega^n_{\psi_{k\tau}}(y) \geqslant -nA_{k\tau}.$$

Hence

$$\operatorname{osc}\psi_{k\tau} \leqslant n(A_0 + A_{k\tau}) + I(\omega_0, \omega_{\psi_{k\tau}}). \tag{34}$$

Since by Theorem 10.7(ii) E_0 is proper on $\mathcal{H}_{c_1}$ in the sense of Tian, if $E_0(\omega, \cdot)$ is uniformly bounded from above on a subset of $\mathcal{H}_{c_1}$ so is $I(\omega, \cdot)$. By the monotonicity of E_0 along the iteration we conclude that $I(\omega, \omega_{\psi_{k\tau}})$ is uniformly bounded independently of k.

To get a uniform bound on the oscillation it remains to bound $A_{k\tau}$. This can be done using a special case of Bando and Mabuchi's Green's function estimate that we now state.

Theorem 6.4. *(See [7, Theorem 3.2].) Let (N, h) be a connected compact closed Riemannian manifold of nonnegative Ricci curvature. Let G_h denote the Green function of $d^{\star h}\circ d + d\circ d^{\star h}$ satisfying $\int_N G(x, y)\,dV_h(y) = 0$ for each $x \in N$ and let $A_h = -\inf_{M\times M} G_h$. Then*

$$A_h \leqslant c_n \frac{\operatorname{diam}(N, h)^2}{\operatorname{Vol}(N, h)},$$

with c_n depending only on n.

Now, along the iteration we have $\operatorname{Ric}\omega_k > (1 - 1/\tau)\omega_k > 0$. By Myers' Theorem [52, p. 245] then

$$\operatorname{diam}(M, \omega_k)^2 \leqslant \pi^2(2n - 1)/(1 - 1/\tau). \tag{35}$$

It follows that

$$\operatorname{osc}\psi_{k\tau} \leqslant C, \tag{36}$$

with C a positive constant independent of k.

We now rewrite (33) as

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega - (1-\frac{1}{\tau})\psi_{k\tau} - \frac{1}{\tau}\psi_{(k-1)\tau}}. \tag{37}$$

Since $\frac{1}{V}\int_M e^{f_\omega}\omega^n = 1$ (see (4)) it follows that the function $(1 - \frac{1}{\tau})\psi_{k\tau} + \frac{1}{\tau}\psi_{(k-1)\tau}$ changes signs. Since $\tau > 1$ then in particular we have

$$(1 - 1/\tau)\sup\psi_{k\tau} + 1/\tau \sup\psi_{(k-1)\tau} \geqslant 0, \tag{38}$$

$$(1 - 1/\tau)\inf\psi_{k\tau} + 1/\tau \inf\psi_{(k-1)\tau} \leqslant 0. \tag{39}$$

The first inequality implies that at least one of the numbers $\sup\psi_{k\tau}, \sup\psi_{(k-1)\tau}$ is nonnegative. Assume without loss of generality that $\sup\psi_{k\tau}\geqslant 0$. Then one of the following cases must arise: (i) $\sup\psi_{k\tau}\geqslant 0\geqslant\inf\psi_{k\tau}$ and $\sup\psi_{(k-1)\tau}\geqslant\inf\psi_{(k-1)\tau}\geqslant 0$, or (ii) $\sup\psi_{k\tau}\geqslant 0\geqslant\inf\psi_{k\tau}$ and $0\geqslant\sup\psi_{(k-1)\tau}\geqslant\inf\psi_{(k-1)\tau}$, or (iii) $\sup\psi_{k\tau}\geqslant\inf\psi_{k\tau}\geqslant 0$ and $\sup\psi_{(k-1)\tau}\geqslant 0\geqslant\inf\psi_{(k-1)\tau}$, or (iv) $\sup\psi_{k\tau}\geqslant\inf\psi_{k\tau}\geqslant 0$ and $0\geqslant\sup\psi_{(k-1)\tau}\geqslant\inf\psi_{(k-1)\tau}$, or (v) $\sup\psi_{k\tau}\geqslant 0\geqslant\inf\psi_{k\tau}$ and $\sup\psi_{(k-1)\tau}\geqslant 0\geqslant\inf\psi_{(k-1)\tau}$. In case (v) both of the functions change signs and so the oscillation estimate implies an L^∞ bound on each one of them. Let us now consider the other cases.

In case (i) we have $\|\psi_{k\tau}\|_{L^\infty}\leqslant C$ with C as in (36). By (39) we have that $\frac{1}{\tau}\inf\psi_{(k-1)\tau}\leqslant -(1-\frac{1}{\tau})\inf\psi_{k\tau}<(1-\frac{1}{\tau})C$, and combined with (36) we have $\|\psi_{(k-1)\tau}\|_{L^\infty}\leqslant\tau C$. In case (ii) we again have $\|\psi_{k\tau}\|_{L^\infty}\leqslant C$. By (38) then $\frac{1}{\tau}\sup\psi_{(k-1)\tau}\geqslant-(1-\frac{1}{\tau})\sup\psi_{k\tau}>-(1-\frac{1}{\tau})C$, and combined with (36) we have $\|\psi_{(k-1)\tau}\|_{L^\infty}\leqslant\tau C$. In case (iii) we have $\|\psi_{(k-1)\tau}\|_{L^\infty}\leqslant C$. By (39) then $(1-\frac{1}{\tau})\inf\psi_{k\tau}\leqslant-\frac{1}{\tau}\inf\psi_{(k-1)\tau}<\frac{1}{\tau}C$, and using (36) we have $\|\psi_{k\tau}\|_{L^\infty}\leqslant\frac{\tau}{\tau-1}C$.

Now consider case (iv). If $\sup\psi_{(k+1)\tau}\geqslant 0\geqslant\inf\psi_{(k+1)\tau}$ the same argument as in case (i) then implies that $\|\psi_{k\tau}\|_{L^\infty}\leqslant\tau C$. So we may assume that $0\geqslant\sup\psi_{(k+1)\tau}\geqslant\inf\psi_{(k+1)\tau}$. Furthermore, we may assume that for some k_0 and all $l\geqslant k_0$ the functions $\psi_{l\tau}$ do not change signs and that for the sequence $\{\psi_{l\tau}\}_{l\geqslant k_0}$ each nonnegative function is followed by a nonpositive function. Note $k_0\geqslant 2$ since the function $\psi_{1\tau}$ itself changes signs as can be seen from the equation $\omega^n_{\psi_{1\tau}}=\omega^n e^{f_\omega-(1-\frac{1}{\tau})\psi_{1\tau}}$ and the fact that $\frac{1}{V}\int_M e^{f_\omega}\omega^n=1$. We now argue inductively. Assume without loss of generality that $k_0=2$. One has $\|\psi_{1\tau}\|_{L^\infty}\leqslant C$. First, assume $\psi_{2\tau}$ is nonnegative. Then by case (iii) $\inf\psi_{2\tau}\leqslant\frac{1}{\tau-1}C$ and $\sup\psi_{2\tau}\leqslant(\frac{1}{\tau-1}+1)C$. Next, $\psi_{3\tau}$ is nonpositive and $(1-\frac{1}{\tau})\sup\psi_{3\tau}\geqslant-\frac{1}{\tau}\sup\psi_{2\tau}$. Hence $\sup\psi_{3\tau}\geqslant-\frac{1}{\tau-1}(\frac{1}{\tau-1}+1)C$ and $-\|\psi_{3\tau}\|_{L^\infty}=\inf\psi_{3\tau}\geqslant-(\frac{1}{\tau-1}(\frac{1}{\tau-1}+1)+1)C$. By induction it follows that $\|\psi_{k\tau}\|_{L^\infty}\leqslant C\sum_{j=0}^{k-1}\frac{1}{(\tau-1)^j}<\frac{\tau-1}{\tau-2}C$, when $\tau>2$. Second, if we assume instead that $\psi_{2\tau}$ is nonpositive then $(1-\frac{1}{\tau})\sup\psi_{2\tau}\geqslant-\frac{1}{\tau}\sup\psi_{1\tau}\geqslant-\frac{1}{\tau}C$ so $\sup\psi_{2\tau}\geqslant-\frac{1}{\tau-1}C$ and $\inf\psi_{2\tau}\geqslant-(\frac{1}{\tau-1}+1)C$. Now $\psi_{3\tau}$ is nonnegative and $(1-\frac{1}{\tau})\inf\psi_{3\tau}\leqslant-\frac{1}{\tau}\inf\psi_{2\tau}$, and so $\sup\psi_{3\tau}\leqslant(\frac{1}{\tau-1}(\frac{1}{\tau-1}+1)+1)C$. As before it follows that when $\tau>2$ we have $\|\psi_{k\tau}\|_{L^\infty}<\frac{\tau-1}{\tau-2}C$. In sum, when $\tau>2$ we have a uniform estimate on $\|\psi_{k\tau}\|_{L^\infty}$, independently of k.

