MATHEMATICS

THE ABSTRACT RIEMANN INTEGRAL AND A THEOREM OF G. FICHTENHOLZ ON EQUALITY OF REPEATED RIEMANN INTEGRALS. Ib

BY

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6. *Darboux's definition of the Riemann integral with respect to a signed measure*

In this section we shall assume, as in section 4, that X is an s-set, i.e. $X \in \Gamma_s$, and that $|\nu|\,(X) \neq 0$. This means in particular that ν is of finite total variation over X, and the set of all partitions of X with sets of Γ is not empty.

For the definition of the upper and lower integral in the case of a signed integral we need the following result.

Theorem 6.1. *Let $\pi = \pi(F_1, \ldots, F_n)$ run through all partitions of X. Then the net $\sum_i \nu^+(F_i)$, where the summation is over those i for which $\nu(F_i) < 0$, and the net $\sum_i \nu^-(F_i)$, where the summation is over those i for which $\nu(F_i) \geqslant 0$, converge to zero in the set of real numbers.*

Proof. We shall prove this only for ν^+, since the proof for ν^- is similar. It is convenient to introduce temporarily the following notation. If $\pi = \pi(F_1, \ldots, F_n)$ is a partition of X, then we write F_i^+ whenever $\nu(F_i) \geqslant 0$ and F_i^- whenever $\nu(F_i) < 0$. For every $\varepsilon > 0$ there exist disjoint sets $A_1, \ldots, A_n$ in Γ such that $\nu(A_i) > 0$ and $\nu^+(X) \leqslant \sum_{i=1}^n \nu(A_i) + \varepsilon/2$. If $A = \bigcup_{i=1}^n A_i$, then $X - A = \bigcup_{j=1}^m B_j$, where $B_j \in \Gamma (j = 1, 2, \ldots, m)$ and the sets B_j are disjoint. Denote by π_ε the partition of X given by $\pi_\varepsilon = \pi(A_1, \ldots$ $\ldots, A_n, B_1, \ldots, B_m)$. Then we have $\nu^+(X) \leqslant \sum_{i=1}^n \nu(A_i) + \sum_j \nu(B_j^+) + \varepsilon/2$. This shows that for every $\varepsilon > 0$ there exists a partition $\pi_\varepsilon = \pi(C_1, \ldots, C_k)$ of X such that $\nu^+(X) \leqslant \sum_i \nu(C_i^+) + \varepsilon/2$. Since $\sum_{i=1}^k \nu^+(C_i) = \nu^+(X) \leqslant \sum_i \nu^+(C_i^+) + \varepsilon/2$, we obtain $\sum_i \nu^+(C_i^-) \leqslant \varepsilon/2$. Let now $\pi = \pi(F_1, \ldots, F_n)$ be a partition of X which is finer than π_ε. Then $\sum_i \nu^+(F_i^-) = \sum\{\nu^+(F_i^-) : F_i^- \subset C_j^+\} +$ $+ \sum\{\nu^+(F_i^-) : F_i^- \subset C_j^-\} \leqslant \sum\{\nu^+(F_i^-) : F_i^- \subset C_j^+\} + \varepsilon/2$. But $\sum\{\nu^+(F_i) :$ $: F_i \subset C_j^+\} \leqslant \nu^+(X) \leqslant \sum_j \nu(C_j^+) + \varepsilon/2 \leqslant \sum\{\nu(F_i^+) : F_i^+ \subset C_j^+\} + \varepsilon/2 \leqslant$

$\leqslant \sum\{\nu^+(F_i^+) : F_i^+ \subset C_j^+\} + \varepsilon/2$. Hence, $\sum\{\nu^+(F_i^-) : F_i^- \subset C_j^+\} \leqslant \varepsilon/2$, i.e. $\sum_i \nu^+(F_i^-) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon$. This completes the proof of the theorem.

We shall now introduce the following additional notation.

