Two-dimensional Inverse Heat Conduction Problem Using a Meshless Manifold Method

G. F. Wei\textsuperscript{a}, H. F. Gao\textsuperscript{b, a,*}

\textsuperscript{a}School of Mechanical and Automotive Engineering, Shandong Polytechnic University, Jinan, 250353, China
\textsuperscript{b}Department of Mechanical Engineering, Shandong Polytechnic, Jinan, 250104, China

Abstract

Inverse heat conduction problems (IHCP) are widely applied to the aerospace, nuclear physics, metallurgy and other fields. The meshless method and the finite element method are main numerical methods to obtain numerical solutions for IHCP. The meshless manifold method (MMM) is based on moving least-square method and the finite cover approximation theory in the mathematical manifold. Comparing with the numerical methods based on mesh, such as finite element method and boundary element method, the MMM only uses cover nodes without having to mash the domain of the problem when the cover function is formed. In this paper, the MMM is used to obtain numerical solutions of two-dimensional IHCP with a source parameter, and the corresponding discretized equations are obtained. A numerical example is given to show the effectiveness of the method. The MMM can also be applied to other inverse problems.

1. Introduction

The inverse heat conduction problem (IHCP), like the vast majority of inverse problems, is known to be ill-posed: besides the possible non-uniqueness of the solution of discretized problem, the results are very sensitive to input data noise. IHCP has numerous important applications in various sciences and engineering. For example, the temperature of a very hot surface is not easily measured directly with sensors. Usually sensors are placed beneath the surface and the temperature of the hot surface is estimated...
by inverse analysis. Other examples of IHCP are the estimation of unknown temperature-dependent thermo-physical parameters of materials from the temperature recordings at the boundary surfaces of the domain [1].

Many researchers have studied the methods to solve the IHCP, and several techniques have been proposed for solving a one-dimensional IHCP [2]. Among the methods proposed for higher dimensional IHCP, boundary element, finite difference, finite elements, meshless and the method of fundamental solutions have been widely adopted for problems in two-dimension [3-4]. However, there is still a need on developing numerical schemes for multi-dimensional IHCP.

Despite the great success of the finite and boundary element methods as effective numerical tools for the solution of boundary value problems on complex domains, the shortcomings of these methods it is still growing interest in the development of new advanced computational methods. Meshless and manifold methods emerge as a competitive alternative to discretization methods. These meshless and manifold methods are becoming popular in rock and soil mechanics. A significant number of such methods have been proposed so far [5]. Recently a novel numerical approach based on meshless manifold method [6] has been proposed.

The meshless manifold method (MMM) is based on moving least-square method and the finite cover approximation theory in the mathematical manifold [7]. In MMM, two cover systems are employed. The mathematical cover system provides the nodes for forming finite covers of the solution domain; and the physical cover system describes geometry of the domain of the problem and the discontinuous surfaces in the domain. The shape functions in this method are formed by the moving least-square method and the finite covers approximation theory, hence the shape functions are not affected by discontinuity of a domain. Therefore crack problems can be treated better. For local problems, the shape functions are more effective than that in other methods.

In MMM, the analyzed domain can be divided into small subdomains with simple geometry. To each of these simple geometric subdomains, the fundamental solutions of some simplified differential operators can then be found. The idea was successfully applied to provide a 2-D fracture problems [9]. In this paper, the MMM is used to solve inverse heat conduction problems in 2-D with a source parameter.

2. Formulation of the problem

Consider a linear inverse heat conduction problem with a source parameter, which in 2-D is described by the governing equation:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + p(t)u + \varphi(x, y, t), \quad 0 \leq x, y \leq 1, \quad 0 \leq t \leq T
\]

Under the following initial-boundary conditions

\[
u(x, y, 0) = f(x, y), \quad 0 \leq x, y \leq 1 \quad (2)
\]

\[
u(0, y, t) = g_o(y, t), \quad 0 \leq t \leq T, \quad 0 \leq y \leq 1 \quad (3)
\]

\[
u(1, y, t) = g_i(y, t), \quad 0 \leq t \leq T, \quad 0 \leq y \leq 1 \quad (4)
\]

\[
u(x, 0, t) = h_o(x, t), \quad 0 \leq t \leq T, \quad 0 \leq x \leq 1 \quad (5)
\]

\[
u(x, 1, t) = h_i(x, t), \quad 0 \leq t \leq T, \quad 0 \leq x \leq 1 \quad (6)
\]

At an interior given point in the domain \( \Omega \), we have

\[
u(x_0, y_0, t) = E(t)
\]

\[
(x_0, y_0) \in (0,1) \times (0,1), \quad 0 \leq t \leq T
\]
where \( f, g_0, g_1, h, \varphi, E \) are known functions, and \( u, p \) are unknown functions. Eqs. (1)-(7) describe a two-dimensional heat conduction problem.