Now when τ is not necessary larger than 2 we will need to normalize $\psi_{k\tau}$, namely put $\tilde\psi_{k\tau}:=\psi_{k\tau}-\frac{1}{V}\int_M\psi_{k\tau}\omega^n$. The estimate (36) implies that $\|\tilde\psi_{k\tau}\|_{L^\infty}\leqslant C$. We now want to show uniform higher derivative estimates for $\tilde\psi_{k\tau}$. The key now is to show that the volume forms $\omega^n_{\tilde\psi_{k\tau}}=\omega^n_{\psi_{k\tau}}$ are uniformly positive and bounded. By (37) that is equivalent to showing that $(1-\frac{1}{\tau})\psi_{k\tau}+\frac{1}{\tau}\psi_{(k-1)\tau}$ is uniformly bounded. We may assume that we are in case (iv) and that each nonnegative function in $\{\psi_{l\tau}\}_{l\geqslant 2}$ is followed by a nonpositive one. We may also assume $\psi_{k\tau}$ is positive and $\psi_{(k-1)\tau}$ is negative, that $B_l:=\|\psi_{l\tau}\|_{L^\infty}$ satisfies $\lim_{\mathbb{N}\ni l\to\infty}B_l=\infty$, and that B_{k-1} is much larger than C, say $B_{k-1}>2519((\tau+1)C+1)$. We have $-\inf\psi_{(k-1)\tau}=B_{k-1}$. Then $-\sup\psi_{(k-1)\tau}\geqslant B_{k-1}-C$. Now from (38) we have $(1-\frac{1}{\tau})\sup\psi_{k\tau}>-\frac{1}{\tau}\sup\psi_{(k-1)\tau}$, namely $\sup\psi_{k\tau}\geqslant\frac{1}{\tau-1}(B_{k-1}-C)$, and so $\inf\psi_{k\tau}\geqslant\frac{1}{\tau-1}(B_{k-1}-C)-C$. From (39) we have that $(1-\frac{1}{\tau})\inf\psi_{k\tau}<-\frac{1}{\tau}\inf\psi_{(k-1)\tau}$, thus $\inf\psi_{k\tau}<\frac{1}{\tau-1}B_{k-1}$, and so $\sup\psi_{k\tau}<\frac{1}{\tau-1}B_{k-1}+C$. Combining these inequalities we have shown that $\frac{1}{\tau-1}(B_{k-1}-C)-C\leqslant\psi_{k\tau}\leqslant\frac{1}{\tau-1}B_{k-1}+C$. Since $-B_{k-1}\leqslant\psi_{(k-1)\tau}\leqslant-B_{k-1}+C$ it follows that

$$-C\leqslant(1-1/\tau)\psi_{k\tau}+1/\tau\,\psi_{(k-1)\tau}\leqslant C,$$

as required. Thus we have shown that $1/C' < \omega^n_{\tilde\psi_{k\tau}}/\omega^n < C'$ for some constant $C' > 0$, independently of k.

Now the Laplacian estimate goes through for $\tilde\psi_{k\tau}$ just as in (28) since that inequality is not sensitive to changing $\psi_{k\tau}$ by a constant, and since, as before, we have a uniform lower bound for the Ricci curvature along the iteration. This gives a uniform estimate on $\operatorname{tr}_{\omega_{\tilde\psi_{k\tau}}}\omega = n - \Delta_{\omega_{\tilde\psi_{k\tau}}}\tilde\psi_{k\tau}$. Using the volume ratio estimate proven in the previous paragraph this implies a uniform estimate on $\operatorname{tr}_\omega \omega_{\tilde\psi_{k\tau}} = n + \Delta_\omega \tilde\psi_{k\tau}$ and subsequently by elliptic regularity also $C^{2,\alpha}$ and higher estimates, as explained in Section 5. We may thus extract from $\{\tilde\psi_{k\tau}\}$ a converging subsequence in $C^{2,\alpha}$ that converges to a smooth limit. Thanks to the monotonicity of E_0 it must converge to a Kähler potential for a Kähler–Einstein metric. Since such a metric is unique [7, Remark 9.3] the same argument as before gives the convergence of the full orbit $\{\omega_{\psi_{k\tau}}\}_{k\geqslant 0}$ of the Ricci iteration. $\square$

7. The Kähler–Ricci flow and the Ricci iteration for a general Kähler class

A natural question is whether on an arbitrary Kähler manifold one may define an iteration scheme generalizing the Ricci iteration. To answer this question of course one first needs to generalize the Ricci flow itself. In this section we recall one such possibility. We end with a conjecture regarding the convergence of this iteration.

A flow on the space of Kähler forms $\mathcal{H}_\Omega$ can be considered as an integral curve of a vector field on this space. A vector field χ on $\mathcal{H}_\Omega$ is an assignment $\omega \mapsto \chi_\omega \in C^\infty(M)/\mathbb{R}$. The Ricci flow describes the dynamics of minus the Ricci potential vector field $-f$. Recall that the vector field f is the assignment $\omega \mapsto f_\omega$ with f_ω defined by $\operatorname{Ric}\omega - \mu\omega = \sqrt{-1}\partial\bar\partial f_\omega$, $\mu \in \mathbb{R}$, where $\mu\Omega = c_1$.

The Ricci iteration in turn can be thought of as describing a piecewise linear trajectory in $\mathcal{H}_\Omega$ induced from the Ricci potential vector field $-f$ and approximating its integral curves.

Motivated by this one is naturally led to extend the definition of the Ricci flow (2) to an arbitrary Kähler manifold simply by defining the flow lines to be integral curves of minus the Ricci potential vector field $-f$ on $\mathcal{H}_\Omega$, with Ω an arbitrary Kähler class. Recall that the Ricci potential is defined in general by $\operatorname{Ric}\omega - H_\omega \operatorname{Ric}\omega = \sqrt{-1}\partial\bar\partial f_\omega$. The resulting flow equation can be written as

$$\begin{aligned} &\frac{\partial\omega(t)}{\partial t} = -\operatorname{Ric}\omega(t) + H_t \operatorname{Ric}\omega(t), \quad t \in \mathbb{R}_+,\\ &\omega(0) = \omega, \end{aligned} \tag{40}$$

for each t for which a solution exists in $\mathcal{H}_\Omega$ (throughout subscripts are meant to indicate that the relevant object corresponds to the metric indexed by that subscript). This flow, introduced by Guan, is part of the folklore in the field although it has not been much studied.[1]

Corresponding to this flow we introduce the following dynamical system on $\mathcal{H}_\Omega$ which generalizes Definition 3.1.

[1] It seems that Guan first considered this flow in unpublished work in the 90's (see references to [34]). After completing this article I also became aware, thanks to G. Székelyhidi, of a recent preprint [35] posted by Guan on his webpage in which this flow is studied. We hope that the elementary discussion in this section is still of some interest even though it was written before learning of [34,35]. For a different but related flow see [62].

Definition 7.1. Given a Kähler form $\omega \in \mathcal{H}_\Omega$ let the time τ Ricci iteration be the sequence of forms $\{\omega_{k\tau}\}_{k\geqslant 0}$, satisfying the equations

$$\omega_{k\tau} = \omega_{(k-1)\tau} + \tau H_{k\tau}\operatorname{Ric}\omega_{k\tau} - \tau\operatorname{Ric}\omega_{k\tau}, \quad k \in \mathbb{N},$$
$$\omega_0 = \omega, \tag{41}$$

for each $k \in \mathbb{N}$ for which a solution exists in $\mathcal{H}_\Omega$.

As in Section 3, setting $k = 1$ and varying τ defines a continuity path that is of independent interest.

An observation that goes back to Calabi characterizes the equilibrium state of the flow and the iteration.

Lemma 7.2. *(See [12, Theorem 1].) The Ricci form of a Kähler metric is a harmonic representative of c_1 with respect to the metric if and only if its scalar curvature is constant.*

Proof. One has

$$n\operatorname{Ric}\omega \wedge \omega^{n-1} = \operatorname{tr}_\omega \operatorname{Ric}\omega\omega^n = s(\omega)\omega^n.$$

Since ω is a harmonic representative of its class, we see that $s(\omega)$ is harmonic, i.e., constant, if and only if $\operatorname{Ric}\omega$ is. □

An infinitesimal automorphism $X \in \operatorname{aut}(M, \mathrm{J})$ naturally induces a vector field ψ^X on $\mathcal{H}_\Omega$ given by

$$\psi^X : \omega \mapsto \psi^X_\omega \in C^\infty(M)/\mathbb{R}, \quad \text{where } \mathcal{L}_X\omega = \sqrt{-1}\partial\bar\partial\psi^X_\omega. \tag{42}$$

Recall the following generalization of the notion of a constant scalar curvature Kähler metric, due to Guan. Alternatively it may be seen as a generalization of the notion of a Kähler–Ricci soliton to an arbitrary class.

Definition 7.3. (See [34].) Let $X \in \operatorname{aut}(M, \mathrm{J})$. A Kähler metric ω will be called a Kähler–Ricci soliton if it satisfies

$$\operatorname{Ric}\omega - H_\omega \operatorname{Ric}\omega = \mathcal{L}_X\omega. \tag{43}$$

Equivalently, if the vector field $\psi^X - f$ on $\mathcal{H}_{[\omega]}$ has a zero at ω.

We remark that the notion of a Ricci soliton for the case of definite first Chern class goes back at least to the work of Friedan [29,30].

Motivated by the results for Kähler–Einstein manifolds we believe the following conjecture should hold.

Conjecture 7.4. *Let (M, J) be a compact closed Kähler manifold, and assume that there exists a constant scalar curvature Kähler metric representing the class Ω. Then for any $\omega \in \mathcal{H}_\Omega$, the Kähler–Ricci flow* (40) *and the Ricci iteration* (41) *exist and converge in an appropriate sense to a constant scalar curvature metric.*

Similarly, we believe an analogous result should hold for Kähler–Ricci solitons (43) using the twisted constructions of Section 9.

8. Another flow and the inverse Ricci operator for a general Kähler class

Our purpose in this section is to explain why the inverse Ricci operator—that appeared as a very singular iterative construction for anticanonically polarized Fano manifolds—is in fact a special case of a more general construction on any Kähler manifold. This gives another application of our approach explained in the Introduction since it involves a discretization of another geometric flow equation.

To that end, given a Kähler form ω let us consider the flow equations

$$\begin{aligned} \frac{\partial \operatorname{Ric}\omega(t)}{\partial t} &= -\operatorname{Ric}\omega(t) + H_t \operatorname{Ric}\omega(t), \quad t \in \mathbb{R}_+, \\ \omega(0) &= \omega, \end{aligned} \tag{44}$$

for each t for which a solution exists.