Notation. *Let f be a bounded real function defined on X and let $\pi = \pi(F_1, \ldots, F_n)$ be a partition of X. By $u_f(\pi, \nu)$ we denote the step function which on $F_i (i = 1, 2, \ldots, n)$ assumes the value* $\sup(f(x) : x \in F_i)$ *whenever* $\nu(F_i) > 0$ *and* $\inf(f(x) : x \in F_i)$ *whenever* $\nu(F_i) \leqslant 0$. *Similarly, $l_f(\pi, \nu)$ is the step function which on F_i assumes the value* $\inf(f(x) : x \in F_i)$ *whenever* $\nu(F_i) > 0$ *and* $\sup(f(x) : x \in F_i)$ *whenever* $\nu(F_i) \leqslant 0$ $(i = 1, 2, \ldots, n)$.

Observe that $u_f(\pi, \nu) \in S(f, \pi)$ and $l_f(\pi, \nu) \in S(f, \pi)$ for all $\pi \in P$. Now the following theorem holds.

Theorem 6.2. *For every bounded real function f defined on X, we have* $\lim_\pi J(u_f(\pi, \nu)) = (J_+)^*(f) - (J_-)_*(f)$ *and* $\lim_\pi J(l_f(\pi, \nu)) = (J_+)_*(f) - (J_-)^*(f)$.

Proof. Let $\pi = \pi(F_1, \ldots, F_n)$ be a partition of X and let $M_i = \sup(f(x) : x \in F_i)$ and let $m_i = \inf(f(x) : x \in F_i)$ $(i = 1, 2, \ldots, n)$. Then it is easy to verify that we have with the notation introduced in the proof of Theorem 6.1

$$J(u_f(\pi, \nu)) = J_+(u_f(\pi)) - J_-(l_f(\pi)) - \sum_i (M_i - m_i)\, \nu^+(F_i^-) - \sum_i (M_i - m_i)\, \nu^-(F_i^+)$$

and

$$J(l_f(\pi, \nu)) = J_+(l_f(\pi)) - J_-(u_f(\pi)) + \sum_i (M_i - m_i)\, \nu^+(F_i^-) + \sum_i (M_i - m_i)\, \nu^-(F_i^+).$$

Hence, the desired result follows immediately from Theorem 4.1, Definition 4.3 and Theorem 6.1.

On the basis of Theorem 6.2 we introduce the following definition.

Definition 6.1 (*Upper and Lower Darboux integrals with respect to a signed measure*). *For every bounded real function f defined on X, the numbers* $\lim_\pi J(u_f(\pi, \nu))$ *and* $\lim_\pi J(l_f(\pi, \nu))$ *are denoted by $J^*(f)$ and $J_*(f)$ respectively, and are called the upper Darboux integral of f with respect to ν and the lower Darboux integral of f with respect to ν respectively.*

From Theorem 6.2 it follows that for every bounded real function f defined on X, we have $J^*(f) = (J_+)^*(f) - (J_-)_*(f)$ and $J_*(f) = (J_+)_*(f) - (J_-)^*(f)$; hence $J_*(f) \leqslant J^*(f)$. Furthermore, if ν is a measure on Γ, then the present definition is the same as Definition 4.3.

The following obvious extension of Theorem 4.2 holds.

Theorem 6.3. *Let f be a bounded real function defined on X. Then $f \in R(X, \Gamma, \nu)$ if and only if $J_*(f) = J^*(f)$, and in this case $J(f) = J_*(f) = J^*(f)$.*

Proof. If $f \in R(X, \Gamma, \nu)$, then it follows from Theorem 5.5 that $f \in R(X, \Gamma, \nu^+)$ and $f \in R(X, \Gamma, \nu^-)$. Hence, by Theorem 4.2, we have

$(J_+)^*(f)=(J_+)_*(f)=J_+(f)$ and $(J_-)^*(f)=(J_-)_*(f)=J_-(f)$. It follows that $J^*(f)=J_*(f)=J(f)=J_+(f)-J_-(f)$.