Consider following variable transformation

\[
    r(t) = \exp\left(-\int p(s)ds\right) 
\]

\[
    \omega(x,y,t) = r(t) u(x,y,t) 
\]

In terms of the variables (8) and (9), Eq. (1) becomes

\[
    \omega_i = \omega_n + r(t) \varphi(x,y,t), \quad 0 \leq x, y \leq 1, \quad 0 < t \leq T 
\]

The initial-boundary conditions become

\[
    \omega(x,y,0) = f(x,y), \quad 0 \leq x, y \leq 1 
\]

\[
    \omega(0,y,t) = r(t) g_0(y,t), \quad (0 \leq t \leq T, \quad 0 \leq y \leq 1) 
\]

\[
    \omega(1,y,t) = r(t) g_1(y,t), \quad (0 \leq t \leq T, \quad 0 \leq y \leq 1) 
\]

\[
    \omega(x,0,t) = r(t) h_0(x,t), \quad (0 \leq t \leq T, \quad 0 \leq x \leq 1) 
\]

\[
    \omega(x,1,t) = r(t) h_1(x,t), \quad (0 \leq t \leq T, \quad 0 \leq x \leq 1) 
\]

\( r(t) \) can be given as

\[
    r(t) = \frac{\omega(x_0,y_0,t)}{E(t)} 
\]

From above variable transformation, the source parameter is eliminated, and we can obtain the fundamental solution to the inverse heat conduction problem.

3. Meshless manifold method for 2-D inverse heat conduction problem

3.1 Meshless Manifold Covers

One key technique used in the construction of the MMM is the finite cover theory in manifold method.

Two cover systems used in the MMM are referred to as the mathematical cover and the physical cover, respectively. The mathematical cover is used to construct approximation functions, whereas the physical cover is used to define element domains. The mathematical cover is often constructed from a simple pattern such as triangle and circle. The construction of a physical cover involves the computations of intersections of the mathematical cover and the boundary of the problem domain. However, a mathematical cover is made up of the influence domain of the node, whereas a physical cover is used to define the boundary and discontinuities in the problem domain in the MMM. And the mathematical cover of each node is a local cover, which is used to construct a local approximation by the moving least-squares (MLS) approximation. The physical cover only constitutes the solving domain of the problem.

![Fig. 1. Node covers of a rectangular domain.](image)
For example, consider a rectangular domain shown in Fig. 1(a), the mathematical cover can be constructed by sub-covers which form the influence domain of the nodes. The influence domains can extend beyond the problem domain. Only nodes from the circles that overlap with the problem domain, which is the physical cover (see Fig. 1(b)), are used. All the nodes, shown in either open or solid circles, are used to obtain local covers.

3.2 MMM for 2-D IHCP

The node distribution with \( n \) covers \((x_i, y_j), i=1,2,...,n \) is used in the problem domain \([0,1] \times [0,1] \). The approximation function \( \omega(x,y,t) \) can be defined as

\[
\omega^k(x,y,t) = \sum_{i=1}^{n} \phi_i(x,y) \lambda_i(t)
\]

where \( \phi_i(x,y) \) is the shape function of MLS approximation, putting Eq. (17) into Eqs. (10) and (11), we obtain

\[
\sum_{i=1}^{n} \phi_i(x,y) \frac{d \lambda_i(t)}{dt} = \sum_{i=1}^{n} \left( \frac{\partial^2 \phi_i(x,y)}{\partial x^2} + \frac{\partial^2 \phi_i(x,y)}{\partial y^2} \right) \lambda_i(t) + \sum_{i=1}^{n} \frac{\phi_i(x_0,y_0)}{E(t)} \varphi(x,y,t) \lambda_i(t)
\]

(18)

\[
\sum_{i=1}^{n} \phi_i(x,y) \lambda_i(0) = f(x,y)
\]

(19)

Eqs. (18) and (19) are valid to each node \((x_i, y_j), i=1,2,...,n \), and can be written as

\[
H \frac{dM}{dt} = (H_1 + C)M
\]

(20)

\[
M(0) = H^{-1}G
\]

(21)

where

\[
M = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))^T
\]

(22)

\[
G = (f(x_1,y_1), f(x_2,y_2), \ldots, f(x_n,y_n))^T
\]

(23)

\[
H = \begin{bmatrix}
\phi_1(x_1,y_1) & \phi_2(x_1,y_1) & \ldots & \phi_n(x_1,y_1) \\
\phi_1(x_2,y_2) & \phi_2(x_2,y_2) & \ldots & \phi_n(x_2,y_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(x_n,y_n) & \phi_2(x_n,y_n) & \ldots & \phi_n(x_n,y_n)
\end{bmatrix}
\]