The following brief and informal discussion comes to motivate this definition. Consider the case when the first Chern class is definite ($\mu \in \mathbb{R} \setminus \{0\}$ with $\Omega = \mu c_1$), or zero (Ω is arbitrary), and take $\omega \in \mathcal{H}_\Omega$. The evolution equation then becomes

$$\frac{\partial \operatorname{Ric}\omega(t)}{\partial t} = -\operatorname{Ric}\omega(t) + \mu\omega(t). \tag{45}$$

Assume momentarily that the flow preserves the Kähler class and that it exists on some time interval $[0, T]$. Then on the level of potentials it can be written as

$$-\Delta_t \dot{\varphi}_t = \log \frac{\omega^n_{\varphi_t}}{\omega^n} + \mu\varphi_t - f_\omega + a_t, \qquad \varphi_0 = \text{const}, \tag{46}$$

or as a Monge–Ampère equation

$$\omega^n_{\varphi_t} = \omega^n e^{f_\omega - \mu\varphi_t - \Delta_t \dot{\varphi}_t - a_t}, \tag{47}$$

with a_t a certain normalizing constant. Set $u := \Delta_t \dot{\varphi}_t$. A time derivative of (46) gives

$$\frac{du}{dt} = -u + \mu G_t u + b_t,$$

with b_t another normalizing constant. One should then first show that

$$\|u\|_{L^\infty(M\times[0,T])} < Ce^{-t},$$

when $\mu \leqslant 0$ and that $\|u\|_{L^\infty(M\times[0,T])} < Ce^{(\mu/\lambda_1(t)-1)t}$, when $\mu > 0$, where $\lambda_1(t)$ is the first nonzero eigenvalue of $-\Delta_t$. The constant C depends a priori on t. Going back to (47) one would then show an a priori estimate $\|\varphi_t\|_{L^\infty(M\times[0,T])} < C_1$, with C_1 depending only on ω, whenever $\mu \leqslant 0$. This then should imply a priori estimates on higher order derivatives under some assumptions. Finally, take a converging subsequence. Along this subsequence λ_1 is uniformly bounded away from zero. Going back to the exponential decay of u we would then apply uniform Schauder estimates to conclude that $\dot{\varphi}_t$ is uniformly decaying. It would then follows that the limit is a Kähler–Einstein metric. By uniqueness of the metric we would then argue, as earlier on, that the flow itself converges exponentially fast to a Kähler–Einstein metric. On the other hand, the case $\mu > 0$ would require more work, quite likely in the spirit of the corresponding result for the Ricci flow [14,76] (cf. also [2,7]).

Motivated by this discussion, we introduce the following dynamical system on $\mathcal{H}_\Omega$ obtained as the time one Euler method for this flow:

$$\operatorname{Ric}\omega_{k+1} = H_k \operatorname{Ric}\omega_k, \quad k \in \mathbb{N},$$
$$\omega_0 = \omega. \tag{48}$$

It can be thought of as describing the dynamics of a generalized inverse Ricci operator. This motivates the following definition, generalizing Definition 6.1 to an arbitrary Kähler manifold.

Definition 8.1. Define the inverse Ricci operator $\operatorname{Ric}_\Omega^{-1} : \mathcal{H}_\Omega \to \mathcal{H}_\Omega$ by letting $\operatorname{Ric}_\Omega^{-1}\omega := \omega_\varphi$ with ω_φ the unique Kähler form in $\mathcal{H}_\Omega$ satisfying $\operatorname{Ric}\omega_\varphi = H_\omega \operatorname{Ric}\omega$. Similary we denote higher order iterates of this operator by $\operatorname{Ric}_\Omega^{-l}$ for each $l \in \mathbb{N}$.

Calabi–Yau manifolds are singled-out as those manifolds for which this operator is a constant map. In general the dynamics of this operator seems intriguing. According to Lemma 7.2 the fixed points of this iteration are constant scalar curvature Kähler metrics. If indeed it were to converge to such a metric when one exists that would imply that the search for canonical metrics in Kähler geometry can be reduced to a repeated application of the Calabi–Yau Theorem coupled with harmonic projections (for the case of general extremal metrics see Remark 8.2 below, while for Kähler–Ricci solitons (Definition 7.3) one may try to use the twisted Ricci operator (see Definition 9.2) on the left-hand side of (48)).

We end this section with two remarks regarding continuity method paths induced from the flow (45), directly continuing the discussion in Section 3. First, it is interesting to note that discretizing this flow for time steps $\tau \in [0, 1]$ gives rise to the well-known continuity path of the Calabi–Yau Theorem (here $\mu = 0$) introduced by Calabi [13, (11)],[2]

$$\operatorname{Ric}\omega_{\varphi_\tau} - \operatorname{Ric}\omega = -\tau \operatorname{Ric}\omega \quad \Longrightarrow \quad e^{\tau f_\omega + d_\tau}\omega^n = \omega_{\varphi_\tau}^n, \tag{49}$$

with $d_\tau = -\log \frac{1}{V}\int_M e^{\tau f_\omega}\omega^n$, $\tau \in [0, 1]$.

The K-energy decreases along this path, however not monotonically in general. This is in contrast to the continuity path arising from the Ricci iteration and fits in well with what we would expect: the former arises from the Euler method (as opposed to the backwards Euler method) and so one does not expect monotonicity, nor convergence for large enough time steps.

Also, we remark that the backwards Euler method of the same evolution equation (45) yields the continuity path

$$\omega_{\varphi_\tau}^n = \omega^n e^{\frac{\tau}{1+\tau}(f_\omega - \mu\varphi)}, \quad \tau \geqslant 0, \tag{50}$$

[2] To obtain this path in the equivalent setting of the search for a Kähler metric with prescribed Ricci form, one considers the flow obtained by replacing the harmonic projection term in (44) by a prescribed form representing c_1.

that coincides in the case $\mu = 1$, after reparametrization, with the continuity path used by Demailly and Kollár alluded to earlier (Remark 3.5).

Remark 8.2. One may also define an analogous iteration whose fixed points are extremal metrics. Once again it arises from discretizing a flow

$$\frac{\partial \operatorname{Ric} \omega(t)}{\partial t} = -\operatorname{Ric} \omega(t) + \Pi_t \operatorname{Ric} \omega(t), \quad t \in \mathbb{R}_+,$$
$$\omega(0) = \omega, \tag{51}$$

where Π is a projection operator such that $\Pi_\omega \operatorname{Ric} \omega = \operatorname{Ric} \omega$ if and only if ω is extremal (see Simanca [62]). One then obtain an iteration given by

$$\operatorname{Ric} \omega_{k+1} = \Pi_k \operatorname{Ric} \omega_k, \quad k \in \mathbb{N},$$
$$\omega_0 = \omega. \tag{52}$$

9. The twisted Ricci iteration and a twisted inverse Ricci operator

When searching for canonical metrics, the presence of continuous symmetries has traditionally required additional analysis. Although the arguments are very similar to the previous sections, there are certain differences. In this section we merely introduce some of the dynamical constructions relevant to this case which we hope to further study elsewhere in the setting of convergence towards Kähler–Einstein metrics with continuous symmetries and Kähler–Ricci solitons (or multiplier Hermitian structures). We also state a monotonicity result that will be used in Section 10.6.

In the presence of holomorphic vector fields one oftentimes modifies the flow equation by a time-dependent family of automorphisms [21,76]. More generally, one may study the dynamics of a perturbation of the vector field $-f$ by an arbitrary vector field χ. Adapting the point of view of either Section 7 or Section 8 yields two ways to obtain discrete dynamics. The following definition corresponds to the former.

Definition 9.1. Given a vector field $\chi : \omega \mapsto \chi_\omega \in C^\infty(M)/\mathbb{R}$ on $\mathcal{H}_\Omega$ define the χ-twisted time τ Ricci iteration to be the sequence of forms $\{\omega_{k\tau}\}_{k\geqslant 0}$ satisfying the equations

$$\omega_{k\tau} = \omega_{(k-1)\tau} + \tau H_{k\tau} \operatorname{Ric} \omega_{k\tau} - \tau \operatorname{Ric} \omega_{k\tau} + \tau\sqrt{-1}\partial\bar\partial \chi_{\omega_{k\tau}}, \quad k \in \mathbb{N},$$
$$\omega_0 = \omega, \tag{53}$$

for each $k \in \mathbb{N}$ for which a solution exists in $\mathcal{H}_\Omega$.

The construction in Section 7 corresponds to the zero vector field. The case $\chi = \psi^X$, with X an infinitesimal automorphism, will be useful when studying convergence towards solitons.

When $\Omega = c_1$, $\tau = 1$ this iteration takes on a special form, giving a certain generalized inverse Ricci operator (cf. Definition 6.1).

Definition 9.2. Given a vector field $\chi : \omega \mapsto \chi_\omega \in C^\infty(M)/\mathbb{R}$ on $\mathcal{H}_{c_1}$ define the χ-twisted Ricci operator $\operatorname{Ric}_\chi : \mathcal{H}_{c_1} \to \mathcal{D}_{c_1}$ by letting $\operatorname{Ric}_\chi \omega := \operatorname{Ric} \omega - \sqrt{-1}\partial\bar\partial \chi_\omega$. Define the χ-twisted inverse Ricci operator $\operatorname{Ric}_\chi^{-1} : \mathcal{H}_{c_1} \to \mathcal{H}_{c_1}$ by letting $\operatorname{Ric}_\chi^{-1} \omega := \omega_\varphi$ whenever there exists a unique Kähler form ω_φ in $\mathcal{H}_{c_1}$ satisfying $\operatorname{Ric}_\chi \omega_\varphi = \omega$. Denote higher-order iterates of these operators by $\operatorname{Ric}_\chi^l$ for $l \in \mathbb{Z}$, setting $\operatorname{Ric}_\chi^0 := \mathrm{Id}$.