Conversely, if $J^*(f)=J_*(f)$, then Theorem 6.2 and Definition 6.1 imply that $(J_+)^*(f)-(J_+)_*(f)=(J_-)_*(f)-(J_-)^*(f)$. Since $(J_+)^*(f)-(J_+)_*(f)\geqslant 0$ and similarly for J_-, we have $(J_+)^*(f)=(J_+)_*(f)$ and $(J_-)^*(f)=(J_-)_*(f)$, and so $f\in R(X, \Gamma, \nu^+)$ and $f\in R(X, \Gamma, \nu^-)$ (Theorem 4.2). Hence by Theorem 5.5, $f\in R(X, \Gamma, \nu)$.

We conclude this section with the following theorem which is similar to Theorem 4.3.

Theorem 6.4. *For every bounded real function f defined on X the following conditions are equivalent.*

(a) $f\in R(X, \Gamma, \nu)$.

(b) *For every pair of elements $s_f, s_f'\in S(f)$, the net $J(s_f(\pi)-s_f'(\pi))$, $\pi\in P$, converges to zero in the set of real numbers.*

(c) *The net $J(s_f(\pi)-s_f'(\pi))$, $\pi\in P$ and $s_f, s_f'\in S(f)$ converges to zero in the set of real numbers uniformly in $s_f, s_f'\in S(f)$.*

(d) *For every pair of elements $v(f), v'(f)\in VS(f)$, the net $J(v(f,\pi)-v'(f,\pi))$, $\pi\in P$, converges to zero in the set of real numbers.*

(e) *The net $J(v(f,\pi)-v'(f,\pi))$, $\pi\in P$ and $v(f)$, $v'(f)\in VS(f)$, converges to zero in the set of real numbers uniformly in $v(f)$, $v'(f)\in VS(f)$.*

Proof. If based on Theorem 6.2 and the fact that for every pair of elements $s_f, s_f'\in S(f)$ we have $|J(s_f(\pi)-s_f'(\pi))|\leqslant J(u_f(\pi,\nu)-l_f(\pi,\nu))$, $\pi\in P$, the proof is completely similar to the proof of Theorem 4.3.

Remark. We leave it to the reader to formulate and prove theorems corresponding to Theorems 4.4 and 4.5.

7. *The spaces R_Ω with respect to a measure.*

In this section we shall discuss a class of Riemann integrable functions for which a generalization of the theorem of Fichtenholz can be proved. The classes $R(X, \Gamma, \mu)$ are too general for this purpose as shown by the following example: Let X be the unit interval in E_1, Γ the σ-algebra of all Lebesgue measurable subsets of this interval, and let μ be Lebesgue measure. Then the class of all Riemann integrable functions is the class of all bounded Lebesgue measurable functions, and the Riemann integral of those functions is the Lebesgue integral. But in section 2 we have shown that the theorems of Fichtenholz cannot be generalized to include the Lebesgue integral.

Let X be a non-empty point set, Γ a semiring of subsets of X and let μ be a measure on X. As in section 4 we shall assume here that X is an s-set or equivalently that $X\in\Gamma_s$ and that $\mu(X)\neq 0$.

Let Ω be a collection of increasing sequences of partitions of X, i.e. every element $\omega\in\Omega$ represents an increasing sequence of elements $\{\pi_{\omega,n} : n\in N\}$ of $P(X, \Gamma, \mu)$.

We introduce now the following definition.

Definition 7.1 (Ω_μ-*integrability*). *Let Ω be a collection of increasing sequences of partitions of X. A bounded real function f defined on X is said to be Ω_μ-integrable over X whenever for every $\omega \in \Omega$ and every pair of elements $s_f, s_f' \in S(f)$, we have* $\lim_{n\to\infty} I(s_f(\pi_{\omega,n}) - s_f'(\pi_{\omega,n})) = 0$.

As a consequence of this definition we have the following theorem.