(24)

\[
H_1 = \begin{bmatrix}
\phi_{1,0}(x_1,y_1) & \phi_{2,0}(x_1,y_1) & \ldots & \phi_{n,0}(x_1,y_1) \\
\phi_{1,0}(x_2,y_2) & \phi_{2,0}(x_2,y_2) & \ldots & \phi_{n,0}(x_2,y_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{1,0}(x_n,y_n) & \phi_{2,0}(x_n,y_n) & \ldots & \phi_{n,0}(x_n,y_n)
\end{bmatrix} + \begin{bmatrix}
\phi_{1,0}(x_1,y_1) & \phi_{2,0}(x_1,y_1) & \ldots & \phi_{n,0}(x_1,y_1) \\
\phi_{1,0}(x_2,y_2) & \phi_{2,0}(x_2,y_2) & \ldots & \phi_{n,0}(x_2,y_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{1,0}(x_n,y_n) & \phi_{2,0}(x_n,y_n) & \ldots & \phi_{n,0}(x_n,y_n)
\end{bmatrix}
\]

(25)

\[
C = \frac{1}{E(t)} \begin{bmatrix}
\phi_1(x_0,y_0)\varphi(x_1,y_1,t) & \phi_2(x_0,y_0)\varphi(x_1,y_1,t) & \ldots & \phi_n(x_0,y_0)\varphi(x_1,y_1,t) \\
\phi_1(x_0,y_0)\varphi(x_2,y_2,t) & \phi_2(x_0,y_0)\varphi(x_2,y_2,t) & \ldots & \phi_n(x_0,y_0)\varphi(x_2,y_2,t) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(x_0,y_0)\varphi(x_n,y_n,t) & \phi_2(x_0,y_0)\varphi(x_n,y_n,t) & \ldots & \phi_n(x_0,y_0)\varphi(x_n,y_n,t)
\end{bmatrix}
\]

(26)

Eqs. (20) and (21) are recognized as the ordinary differential equations, time discretization can be given for Eq. (20)

\[
H \frac{M^{(i+1)} - M^{(i)}}{\Delta t} = (H_1 + C) \frac{M^{(i+1)} + M^{(i)}}{2}
\]

(27)
then

\[(2H - \Delta t(H_1 + C))M^{k+1} = (2H + \Delta t(H_1 + C))M^k\]  

From initial condition and boundary conditions, we can obtain \(M^{(1)}, M^{(2)}, \ldots, M^{(k)}\) as

\[M^{(k)} = M(t_k) = (\lambda_1(t_k), \lambda_2(t_k), \ldots, \lambda_n(t_k))\]

4. Numerical examples

A numerical example is presented to demonstrate the MMM for IHCP in this paper.

Consider IHCP (1)-(7), where

\[f(x,y) = \sin(\pi x)\cos(\pi y)\]  
\[\varphi(x,y,t) = (2\pi^2 - t^2 - 2)\exp(-t)\sin(\pi x)\cos(\pi y)\]
\[E(t) = \frac{1}{2}\exp(-t)\]
\[(x_o, y_o) = (0.25, 0.25)\]

A square with homogeneous material properties is analyzed (Fig. 2). The case of boundary conditions is considered to verify the efficiency and accuracy of the proposed method in solving the IHCP. A source parameter is considered, i.e. \(p(t) = t^2 + 1\). In Fig.3 boundary conditions are prescribed on the top, bottom sides and left lateral side of the square. For the analysis by the MMM, 121 node covers are considered on the boundary and inside the investigated domain. A regular node cover distribution is used. weighted function is formed by MLS, and the cover displacement function is linear basis. The numerical solution and the exact solution of the source parameter are shown in Fig. 3. Fig.4 gives the numerical solution and the exact solution at \(t=0.1, 0.3, 0.5, 0.7, 0.9\) in \(y=0.6\). Fig.5 gives the numerical solution and the exact solution at \(t=0.1, 0.3, 0.5, 0.7, 0.9\) in \(x=0.5\).

For the purpose of error estimation, we calculate the relative error for temperature defined as

\[r_t = \frac{|u_{\text{num}} - u_{\text{exact}}|}{u_{\text{exact}}}\]

The relative error \(r_t\) for MMM is 0.0052 and 0.0048 for Fig. 4 and Fig.5. The results show that the numerical solutions are excellent agreement with analytical ones.
5. Conclusions

The meshless manifold method based on the MLS approximation for the mathematical cover together with the finite cover theory is presented for inverse heat conduction problems in 2-D. It shows that the precision of the method given in this paper is satisfactory, and the convergence speed of iteration is very rapid.

In the MMM in this paper, the MLS approximation is used, the algebra equation system can be solved with fewer unknown coefficients. Then for an arbitrary point in the domain, we need fewer nodes with domains of influence that cover the point, and thus we also require fewer nodes in the whole domain.

Comparing with the numerical methods based on mesh, such as finite element method and boundary element method, the MMM only uses cover nodes without having to mash the domain of the problem when the cover function is formed.

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