Recall that the Bakry–Émery Ricci form associated to a pair $(\omega, a) \in \mathcal{H}_\Omega \times C^\infty(M)$ is the form $\operatorname{Ric} \omega - \sqrt{-1}\partial\bar\partial a$, that can viewed as the Ricci form of the Kähler manifold (M, J) equipped with a Kähler form whose top exterior product equals $e^{2\pi a}\omega^n$ [4, (4b)]. The twisted Ricci operator is thus an assignment of a Bakry–Émery Ricci form to each Kähler form determined by a vector field on $\mathcal{H}_\Omega$. The simplest examples include the zero vector field and the Ricci potential vector field that yield the Ricci operator and the identity operator, respectively. Note that the fixed points of the twisted Ricci operator are for certain choices of χ the multiplier Hermitian structures defined by Mabuchi [46]. The twisted inverse Ricci operator is not defined for general χ, however it is for some geometrically significant vector fields. Assume that X belongs to a reductive Lie subalgebra of $\operatorname{aut}(M, \mathrm{J})$ and that the one-parameter subgroup $T_{\mathrm{J}X}$ generated by $\mathrm{J}X$ is a compact torus in $\operatorname{Aut}(M, \mathrm{J})$. When $\chi = \psi^X$ the operator $\operatorname{Ric}_{\psi^X}^{-1}$ restricted to $\mathcal{H}_{c_1}(T_{\mathrm{J}X})$ exists and is well-defined according to a theorem of Zhu [81]. More generally, this is still true when χ is a smooth function of ψ^X under some assumptions [46].

First, continuing the discussion of Section 3 (Remark 3.6) observe that when $\Omega = c_1$ and $\chi = \psi^X$ the continuity method path obtained by setting $k = 1$ and letting τ vary in the segment $[1, \infty)$ coincides with the Tian–Zhu continuity path [74, (1.4)]

$$\omega_{\varphi_s}^n = \omega^n e^{f_\omega - \psi^X_{\omega\varphi_s} - s\varphi_s}, \quad s \in [0, 1], \tag{54}$$

via the reparametrization $s = 1 - \frac{1}{\tau}$, discretizing the ψ^X-twisted Kähler–Ricci flow [76, (4.4)]

$$\begin{aligned} \frac{\partial \omega(t)}{\partial t} &= -\operatorname{Ric} \omega(t) + \omega(t) + \mathcal{L}_X \omega(t), \quad t \in \mathbb{R}_+, \\ \omega(0) &= \omega \in \mathcal{H}_{c_1}(T_{\mathrm{J}X}). \end{aligned} \tag{55}$$

In fact, more generally Mabuchi's continuity path [46, (5.1.4)] in the context of multiplier Hermitian structures is obtained in the same manner from (53) as a result of discretizing the corresponding twisted Kähler–Ricci flow.

We now discuss briefly the special case of Kähler–Ricci solitons. This is mainly done for the sake of concreteness since, due to the work of Mabuchi, the relevant computations go through also for general multiplier Hermitian structures.

In their study of Kähler–Ricci solitons on Fano manifolds Tian and Zhu introduced a twisted version of the functional E_0 [75]. To define it we first recall some relevant facts [31, §2.4], [46,74]. Given $X \in \operatorname{aut}(M, \mathrm{J})$, let $L_\omega^{\psi^X}$ denote the elliptic operator $L_\omega^{\psi^X}\phi := \Delta_\omega \phi + X\phi$. This operator is self-adjoint with respect to the $L^2(M, e^{\psi^X_\omega}\omega^n)$ inner product denoted by $\langle \cdot, \cdot \rangle_{\psi^X}$. The vector field ψ^X on $\mathcal{H}_{c_1}$ induces a vector field on the space of Kähler potentials (that we still denote by the same notation) by decreeing that $\frac{1}{V}\int_M e^{\psi^X_\omega}\omega^n = 1$ for each $\omega \in \mathcal{H}_{c_1}$. One then has $\psi^X_{\omega_\varphi} = \psi^X_\omega + X\varphi$ since $\frac{d}{dt}\frac{1}{V}\int_M e^{\psi^X_\omega + X(t\nu)}\omega^n_{t\nu} = \langle L^{\psi^X}_{\omega_{t\nu}}\nu, 1\rangle_{\psi^X} = 0$. Define a functional on $\mathcal{H}_{c_1}(T_{JX}) \times \mathcal{H}_{c_1}(T_{JX})$ by

$$E_0^{\psi^X}(\omega, \omega_\varphi) = \frac{1}{V}\int_{[0,1]} \big\langle \dot\varphi_t, L^X_{\omega_{\varphi_t}}\big(\psi^X_{\omega_{\varphi_t}} - f_{\omega_{\varphi_t}}\big)\big\rangle_{\psi^X}\, dt, \tag{56}$$

where $\omega_{\varphi_0} = \omega$ and $\omega_{\varphi_1} = \omega_\varphi$. This functional is well-defined independently of a choice of path and is exact. Its critical points are Kähler–Ricci solitons.

Lemma 9.3. *Assume that X belongs to a reductive Lie subalgebra of* aut(M, J) *and that the one-parameter subgroup T_{JX} generated by* JX *is a compact torus in* Aut(M, J). *The functional $E_0^{\psi^X}$ is monotonically decreasing along the ψ^X-twisted time τ Ricci iteration for each $\tau > 0$ for which the iteration exists.*

Proof. Let $\omega \in \mathcal{H}_{c_1}(T_{JX})$. Whenever the ψ^X-twisted time τ Ricci iteration exists the same is true for smaller time step iterations. Hence the continuity path

$$\omega^n_{\varphi_t} = \omega^n e^{f_\omega - \psi^X_{\omega_{\varphi_t}} + (\frac{1}{t}-1)\varphi_t}, \quad t \in [0, \tau], \tag{57}$$

exists. Differentiating Eq. (57) gives $(L^{\psi^X}_{\omega_{\varphi_t}} + 1 - \frac{1}{t})\dot\varphi_t = -\frac{1}{t^2}\varphi_t$. Hence one has

$$\begin{aligned} E_0^{\psi^X}(\omega_0, \omega_\tau) &= \frac{1}{V}\int_{[0,\tau]} \frac{1}{t}\big\langle \dot\varphi_t, L^{\psi^X}_{\omega_{\varphi_t}}\varphi_t\big\rangle_{\psi^X}\, dt \\ &= -\frac{1}{V}\int_{[0,\tau]} t\left\langle L^{\psi^X}_{\omega_{\varphi_t}}\dot\varphi_t, \left(L^{\psi^X}_{\omega_{\varphi_t}} + 1 - \frac{1}{t}\right)\dot\varphi_t\right\rangle_{\psi^X} dt \leqslant 0. \end{aligned}$$

When $\tau \leqslant 1$ the last inequality is a just a consequence of the ellipticity of $L^{\psi^X}_{(\cdot)}$. When $\tau > 1$ it follows since $L^{\psi^X}_{\omega_{\varphi_t}} + 1 - \frac{1}{t}$ is still elliptic [74, Lemma 2.2(ii)]. □

10. Some applications

In this section we describe several applications of the Ricci iteration and the inverse Ricci operator to some classical objects and problems in Kähler and conformal geometry.

10.1. The Moser–Trudinger–Onofri inequality on the Riemann sphere and its higher dimensional analogues

We recall some notions from [58] and explain how the results there on the Moser–Trudinger–Onofri inequality can be rephrased in terms of the inverse Ricci operator. This sheds new light on our discussion there and at the same time expands it (this was omitted from [58] for the sake of brevity).

Let $\omega_{\mathrm{FS},c}$ denote the Fubini–Study form of constant Ricci curvature c on (S^2, J), the Riemann sphere, given locally by

$$\omega_{\mathrm{FS},c} = \frac{\sqrt{-1}}{c\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}.$$

Here $V = \int_{S^2}\omega_{\mathrm{FS},c} = c_1([M])/c = 2/c$. For $c = 1/2\pi$ it is induced from restricting the Euclidean metric on $\mathbb{R}^3$ to the radius 1 sphere. Denote by $W^{1,2}(S^2)$ the space of functions on S^2 that are square-summable and so is their gradient (with respect to some Riemannian metric). The Moser–Trudinger–Onofri inequality states:

Theorem 10.1. *(See [47,50,77].) For $\omega = \omega_{\mathrm{FS},2/V}$ and any function φ on S^2 in $W^{1,2}(S^2)$ one has*

$$\frac{1}{V}\int_{S^2} e^{-\varphi+\frac{1}{V}\int_{S^2}\varphi\omega}\omega \leqslant e^{\frac{1}{V}\int_{S^2}\frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi}. \tag{58}$$

Equality holds if and only if ω_φ is the pull-back of ω by a Möbius transformation.

Several proofs of this classical result have been given in the literature, and we list here the ones we are aware of, chronologically: Onofri [50], Hong [37], Osgood–Phillips–Sarnak [51], Beckner [8], Carlen and Loss [15,16], Ghigi [33], Li and Zhu [41] (for more background we refer to Chang [17]). All of these proofs use crucially some symmetrization/rearrangement arguments that reduce the problem to a single dimension. Previously we gave a new proof of this inequality coming from Kähler geometry [58]. At the same time we also formulated an optimal (in a sense to be clarified below) extension of it to higher-dimensional Kähler–Einstein manifolds of positive scalar curvature, extending the work of Ding and Tian.

A function satisfies (58) if and only if

$$F_1(\omega,\omega_\varphi) = \frac{1}{V}\int_{S^2}\frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi - \frac{1}{V}\int_{S^2}\varphi\omega - \log\frac{1}{V}\int_{S^2}e^{-\varphi}\omega \geqslant 0.$$

This functional was studied already by Berger and Moser [9,47]. Moser, extending work of Trudinger, showed that $F_1(\omega,\omega_\varphi) \geqslant -C$. Then, Onofri showed that $C = 0$ and characterized the cases of equality.