Theorem 7.1. *Let Ω be a collection of increasing sequences of partitions of X, and let f be a bounded real function defined on X. If f is Ω_μ-integrable over X, then f is Riemann integrable over X with respect to μ and $I(f) = \lim_{n\to\infty} I(s_f(\pi_{\omega,n}))$ for all $\omega \in \Omega$ and all $s_f \in S(f)$.*

Proof. If f is Ω_μ-integrable over X, then $\lim_{n\to\infty} I(u_f(\pi_{\omega,n} - l_f(\pi_{\omega,n})) = 0$ for all $\omega \in \Omega$. Since $I(l_f(\pi_{\omega,n})) \leqslant I_*(f) \leqslant I^*(f) \leqslant I(u_f(\pi_{\omega,n}))$, $\omega \in \Omega$, $n \in N$, we obtain that $I_*(f) = I^*(f)$. Hence, the desired result follows from Theorem 4.2.

We introduce now the following additional notation.

Notation. *Let Ω be a collection of increasing sequences of partitions of X. The class of all bounded real functions which are Ω_μ-integrable over X will be denoted by $R_\Omega(X, \Gamma, \mu)$, or shortly by R_{Ω_μ}.*

Remark. It follows from Theorem 7.1 that R_{Ω_μ} is a subset of $R(X, \Gamma, \mu)$. It is easy to see, however, that it is a linear subset of R. Furthermore, from the obvious inequalities: $u_{f^+}(\pi) - l_{f^+}(\pi) \leqslant u_f(\pi) - l_f(\pi)$ and $u_{f^-}(\pi) - l_{f^-}(\pi) \leqslant u_f(\pi) - l_f(\pi)$, $\pi \in P$, it follows immediately that R_{Ω_μ} is also a Riesz space, i.e. R_{Ω_μ} is a Riesz subspace of R.

We conclude this section with the following theorem which is similar to Theorem 4.4.

Theorem 7.2. *Let Ω be a collection of increasing sequences of partitions of X and let f be a bounded real function defined on X. Then the following conditions are equivalent.*

(a) $f \in R_\Omega(X, \Gamma, \mu)$.

(b) *For every $\omega \in \Omega$, $\lim_{n\to\infty} I(s_f(\pi_{\omega,n}) - s_f'(\pi_{\omega,n})) = 0$ for all pairs $s_f, s_f' \in S(f)$.*

(c) *For every $\omega \in \Omega$, $\lim_{n\to\infty} I(s_f(\pi_{\omega,n}) - s_f'(\pi_{\omega,n})) = 0$, uniformly in $s_f, s_f' \in S(f)$.*

(d) *For every $\omega \in \Omega$, $\lim_{n\to\infty} I(v(f, \pi_{\omega,n}) - v'(f, \pi_{\omega,n})) = 0$ for all pairs $v(f)$, $v'(f) \in VS(f)$.*

(e) *For every $\omega \in \Omega$, $\lim_{n\to\infty} I(v(f, \pi_{\omega,n}) - v'(f, \pi_{\omega,n})) = 0$, uniformly in $v(f)$, $v'(f) \in VS(f)$.*

Proof. The proof is left to the reader since it is similar to the proof of Theorem 4.4.

8. *Spaces R_Ω with respect to a signed measure*

As in section 6, let X be a non-empty set, Γ a semiring of subsets of X such that $X \in \Gamma_s$, and let ν be a signed measure defined on Γ such that $|\nu|\,(X) \neq 0$.

Definition 8.1 (*Ω_ν-integrability*). *Let Ω be a collection of increasing sequences of partitions of X. A bounded real function f defined on X is said to be Ω_ν-integrable over X whenever for every $\omega \in \Omega$ and every pair of elements $s_f, s_f' \in S(f)$, we have $\lim_{n\to\infty} J\{(s_f(\pi_{\omega,n}) - s_f'(\pi_{\omega,n}))\chi_A\} = 0$ for all $A \in \Gamma$.*

In analogy to Theorem 7.1 we have the following result.