Aubin first suggested a connection between the classical inequality (58) and Kähler–Einstein manifolds [2, (4)]. Following this, Ding [24] showed how to generalize the functional F_1 to higher-dimensional Fano manifolds—see Eq. (17)—using Aubin's functional J. Using this observation, and modifying the proof of a fundamental result of Bando and Mabuchi concerning the boundedness of the K-energy, Ding and Tian proved Theorem 10.1 for those functions that belong to the subspace $\mathcal{H}_\omega \subset W^{1,2}(S^2)$. We state both results and their corollary. The corollary is Ding–Tian's restricted generalization (this adjective is meant to emphasize that this corollary generalized a restricted version of the classical inequality) of the Moser–Trudinger–Onofri inequality to higher-dimensional Kähler–Einstein manifolds.

Theorem 10.2. *(See [7, Theorem A], [5, Theorem 1], [25, Theorem 1.1].) Let* (M, J, ω) *be a Kähler–Einstein Fano manifold. Then* $E_0(\omega, \omega_\varphi)$, $F_1(\omega, \omega_\varphi) \geqslant 0$ *for all* $\omega_\varphi \in \mathcal{H}_{c_1}$ *with equality if and only if* $\omega_\varphi = h^\star\omega$ *with* $h \in \mathrm{Aut}(M, \mathrm{J})_0$.

Corollary 10.3. *(See [25].) Let* (M, J, ω) *be a Kähler–Einstein Fano manifold with* $\omega \in \mathcal{H}_{c_1}$. *Then for each* $\varphi \in \mathcal{H}_\omega$ *holds*

$$\frac{1}{V}\int_M e^{-\varphi+\frac{1}{V}\int_M \varphi\omega^n}\omega^n \leqslant e^{J(\omega,\omega_\varphi)}. \tag{59}$$

Equality holds if and only if ω_φ *is the pull-back of* ω *by a holomorphic transformation.*

One should note that the subspace of Kähler potentials can be considered as a rather small "ball" sitting inside $C^\infty(M)(S^2) \subset W^{1,2}(S^2)$ since in general a large enough multiple of an element of $\mathcal{H}_\omega$ will no longer belong to $\mathcal{H}_\omega$. Following the work of Ding and Tian it remained an open problem how to extend their techniques and provide a complex-geometric proof of the Moser–Trudinger–Onofri inequality. The key hurdle in proving Theorem 10.1 is to extend the argument to the set $C^\infty(M) \setminus \mathcal{H}_\omega$ which a priori has no clear Kähler geometric significance as it represents indefinite forms rather than Kähler forms.

Alternatively, what is missing is a geometric interpretation of the Berger–Moser–Ding functional F_1. The following result is the key ingredient in our proof of (58) [58, Lemma 2.4].

Proposition 10.4. *Let* $\Omega = c_1$. *The following relation holds*

$$\big(\mathrm{Ric}^{-1}\big)^\star E_n = F_1, \quad \text{on } \mathcal{H}_{c_1} \times \mathcal{D}_{c_1}.$$

This provides a geometric interpretation for F_1. Indeed, the functional E_n is the potential for the Laplacian of the determinant of the Ricci tensor, considered as a 1-form on $\mathcal{H}_\Omega$, i.e., $dE_n(\omega, \omega_\varphi) = \Delta_{\omega_\varphi}(\frac{(\mathrm{Ric}\,\omega_\varphi)^n}{\omega_\varphi^n})\omega_\varphi^n$.

Hence, this result explains the geometric meaning the set $C^\infty(M) \setminus \mathcal{H}_\omega$ plays in the Moser–Trudinger–Onofri inequality. Namely, a function will satisfy this inequality if and only if it represents the Ricci form of a Kähler metric whose Ricci energy E_n is nonnegative with respect to a Kähler–Einstein metric. It now becomes important to understand the set $\mathcal{A}_n = \{\omega_\varphi \in \mathcal{H}_{c_1}\colon E_n(\omega, \omega_\varphi) \geqslant 0\}$, defined in Section 4. Naturally, we introduce the following definition.

Definition 10.5. Let the Moser–Trudinger–Onofri neighborhood of $\mathcal{H}_\omega$ be the subset

$$MTO_n = \big\{\varphi \in C^\infty(M)\colon \varphi \text{ satisfies (59) on the Fano manifold } (M, \mathrm{J}),\ \dim_{\mathbb{C}} M = n\big\}. \tag{60}$$

We are now in a position to state our generalization of Corollary 10.3 that is optimal in higher dimensions as well as some practical bounds. The result says that the Moser–Trudinger–Onofri inequality holds in higher dimensions on a canonically defined set MTO_n that is strictly larger than the space of Kähler potentials $\mathcal{H}_\omega$ and is geometrically related to Ricci curvature.

Theorem 10.6. *Let* (M, J, ω) *be a Kähler–Einstein Fano manifold.*

(i) *The generalized Moser–Trudinger–Onofri inequality* (59) *holds precisely on the set* $MTO_n = \mathrm{Ric}(\mathcal{A}_n)$. *Furthermore,* $\mathcal{H}_{c_1} \subsetneqq MTO_n \subseteq \mathcal{D}_{c_1}$.

(ii) *Define the sets* $\mathcal{B}_k := \{\omega_\varphi \in \mathcal{H}_{c_1}\colon\ I_k(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi) \geqslant 0\}$. *Then one has* $\mathcal{H}_{c_1}^+ \subsetneqq \mathcal{B}_k \subseteq \mathcal{A}_k$.

(iii) *One has* $\mathcal{A}_1 = \mathcal{B}_1 = \mathcal{H}_{c_1}$, $\mathcal{B}_2 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1}\colon\ \mathrm{Ric}\,\omega_\varphi + 2\omega_\varphi \geqslant 0\}$, $\mathcal{B}_3 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1}\colon\ \mathrm{Ric}\,\omega_\varphi + \omega_\varphi \geqslant 0\}$, *and for each k one may readily obtain an explicit bound on the set* $\mathcal{B}_k$, *hence on* $\mathcal{A}_k$, *in terms of a lower bound on the Ricci curvature using* (16). *In particular there exist* $c_n > 0$ *depending only on n such that,*

$$MTO_n \supseteq \left\{\varphi \in C^\infty(M)\colon\ \omega_\varphi \geqslant -c_n\, \mathrm{Ric}^{-1}\omega_\varphi\right\},$$

and, e.g., $c_1 = \infty$, $c_2 \geqslant 2$, $c_3 \geqslant 1$.

Proof. The first statement of (i) follows from Proposition 10.4 while the second part will follow from (ii). The statement (ii) itself follows from Proposition 4.1 and the fact that I_k is nonnegative on $\mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$. To see that $\mathcal{B}_k$ contains also points outside $\mathcal{H}_{c_1}^+$ as well as to prove (iii) one performs a direct computation in terms of a lower bound on the eigenvalues of $\mathrm{Ric}\,\omega_\varphi$ with respect to ω_φ. For example, since I_1 is nonnegative on $\mathcal{D}_{c_1} \times \mathcal{D}_{c_1}$ we obtain $MTO_1 = \mathcal{D}_{c_1}$, since $I_2(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi) \geqslant 0$ when $\mathrm{Ric}\,\omega_\varphi \geqslant -2\omega_\varphi$ we obtain $\{\varphi \in C^\infty(M)\colon\ \omega_\varphi \geqslant -2\,\mathrm{Ric}^{-1}\omega_\varphi\} \subseteq MTO_2$, and similarly $\{\varphi \in C^\infty(M)\colon\ \omega_\varphi \geqslant -\mathrm{Ric}^{-1}\omega_\varphi\} \subseteq MTO_3$. $\square$

As a corollary we are now able to provide a complex-geometric proof of the classical Moser–Trudinger–Onofri inequality.

Proof of Theorem 10.1. Observe that $MTO_1 = \mathrm{Ric}(\mathcal{A}_1) = \mathrm{Ric}(\mathcal{H}_{c_1}) = \mathcal{D}_{c_1}$. The last equality requires solving the equation[3] $\mathrm{Ric}\,\omega_\varphi = \omega_\psi$ for φ, equivalently Poisson's equation $\Delta_\omega\varphi = e^{f_\omega - \psi} - 1$. $\square$

10.2. An analytic characterization of Kähler–Einstein manifolds and an analytic criterion for almost-Kähler–Einstein manifolds

In the first part of this subsection we explain how the inverse Ricci operator can be used to solve two problems concerning energy functionals on the space of Kähler forms. We hope this sheds new light on the solution of these problems that we gave previously [58].

Chen and Tian's generalization of Mabuchi's Kähler energy, E_0, and of Bando and Mabuchi's Ricci energy, E_n, to a family of functionals $\{E_k\}_{k=0}^n$ (see Section 4 for definitions) naturally raised the question of whether Tian's analytic characterization of Kähler–Einstein manifolds in terms of E_0 generalizes to these functionals. In addition it raised the question whether Bando and Mabuchi's criterion for almost-Kähler–Einstein manifolds in terms of E_0 generalizes to these functionals. These questions were also independently raised by Chen [19, p. 37], [20, §1.3]. We now recall both of these fundamental results and explain how to generalize them. This provides an answer to these questions. It shows that the answer is both "yes" and "no": these criteria

[3] This is the classical $n = 1$ version of the Calabi–Yau Theorem whose proof goes back at least to Wallach and Warner [78].

extend to the other functionals $\{E_k\}$, however they fail to extend in an identical manner. The subtlety comes from the appearance of the inverse Ricci operator as we will see below.

Theorem 10.7. *Let* (M, J) *be a Fano manifold.*

(i) [5,7,25] *If either* F_1 *or* E_0 *is bounded from below on* $\mathcal{H}_{c_1}$ *then for each* $\epsilon > 0$ *there exists a Kähler metric* $\omega_\epsilon \in \mathcal{H}_{c_1}$ *satisfying* $\operatorname{Ric}\omega_\epsilon > (1-\epsilon)\omega_\epsilon$.
(ii) [70,71,73] *Assume that* $\operatorname{Aut}(M, \mathrm{J})$ *is finite.*[4] *Then the properness of* F_1 *(or* E_0*) on* $\mathcal{H}_{c_1}$ *is equivalent to the existence of a Kähler–Einstein metric.*

Our strategy in extending these results to the functionals $\{E_k\}_{k=0}^n$ was: (a) first prove a new formula that expresses E_k as the sum of E_0 and another new exact energy functional $(\omega, \omega_\varphi) \mapsto I_k(\omega_\varphi, \operatorname{Ric}\omega_\varphi) - I_k(\omega, \operatorname{Ric}\omega)$ (Proposition 4.1) and use it to show

$$\begin{aligned} F_1 \text{ bounded from below on } \mathcal{H}_{c_1} \quad &\Rightarrow \quad E_0 \text{ bounded from below on } \mathcal{H}_{c_1} \\ &\Rightarrow \quad E_1 \text{ bounded from below on } \mathcal{H}_{c_1} \\ &\Rightarrow \quad E_2 \text{ bounded from below on } \mathcal{H}_{c_1}^+ \\ &\quad \vdots \\ &\Rightarrow \quad E_n \text{ bounded from below on } \mathcal{H}_{c_1}^+. \end{aligned}$$

Note that here we use the fact that $I_1(\omega, \cdot)$ is bounded from below on $\mathcal{D}_{c_1}$ while $I_k(\omega, \cdot)$, $k \geqslant 2$, are only bounded from below on $\mathcal{H}_{c_1}$.

(b) Next use Proposition 10.4 to conclude:

$$E_n \text{ bounded from below on } \mathcal{H}_{c_1}^+ \quad \Rightarrow \quad F_1 \text{ bounded from below on } \mathcal{H}_{c_1}.$$

(c) Finally, some additional arguments were needed in order to prove that the properness of E_n on $\mathcal{H}_{c_1}^+$ implies the existence of a Kähler–Einstein metric.

We can now state the extension of the theorems of Bando–Mabuchi and Tian to the energy functionals $\{E_k\}$. The case $k = 1$ was proven before by Chen–Li–Wang and Song–Weinkove in a different manner [20,64].

Theorem 10.8. *Let* (M, J) *be a Fano manifold.*

(i) *If either* F_1 *or* E_k *(for some* $k \in \{0, \ldots, n\}$*) is bounded from below on* $\mathcal{H}_{c_1}^+$ *then for each* $\epsilon > 0$ *there exists a Kähler metric* $\omega_\epsilon \in \mathcal{H}_{c_1}$ *satisfying* $\operatorname{Ric}\omega_\epsilon > (1-\epsilon)\omega_\epsilon$.
(ii) *Assume that* $\operatorname{Aut}(M, \mathrm{J})$ *is finite.*[5] *Then the properness of* F_1 *or of* E_k *(for some* $k \in \{0, \ldots, n\}$*) on* $\mathcal{H}_{c_1}^+$ *is equivalent to the existence of a Kähler–Einstein metric.*

It is important to note that the appearance of the inverse Ricci operator in step (b) was a crucial ingredient. The discrepancy between the behavior of F_1, E_0, E_1 and that of the functionals

[4] In the general case a slightly more involved statement holds (see [58] for details).
[5] See footnote to Theorem 10.7.

$E_2, \ldots, E_n$ can be explained using the time one Ricci iteration: the first three are unconditionally monotone along the iteration, while for the latter $n-1$ this is true if and only if one assumes that the initial point lies in $\mathcal{B}_k$, and $\mathcal{H}_{c_1}^{+} \subsetneqq \mathcal{B}_k \subseteq \mathcal{A}_k \subsetneqq \mathcal{H}_{c_1}$ (Proposition 4.2). Furthermore, along the first step of the iteration the functionals E_k may increase by an arbitrary amount! To be precise, for any Fano manifold (Kähler–Einstein or not) we have the following result [58][6]: The Ricci energy E_n is bounded from below on $\mathcal{H}_{c_1}$ if and only if $n=1$. We conclude that the assumption in Theorem 10.8(ii) is essential and cannot be weakened from $\mathcal{H}_{c_1}^{+}$ to $\mathcal{H}_{c_1}$. This explains our remark earlier on the subtlety present when $k \geqslant 2$.

Previously, several authors (for references see [58]) have proven the four implications on the equivalence of the boundedness from below of F_1, E_0 and E_1. Often they appealed to results on the Ricci flow. This suggests that in this context the Ricci iteration rather than the flow is perhaps more suited.

Remark 10.9. In light of the discussion above it would be interesting to know whether there exist initial conditions in $\mathcal{H}_{c_1}$ for which E_k, $k \geqslant 2$, increases by an arbitrary amount along the Ricci flow restricted to the time interval [0, 1].

Remark 10.10. For a relation between the functionals E_k and GIT (geometric invariant theory) in light of Theorem 10.8(ii) see [57, §§4.4.5].

10.3. A new Moser–Trudinger–Onofri inequality on the Riemann sphere and a family of energy functionals

In the first part of this subsection we prove results that improve on the restricted generalized Moser–Trudinger–Onofri inequality (Corollary 10.3) in a different direction than that explored in Section 10.1 (Theorem 10.6). Namely, we show that the inequality holds even when one adds certain negative terms to the exponent on the right-hand side. This is done by expressing the excess in the inequality in geometric terms, namely in terms of the inverse Ricci operator. This is different from Tian's approach to a strenghtened inequality on Kähler–Einstein manifolds [71, Theorem 6.21] and in particular involves sharp constants and a precise characterization of the case of equality. In the future we hope to address the relation between these two approaches. In the second part we introduce a family of energy functionals and explain their relation to the improved inequality.

We now state the main result of this subsection.

Theorem 10.11. *Let (M, J, ω) be a Fano Kähler–Einstein manifold. Then for each $\varphi \in MTO_n$ holds*

$$\frac{1}{V}\int_M e^{-(\varphi-\frac{1}{V}\int_M\varphi)}\omega^n \leqslant e^{J(\omega,\omega_\varphi)-\sum_{j=1}^{\infty} J(\mathrm{Ric}^{-j}\omega_\varphi,\mathrm{Ric}^{-j+1}\omega_\varphi)}. \tag{61}$$

[6] We believe that the same result should hold for E_k for each $2 \leqslant k \leqslant n$.

Each of the terms in the sum is nonnegative precisely when $\omega_\varphi \in \mathrm{Ric}(\mathcal{B}_n) \supsetneqq \mathcal{H}_{c_1}$.[7] *Equality holds if and only if ω_φ is the pull-back of ω by a holomorphic transformation.*

Recall that Theorem 10.6 strengthened the restricted generalized Moser–Trudinger–Onofri inequality (Corollary 10.3) by optimally enlarging the set of functions on which it holds to a set strictly containing $\mathcal{H}_\omega$. Theorem 10.11 further strengthens Theorem 10.6: it shows that the sets MTO_n (see (60)) are characterized by an inequality stronger than (59). A version of this result holds also under the assumption that the K-energy is bounded from below. For simplicity we only state the result in the Kähler–Einstein setting.

Proof. By definition $F_1(\omega, \omega_\varphi) \geqslant 0$ for each $\varphi \in MTO_n$. Observe that by Theorem 10.6(i) it follows that Ric^{-1} preserves MTO_n. Therefore for each $l \in \mathbb{N}$,

$$F_1\big(\omega, \mathrm{Ric}^{-l}\,\omega_\varphi\big) \geqslant 0, \quad \forall \varphi \in MTO_n.$$

By exactness of F_1 we obtain

$$F_1(\omega, \omega_\varphi) + F_1\big(\omega_\varphi, \mathrm{Ric}^{-1}\,\omega_\varphi\big) + \cdots + F_1\big(\mathrm{Ric}^{-l+1}\,\omega_\varphi, \mathrm{Ric}^{-l}\,\omega_\varphi\big) \geqslant 0,$$

that is,

$$F_1(\omega, \omega_\varphi) \geqslant \sum_{j=1}^{l} F_1\big(\mathrm{Ric}^{-j}\,\omega_\varphi, \mathrm{Ric}^{-j+1}\,\omega_\varphi\big). \tag{62}$$

Now, using (14)–(15) and (24) one has

$$F_1(\omega, \omega_\varphi) = J(\omega, \omega_\varphi) - \frac{1}{V}\int_M \varphi\omega^n - \log\frac{1}{V}\int_M e^{f_\omega - \varphi}\omega^n. \tag{63}$$

It follows that for any $\alpha \in \mathcal{H}_{c_1}$ holds

$$F_1(\alpha, \mathrm{Ric}\,\alpha) = J(\alpha, \mathrm{Ric}\,\alpha) - \frac{1}{V}\int_M f_\alpha \alpha^n. \tag{64}$$

Combining (62)–(64), and letting l tend to infinity, yields

$$\frac{1}{V}\int_M e^{-(\varphi - \frac{1}{V}\int_M \varphi)}\omega^n \leqslant e^{J(\omega,\omega_\varphi) - \sum_{j=1}^{\infty} J(\mathrm{Ric}^{-j}\,\omega_\varphi, \mathrm{Ric}^{-j+1}\,\omega_\varphi)} \cdot e^{\sum_{j=1}^{\infty} \frac{1}{V}\int_M f^{(j)}_{\omega_\varphi} (\mathrm{Ric}^{-j}\,\omega_\varphi)^n}, \tag{65}$$

[7] More precisely, each of the terms with $j \geqslant 2$ is nonnegative for all $\varphi \in MTO_n$, while the term with $j = 1$ is nonnegative precisely when $\omega_\varphi \in \mathrm{Ric}(\mathcal{B}_n)$. Therefore, after possibly omitting the first term in the sum, (61) is an improvement over (59) for all $\varphi \in MTO_n$ and not just for the subset $\mathrm{Ric}(\mathcal{B}_n) \subseteq MTO_n$ (both strictly contain $\mathcal{H}_{c_1}$).

where $f^{(j)}$ is the push-forward of the vector field f under Ric^{-j}. Since by (4) the second term in (64) is nonnegative the desired inequality now follows from (65).