Theorem 8.1. *Let Ω be a collection of increasing sequences of partitions of X, and let f be a bounded real function defined on X. If f is Ω_ν-integrable over X, then f is Riemann integrable over X with respect to ν.*

Proof. Observe that the definition of Ω_ν-integrability implies that for every $\omega \in \Omega$, $\lim_{n\to\infty} J\{(u_f(\pi_{\omega,n}) - l_f(\pi_{\omega,n}))\chi_A\} = 0$ for all $A \in \Gamma$. Since $l_f(\pi) \leqslant f \leqslant u_f(\pi)$ for all $\pi \in P$, and $\pi \leqslant \pi'$ implies that $l_f(\pi) \leqslant l_f(\pi')$ and $u_f(\pi') \leqslant u_f(\pi)$ (Theorem 4.1 (a)), the desired result follows from Definition 5.1.

Notation. *The class of all Ω_ν-integrable functions on X will be denoted by $R_\Omega(X, \Gamma, \nu)$, or shortly by R_{Ω_ν}.*

Theorem 8.2. *Let Ω be a collection of increasing sequences of partitions of X. A bounded real function f defined on X is Ω_ν-integrable over X if and only if f is Ω_{ν^+}-integrable over X and Ω_{ν^-}-integrable over X, and in this case we have for every $\omega \in \Omega$ that $\lim_{n\to\infty} J(s_f(\pi_{\omega,n})) = J(f)$ for all $s_f \in S(f)$.*

Proof. If $f \in R_{\Omega_{\nu^+}}$ and if $f \in R_{\Omega_{\nu^-}}$, then it is obvious that $f \in R_{\Omega_\nu}$, and in that case it follows from Theorem 7.1 that $J_+(f) - J_-(f) = \lim_{n\to\infty} J_+(s_f(\pi_{\omega,n})) - \lim_{n\to\infty} J_-(s_f(\pi_{\omega,n})) = \lim_{n\to\infty} J(s_f(\pi_{\omega,n}))$ for every $\omega \in \Omega$ and for every $s_f \in S(f)$. Hence, by Theorem 5.5, $J(f) = \lim_{n\to\infty} J(s_f(\pi_{\omega,n}))$ for every $\omega \in \Omega$ and every $s_f \in S(f)$.

Conversely, assume that f is Ω_ν-integrable over X. Then

$$\lim_{n\to\infty} J\{(u_f(\pi_{\omega,n}) - l_f(\pi_{\omega,n}))\chi_A\} = 0$$

for all $A \in \Gamma$ and every $\omega \in \Omega$. Since the sequence of non-negative step functions $\{u_f(\pi_{\omega,n}) - l_f(\pi_{\omega,n}) : n \in N\}$, $\omega \in \Omega$, is decreasing in n, it follows from Theorem 5.4 that $\lim_{n\to\infty} J_+(u_f(\pi_{\omega,n}) - l_f(\pi_{\omega,n})) = 0$ for every $\omega \in \Omega$, and similarly for J_-. We conclude that $f \in R_{\Omega_{\nu^+}}$ and $f \in R_{\Omega_{\nu^-}}$. Furthermore, $J_+(f) = \lim_{n\to\infty} J_+(s_f(\pi_{\omega,n}))$ and $J_-(f) = \lim_{n\to\infty} J_-(s_f(\pi_{\omega,n}))$, $\omega \in \Omega$ and $s_f \in S(f)$, so $J(f) = \lim_{n\to\infty} J(s_f(\pi_{\omega,n}))$, $\omega \in \Omega$, $s_f \in S(f)$.

Remark. From Theorem 7.1 it follows that Theorem 8.2 implies Theorem 8.1. Furthermore, this theorem shows that R_{Ω_ν} and $R_{\Omega_{|\nu|}}$ contain the same elements, so R_{Ω_ν} is also a Riesz space.

We conclude this section with the following theorem.

Theorem 8.3. *Let Ω be a collection of increasing sequences of partitions of X. A bounded real function f defined on X is Ω_ν-integrable over X if and* *only if for every* $\omega \in \Omega$ *we have* $\lim_{n\to\infty} J\{(v(f, \pi_{\omega,n}) - v'(f, \pi_{\omega,n}))\chi_A\} = 0$ *for all* $A \in \Gamma$ *and* $v(f)$, $v'(f) \in VS(f)$, *and in that case* $J(f) = \lim_{n\to\infty} J(v(f, \pi_{\omega,n}))$, $\omega \in \Omega$ *and* $v(f) \in VS(f)$.