The last statement follows from the fact that $I_n = J$, Proposition 10.4, and the definition of $\mathcal{B}_n$ (21). □

In the case of the Riemann sphere S^2, Theorem 10.6(iii) implies $\mathrm{Ric}(\mathcal{B}_1) = \mathrm{Ric}(\mathcal{A}_1) = MTO_1 = C^\infty(M)(S^2)$. Therefore we have the following improvement of the classical Moser–Trudinger–Onofri inequality (Theorem 10.1). For notation we refer to Section 10.1.

Corollary 10.12. *Denote by $(S^2, \mathrm{J}, \omega = \omega_{\mathrm{FS},2/V})$ a round sphere of volume V. For any function φ on S^2 in $W^{1,2}(S^2)$ one has*

$$\frac{1}{V}\int_{S^2} e^{-\varphi+\frac{1}{V}\int_{S^2}\varphi\omega}\omega \leqslant e^{\frac{1}{V}\int_{S^2}\frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi-\sum_{j=1}^{\infty}J(\mathrm{Ric}^{(-j)}\omega_\varphi,\mathrm{Ric}^{(-j+1)}\omega_\varphi)}. \tag{66}$$

Each of the terms in the sum is nonnegative, and equals zero if and only if ω_φ is obtained from ω by a Möbius transformation. This also characterizes when equality holds in (66).

Note that the smoothing property of the iteration (see page 1532) implies that the extra terms in the sum are meaningful under the assumption $\varphi \in W^{1,2}(S^2)$. In essence, Corollary 10.12 gives the lower order terms for the classical Moser–Trudinger–Onofri inequality. The first term that was classically known $\frac{1}{V}\int_{S^2}\frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi = J(\omega,\omega_\varphi)$ is a $W^{1,2}$-seminorm term while the lower order terms $J(\mathrm{Ric}^{(-j)}\omega_\varphi, \mathrm{Ric}^{(-j+1)}\omega_\varphi)$ are essentially $W^{-(2j-1),2}$-seminorm terms. It would be very interesting to understand whether more generally other critical Sobolev inequalities admit such lower order term corrections.

Motivated by Proposition 10.4 we define the following family of energy functionals. For each $k \in \{0,\dots,n\}$ and $l \in \mathbb{N}\cup\{0\}$ let $E_{k,l}$ denote the pull-back by Ric^{-l} of the Chen–Tian functional E_k (see (18)). That is

$$E_{k,l}(\omega,\omega_\varphi) = E_k\big(\mathrm{Ric}^{-l}\omega, \mathrm{Ric}^{-l}\omega_\varphi\big). \tag{67}$$

For example, $E_{n,1} = F_1$, and

$$E_{k,1}(\omega,\omega_\varphi) = F_1(\omega,\omega_\varphi) - (J-I_k)\big(\mathrm{Ric}^{-1}\omega_\varphi,\omega_\varphi\big) + (J-I_k)\big(\mathrm{Ric}^{-1}\omega,\omega\big), \quad k = 0,\dots,n. \tag{68}$$

In light of this, Theorem 10.11 is seen to be a corollary of the following inequality:

$$E_{n,l+1}(\omega,\,\cdot\,) = \big(\mathrm{Ric}^{-l}\big)^\star F_1\big|_{\{\omega\}\times MTO_n} \geqslant 0, \quad \forall l \in \mathbb{N}. \tag{69}$$

Finally, we remark that in light of Definition 8.1 and Proposition 10.4 one may also extend the definition of Ding's functional to an arbitrary Kähler manifold and class.

10.4. Construction of Nadel-type obstruction sheaves

Up until this point we have scarcely concerned ourselves with the behavior of the various dynamical systems constructed in the absence of a fixed point. In this subsection we show that in this situation, and in the Fano setting, the Ricci iteration will produce Nadel-type obstruction sheaves, similarly to the continuity method and the Ricci flow. The basic references for this subsection are Demailly–Kollár [23] and Nadel [48].

Let $PSH(M,\mathrm{J},\omega)\subseteq L^1_{\mathrm{loc}}(M)$ denote the set of ω-plurisubharmonic functions. For $\varphi\in PSH(M,\mathrm{J},\omega)$ define the multiplier ideal sheaf associated to φ as the sheaf $\mathcal{I}(\varphi)$ defined for each open set $U\subseteq M$ by local sections

$$\mathcal{I}(\varphi)(U)=\bigl\{h\in\mathcal{O}_M(U)\colon\ |h|^2e^{-\varphi}\in L^1_{\mathrm{loc}}(M)\bigr\}. \tag{70}$$

Such sheaves are coherent. Such a sheaf is called proper if it is neither zero nor the structure sheaf $\mathcal{O}_M$.

Nadel showed that in the absence of a Kähler–Einstein metric the continuity method (12) will produce a certain family of multiplier ideal sheaves. Phong, Šešum and Sturm showed that certain multiplier ideal sheaves can be obtained also from the Ricci flow

$$\omega^n_{\varphi_t}=\omega^n e^{f_\omega-\varphi_t+\dot\varphi_t},\qquad \varphi(0)=\mathrm{const}. \tag{71}$$

Denote by $\lfloor x\rfloor$ the largest integer not larger than x.

Theorem 10.13. *(See [53].) Let (M,J) be a Fano manifold not admitting a Kähler–Einstein metric. Let $\gamma\in(1,\infty)$ and let $\omega\in\mathcal{H}_{c_1}$. Then there exists an initial condition $\varphi(0)$ and a subsequence $\{\varphi_{t_j}\}_{j\geqslant 0}$ of solutions of* (71) *such that* $\lim_{j\to\infty}\varphi_{t_j}=\varphi_\infty\in PSH(M,\mathrm{J},\omega)$ *and $\mathcal{I}(\gamma\varphi_\infty)$ is a proper multiplier ideal sheaf satisfying*

$$H^r\bigl(M,\mathcal{I}(\gamma\varphi_\infty)\otimes K_M^{-\lfloor\gamma\rfloor}\bigr)=0,\quad \forall r\geqslant 1. \tag{72}$$

Their proof relies on some of Perelman's estimates for the Ricci flow as well as the following theorem of Kołodziej.

Theorem 10.14. *(See [40].) Let $F\in L^p(M,\omega)$, $p>1$, be a positive continuous function with $\frac{1}{V}\int_M F\omega^n=1$. There exists a bounded solution φ to the equation $\omega^n_\varphi=F\omega^n$ on M which satisfies* $\operatorname{osc}\varphi\leqslant C$ *with C depending only on $\|F\|_{L^p(M,\omega)}$, p and (M,ω).*

Let $\tau=1/\mu=1$. The following result is a discrete analogue of Theorem 10.13. Its very simple proof compared to that of the analogous result for the Ricci flow is one of our main motivations for including it here. Moreover, the sheaves produced in this way are essentially computable (see the next subsection).

Theorem 10.15. *Let (M,J) be a Fano manifold not admitting a Kähler–Einstein metric. Let $\gamma\in(1,\infty)$ and let $\omega\in\mathcal{H}_{c_1}$. Then there exists a subsequence $\{\psi_{j_k}\}_{k\geqslant 1}$ of solutions of* (5) *such that* $\lim_{k\to\infty}\psi_{j_k}=\psi_\infty\in PSH(M,\mathrm{J},\omega)$ *and $\mathcal{I}(\gamma\psi_\infty)$ is a proper multiplier ideal sheaf satisfying* (72).

Proof. Indeed, since the iteration takes the form

$$\omega_{\psi_{l+1}}^n = \omega^n e^{f_\omega - \psi_l}, \quad l \in \mathbb{N}, \tag{73}$$

Theorem 10.14 can be directly applied (observe that from (4) an estimate on $\operatorname{osc}\psi_l$ implies one on $\|\psi_l\|_{L^\infty(M)}$) to construct sheaves with $\gamma > 1$, making use of Proposition 4.2 (for more details see [59, §2 (iv)]). □

Remark 10.16. One should also be able to construct multiplier ideal sheaves for the Ricci iteration with other time steps and for exponents in the range $(n/(n+1), 1)$ much the same as the continuity method sheaves constructed by Nadel as well as the analogous ones constructed in [59] for the Ricci flow (we hope to discuss this in more detail elsewhere; note that the latter construction strenghthened Theorem 10.13).

We now complete the proof of Theorem 3.3.

Proof of Theorem 3.3 *(concerning initial data in $\mathcal{H}_{c_1}(G)$).* After Sections 5 and 6 it remains only to prove the last statement of Theorem 3.3. Assume without loss of generality as in Section 6 that $\mu = 1$. Assume that for some compact subgroup $G \subset \operatorname{Aut}(M, \mathrm{J})$ one has $\alpha_G(M) > 1$ and let $\omega \in \mathcal{H}_\Omega(G)$ (we are assuming as in Theorem 3.3 that $\operatorname{Aut}(M, \mathrm{J})$ is finite, however the statement on G-invariant initial data still holds for general compact G with the proof given here if we replace "converges" by "subconverges"). We consider the time one Ricci iteration $\{\omega_{\psi_k} := \operatorname{Ric}^{-k}\omega\}_{k\geqslant 0}$, satisfying Eqs. (73). From (4) it follows that $\sup\psi_k \geqslant 0$. Fix p such that $p \in (1, \alpha_G(M))$. Then $C := \sup_{\varphi\in\mathcal{H}_\omega(G)} \int_M e^{-p(\varphi - \sup\varphi)}\omega^n < \infty$, and

$$\int_M \left(e^{f_\omega - \psi_k}\right)^p \omega^n \leqslant e^{p\sup f_\omega} \int_M e^{-p\psi_k}\omega^n \leqslant e^{p\sup f_\omega} \int_M e^{-p(\psi_k - \sup\psi_k)}\omega^n \leqslant C e^{p\sup f_\omega}.$$

It follows from Theorem 10.14 that $\operatorname{osc}\psi_k$ and hence $\|\psi_k\|_{L^\infty}$ are uniformly bounded, independently of k. The same arguments as in Sections 5 and 6 now imply uniform higher derivative estimates on ψ_k and convergence of the Ricci iteration to the Kähler–Einstein metric. □

Call Fano manifolds that admit a compact group $G \subset \operatorname{Aut}(M, \mathrm{J})$ for which $\alpha_G(M) > 1$ exceptional. Such manifolds are expected to exist although they have not yet been systematically studied. Recent work of Cheltsov and Heier indicates that $\mathbb{P}^2 \# 4\overline{\mathbb{P}^2}$ is such a manifold with $\alpha_G(M) = 2$ for G equal to the automorphism group.