Proof. If f is Ω_ν-integrable over X, then the condition of the theorem follows immediately from Definition 8.1 and Theorem 8.2.

Conversely, the condition of the theorem implies, using Theorem 5.4 (observe $|v(f, \pi_{\omega,n}) - v'(f, \pi_{\omega,n})| \leqslant 2M$, where $M = \sup(|f(x)| : x \in X)$, for all $\omega \in \Omega$ and $v(f)$, $v'(f) \in VS(f)$), that $\lim_{n\to\infty} J_+(v(f, \pi_{\omega,n})) - v'(f, \pi_{\omega,n})) = 0$ and $\lim_{n\to\infty} J_-(v(f, \pi_{\omega,n}) - v'(f, \pi_{\omega,n})) = 0$, $\omega \in \Omega$, $v(f)$, $v'(f) \in VS(f)$. Hence, Theorem 7.2 implies that f is Ω_{ν^+}-integrable over X and also Ω_{ν^-}-integrable over X. We conclude from Theorem 8.2 that f is Ω_ν-integrable over X. The remainder of the theorem follows then also from Theorem 8.2.

Remark. The reader may easily verify that there does not always exist sets Ω of increasing sequences of partitions of X such that $R_{\Omega_\nu} = R$. In the theory of the familar Riemann integral in Euclidean spaces it can be shown that there exists Ω such that $R_\Omega = R$ (see section 10). It seems therefore natural to ask for sufficient conditions, or perhaps necessary and sufficient conditions, on Γ in order that $R_\Omega = R$ for some Ω. We have not been able to solve this problem in any satisfactory way.

9. *A generalization of the theorem of Fichtenholz*

Let X and Y be two non-empty point sets, and let Γ_x and Γ_y be semirings of subsets of X and Y respectively. Furthermore, we assume that ν_x and ν_y are signed measures defined on Γ_x and Γ_y respectively, and that $X \in (\Gamma_x)_s$, $Y \in (\Gamma_y)_s$, $|\nu_x|(X) \neq 0$ and $|\nu_y|(Y) \neq 0$. We shall denote the Riemann integral with respect to ν_x and ν_y by J_x and J_y respectively.

Then the following generalization of the theorem of Fichtenholz holds.

Theorem 9.1. *Let* Ω_x *and* Ω_y *be collections of increasing sequences of partitions of* X *and* Y *respectively. If* $f = f(x, y)$, $(x, y) \in X \times X$, *is a real function defined on* $X \times Y$, *satisfying the following conditions*:

(i) f *is bounded on* $X \times Y$, *i.e. there exists a constant* $M > 0$ *such that* $|f(x, y)| \leqslant M$ *for all* $(x, y) \in X \times Y$,

(ii) *for all* $y \in Y$, $f(x, y) = f_y(x)$ $(x \in X)$ *is* Ω_{x,ν_x}*-integrable over* X,

(iii) *for all* $x \in X$, $f(x, y) = f_x(y)$ $(y \in Y)$ *is* Ω_{y,ν_y}*-integrable over* Y, *then* $\varphi(x) = J_y(f_x)$ $(x \in X)$ *and* $\psi(y) = J_x(f_y)$ $(y \in Y)$ *are* Ω_{x,ν_x}*-integrable over* X *and* Ω_{y,ν_y}*-integrable over* Y *respectively, and* $J_x(\varphi) = J_y(\psi)$, *i.e. repeated integrals* $J_x(J_y(f))$ *and* $J_y(J_x(f))$ *exist and are equal.*

Proof. We shall prove that φ is Ω_{x,ν_x}-integrable over X. The proof will be based on Theorem 8.3. For this purpose, let $v(\varphi)$, $v'(\varphi) \in VS(\varphi)$ and let $A \in \Gamma_x$. If π is a partition of X, then, since $v(\varphi, \pi)$ is functional and linear in φ, we have that $J_x(v(\varphi, \pi))=J_x(v(J_y(f_x(y), \pi)))=J_x(J_y(v(f_x(y),\pi)))= =J_y(J_x(v(f_x(y), \pi)))$ (the latter equality follows from the fact that $J_y(v(f_x(y),\pi))$ is a step function). Hence, if $\omega_x \in \Omega_x$, then

$$J_x\{(v(\varphi, \pi_{\omega_x}, n)-v'(\varphi,\pi_{\omega_x}, n))\chi_A\}=J_y(J_x[\{v(f_y(x), \pi_{\omega_x}, n)-v'(f_y(x),\pi_{\omega_x}, n)\}\chi_A]).$$

If we put

$$J_x[\{v(f_y(x), \pi_{\omega_x}, n)-v'(f_y(x), \pi_{\omega_x}, n)\}\chi_A]=g_n(y), \quad y \in Y$$

and $n \in N$, then by condition (ii) of this theorem and Theorem 8.3 we have $\lim g_n(y)=0$ for all $y \in Y$. Also, $|g_n| \leqslant 2M|\nu_y|(Y)$, and hence, by Arzelà's Theorem (Theorem 3.7) $\lim_{n\to\infty} J_y(g_n)=0$, i.e. φ is Ω_{x,ν_x}-integrable over X. In the same way we can show that ψ is Ω_{y,ν_y}-integrable over Y. In order to prove that the Riemann integrals of φ and ψ are equal, let $h_n(y)=J_x\{v(f_y(x), \pi_{\omega_x}, n)\}$, $n \in N$, $y \in Y$ and $v \in VS$. Then $\lim h_n(y)=\psi(y)$ for all $y \in Y$ (Theorem 8.3) and $|h_n| \leqslant M|\nu_y|(Y)$. It follows from Arzelà's theorem that $\lim_{n\to\infty} J_y(h_n)=J_y(\psi)$. Since $J_y(h_n)=J_x(v(\varphi, \pi_{\omega_x}, n))$ and φ is Ω_{x,ν_x}-integrable over X, we conclude from Theorem 8.3 that

$$\lim_{n\to\infty} J_y(h_n)=J_x(\varphi), \text{ so } J_y(\psi)=J_x(\varphi).$$

This completes the proof of the theorem.

Remark. It is easy to see that the theorem can be generalized to products of more than two spaces. Indeed, with obvious meaning for the symbols employed, if $f=f(x_1, \ldots, x_n)$ is a real function defined on $X_1 \times \ldots \times X_n$ such that f is bounded and $f(x_1, \ldots, x_n)=f_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n}(x_i)$ is $\Omega_{x_i,\nu_{x_i}}$-integrable over X_i $(i=1, 2, \ldots, n)$, then the $n!$-integrals

$$J_{x_{i_1}}(J_{x_{i_2}}(\ldots(J_{x_{i_n}}(f))\ldots)$$

all exist in the sense of Ω-integrability and are equal.

10. *Applications*

(a) *The Riemann–Stieltjes integral in Euclidean space $E_k (k \geqslant 1)$.* Let $X=E_k$ and let Γ be the semiring consisting of ϕ and all the *cells* $A=(a_1, b_1; \ldots; a_k, b_k]=$

$$\{x : x=(x_1, \ldots, x_k) \in E_k \text{ and } -\infty < a_i < x_i \leqslant b_i < +\infty, i=1, 2, \ldots, k\}.$$

Assume that μ is a measure on Γ such that $\mu(A)<\infty$ for all $A \in \Gamma$.

Measures defined on this semiring can be characterized as follows.