10.5. Relation to balanced metrics

In this paragraph we describe an immediate corollary of the work of Donaldson. It gives with no further work an algorithm for computing Kähler–Einstein metrics using balanced metrics: Given a polarized Hodge manifold (X, L) and a volume form ν Donaldson [28] constructs a sequence of pull-backs of Fubini–Study metrics in $\mathcal{H}_{c_1(L)}$ induced from Kodaira embeddings that converge to a solution of the Calabi–Yau equation $\omega_\varphi^n = \nu$. Since in the Fano case our time one Ricci iteration consists precisely of solving a Calabi–Yau equation at each iteration we see that repeated application of Donaldson's constructions approximates the Ricci iteration and in this sense provides a quantization of the Ricci flow.

Another consequence is the possibility to numerically construct Nadel-type sheaves on Fano manifolds admitting no Kähler–Einstein metrics, by Section 10.4.

Note that more generally one may approximate in the same manner the orbits of the iteration given by the inverse Ricci operator (Definition 8.1) on an arbitrary Kähler manifold with $\Omega \in H^2(M, \mathbb{Z})$.

Finally, it would be interesting to find more relations between discretizations of other geometric flows and iteration schemes involving Bergman metrics.

10.6. A question of Nadel

As explained in Section 2 one of the original motivations for our work was a question raised by Nadel [49]: Given $\omega \in \mathcal{H}_{c_1}$ define a sequence of metrics $\omega, \operatorname{Ric}\omega, \operatorname{Ric}(\operatorname{Ric}\omega), \ldots,$ as long as positivity is preserved; what are the periodic orbits of this dynamical system? The cases $k = 2, 3$ in the following theorem are due to Nadel.

Theorem 10.17. *Let (M, J, ω) be a Fano manifold and assume that $\operatorname{Ric}^l \omega = \omega$ for some $l \in \mathbb{Z}$. Then ω is Kähler–Einstein.*

Proof. The theorem follows from Proposition 4.2. Indeed, note that the nonexistence of periodic fixed points of negative order implies that of positive order, and vice versa. Therefore assume that for some $\omega \in \mathcal{H}_{c_1}$ and some $l \in \mathbb{N}$ one has $\operatorname{Ric}^{-l} \omega = \omega$. By the cocycle condition we thus have

$$0 = E_0\big(\omega, \operatorname{Ric}^{-l} \omega\big) = \sum_{i=0}^{l-1} E_0\big(\operatorname{Ric}^{-i} \omega, \operatorname{Ric}^{-i-1} \omega\big). \tag{74}$$

By Proposition 4.2 one has

$$E_0\big(\operatorname{Ric}^{-i} \omega, \operatorname{Ric}^{-i-1} \omega\big) < 0,$$

unless $\operatorname{Ric}^{-i} \omega = \operatorname{Ric}^{-i-1} \omega$. Therefore each of the terms in (74) vanishes and ω is Kähler–Einstein. □

Moreover, from the proof we have the following stronger conclusion:

Corollary 10.18. *Let (M, J, ω) be a Fano manifold with trivial Futaki character. Assume that $\operatorname{Ric}^l \omega = h^\star \omega$ for some $l \in \mathbb{Z}$ and some $h \in \operatorname{Aut}(M, \mathrm{J})_0$. Then $h = \mathrm{id}$ and ω is Kähler–Einstein.*

Lemma 9.3 implies the following natural generalization of Theorem 10.17 to the setting of solitons.

Corollary 10.19. *Let (M, J, ω) be a Fano manifold and assume that X belongs to a reductive Lie subalgebra of $\operatorname{aut}(M, \mathrm{J})$ and that the one-parameter subgroup $T_{\mathrm{J}X}$ generated by $\mathrm{J}X$ is a compact torus in $\operatorname{Aut}(M, \mathrm{J})$. Let $\omega \in \mathcal{H}_{c_1}(T_{\mathrm{J}X})$. Assume that $\operatorname{Ric}^l_{\psi_X} \omega = \omega$ for some $l \in \mathbb{Z}$. Then ω is a Kähler–Ricci soliton.*

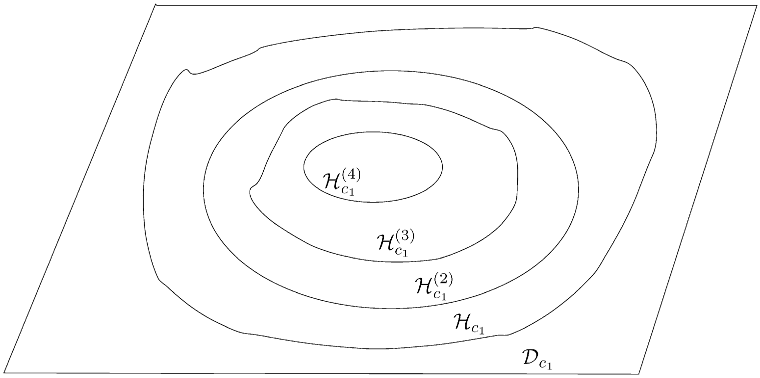

Fig. 1. Nested subspaces inside the space of closed forms representing the first Chern class on a Fano manifold.

In addition, under the assumption that the Tian–Zhu character [75] is trivial one has a statement analogous to Corollary 10.18. Also, as noted in Section 9, and using a generalized character introduced by Futaki [32], this result extends to the setting of multiplier Hermitian structures.

To conclude this subsection we remark that what now becomes apparent is that Nadel's iteration scheme is precisely the Euler method for the conjugate Ricci flow and is thus dual to our iteration that corresponds to the backwards Euler method for the Ricci flow.

Remark 10.20. In light of Theorem 10.17 perhaps it would be interesting to re-examine Nadel's generalized maximum principle which was used to provide a completely different proof for the cases $k = 2, 3$.

10.7. The Ricci index and a canonical nested structure on the space of Kähler metrics

In this subsection we describe a new canonical structure inherent in the space of Kähler forms determined by the complex structure and the Kähler class alone.

Consider first the case of a Fano manifold. As we saw earlier the iteration of the inverse Ricci operator on $\mathcal{H}_{c_1}$ has the advantage of possessing infinite orbits starting at any initial point. The Ricci operator on the other hand lacks this property, according to the Calabi–Yau theorem. This motivates the following definition.

Definition 10.21. Let (M, J) be a Fano manifold. For each $l \in \mathbb{N} \cup \{0\}$ denote by $\mathcal{H}_{c_1}^{(l)}$ the domain of definition of Ric^l.

One has

$$\mathcal{D}_{c_1} = \mathcal{H}_{c_1}^{(0)} \supset \mathcal{H}_{c_1} = \mathcal{H}_{c_1}^{(1)} \supset \mathcal{H}_{c_1}^{(2)} = \mathcal{H}_{c_1}^{+} \supset \cdots \supset \mathcal{H}_{c_1}^{(l)} \supset \cdots. \tag{75}$$

In other words, we may define on $\mathcal{H}_{c_1}$ an integer-valued function

$$\omega \mapsto r(\omega), \tag{76}$$

where $r(\omega)$ is the unique positive integer satisfying $\omega \in \mathcal{H}_{c_1}^{(r(\omega))} \setminus \mathcal{H}_{c_1}^{(r(\omega)+1)}$ (see also Fig. 1). When no such number exists we set $r(\omega) = \infty$. We call the function $r : \mathcal{H}_{c_1} \to \mathbb{N}$ the Ricci index. The number $r(\omega)$ is a Riemannian invariant of the manifold (M, J, ω). It may also be defined for general Riemannian manifolds however it seems hard to study in such generality.

One may extend such a construction to a general Kähler manifold in at least two ways, using either the Ricci iteration or the inverse Ricci operator. Choosing the latter we obtain the following extension of Definition 10.21.

Definition 10.22. Let (M, J) be a Kähler manifold and let Ω denote a Kähler class. For each $l \in \mathbb{N}$ denote by $\mathcal{H}_\Omega^{(l)}$ the image of $\mathcal{H}_\Omega$ under $\mathrm{Ric}_\Omega^{-l+1}$.

Several natural questions arise. What is $\mathcal{H}_\Omega^{(\infty)} := \bigcap_{l=1}^{\infty} \mathcal{H}_\Omega^{(l)}$? How to asymptotically relate the Ricci index to the time parameter of the Ricci flow? Also, how to relate the Ricci index, on the one hand, to the metric structure on the space $\mathcal{H}_\Omega$ [27,45,61] defined by $\langle \mu, \nu \rangle_\omega = \frac{1}{V} \int_M \mu\nu\omega^n$, $\forall \mu, \nu \in T_\omega \mathcal{H}_\Omega \cong C^\infty(M)/\mathbb{R}$ and, on the other hand, to sublevel sets of Calabi's energy and Mabuchi's K-energy? Finally, what is the relation between the Ricci index and positivity?

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The present article was posted (on arxiv.org) on September 7, 2007. Several days later a preprint was posted on arxiv. Parts of that preprint (related to, among other things, the definition of the inverse Ricci operator, answering Nadel's question, discussion on energy functionals, definition of an iteration on noncanonical classes on a Fano manifold) appear to be nearly identical to my August 2005 manuscript [55] that I communicated in July 2006 to the author of that posted preprint. In addition, I presented in detail at Imperial College in December 2006 the idea of relating my discrete constructions to discretization of the Ricci flow.

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