Theorem 10.1. *Let g be a real function defined on $E_k (k \geqslant 1)$ such that* (i) *$g(x)=g(x_1, \ldots, x_k)=0$ $(x \in E_k)$ if one or more of the coordinates of x are zero,* (ii) *$g(x)$ is right continuous, i.e. $\lim g(x_1+\varepsilon_1, x_2+\varepsilon_2, \ldots, x_k+\varepsilon_k)= =g(x_1, \ldots, x_k)$ as ε_i tends to zero through strictly positive values for every $i=1, 2, \ldots, k$,* (iii) *if $a=(a_1, \ldots, a_k)$ and $b=(b_1, \ldots, b_k)$ are two points of E_k such that $a_i<b_i$ $(i=1, 2, \ldots, k)$, then $\sum_c(-1)^{n(c)}g(c) \geqslant 0$, where $c=(c_1, \ldots, c_k)$*

is a point of E_k the coordinates c_i of which are either a_i or b_i ($i=1, 2, \ldots, k$) and $n(c)$ is the number of a_i among the c_i. Then,

$$\textit{if } A \in \Gamma,\ A=(a_1, b_1; \ldots; a_k, b_k],\ \mu(A)=\sum_c (-1)^{n(c)} g(c)$$

is a measure on Γ such that $\mu(A)$ is finite for all $A \in \Gamma$. Conversely, if μ is a finite measure on Γ, then there exists a function g on E_k with properties (i) *to* (iii) *such that $\mu(A)=\sum_c (-1)^{n(c)} g(c)$ for all $A \in \Gamma$, where $A=(a_1, b_1; \ldots; a_k, b_k]$.*

Proof. If g is a function defined on E_k satisfying the conditions of the theorem, then it is easy to verify that $\mu(A)=\sum_c (-1)^{n(c)} g(c)$, $A=(a_1, b_1; \ldots; a_k, b_k] \in \Gamma$ is a measure on Γ, provided we put $\mu(\phi)=0$.

Conversely, if μ is a measure on Γ, then g can be defined as follows: put $g(x)=0$ if one or more of the coordinates of x are equal to zero, and put $g(x)=(-1)^{m(x)}\mu(A)$ if none of the coordinates of x are zero, where $m(x)$ is the number of $x_i<0$ and $\mu(A)$ is the measure of the cell $\{y : \min(0, x_i)<y_i\leqslant \max(0, x_i), i=1, 2, \ldots, k\}$. The verification of the fact that g satisfies the conditions of the theorem is left to the reader.

Examples of functions g satisfying conditions (i) to (iii) of Theorem 10.1 are the following: (i) $g(x)=g(x_1, \ldots, x_k)=x_1 x_2 \ldots x_k$ which gives rise to the Euclidean volume of an interval in E_k, (ii) $g(x)=g(x_1, \ldots, x_k)=g_1(x_1)\, g_2(x_2) \cdot \ldots \cdot g_k(x_k)$, where $g_1, \ldots, g_k$ are k increasing functions of a real variable vanishing at the origin and right continuous.

If ν is a signed measure on Γ, then by applying the Jordan decomposition $\nu=\nu_+-\nu_-$ we can obtain a similar characterization for ν.

If $k=1$, then Theorem 10.1 expresses the well-known fact that the family of all measures on the semiring of the cells is in one-to-one correspondence with the family of right continuous increasing functions which vanish at the origin.

Perhaps it is of interest to point out that if we take a larger semiring of intervals, say for instance the semiring of all intervals, then the characterization theorem becomes more complicated. The reader may convince himself of this fact by examining the case $k=1$.

Let A be an interval the closure of which is the interval $[a_1, b_1; \ldots; a_k, b_k]$; A is called *degenerate* if $a_i=b_i$ for at least one value of i ($i=1, 2, \ldots, k$).

We introduce now the following definition.

Definition 10.1. *A non-degenerate interval A in E_k ($k\geqslant 1$) is called an interval of continuity of μ if, for every $\varepsilon>0$, there exist cells $A_1, A_2 \in \Gamma$ such that* (i) *the closure of A_1 is contained in the interior of A,* (ii) *the closure of A is contained in the interior of A_2,* (iii) *$\mu(A_2)-\mu(A_1)<\varepsilon$. A degenerate interval A in $E_k (k\geqslant 1)$ is called an interval of continuity of μ, if, for every $\varepsilon>0$, there exists a cell B such that the closure of A is contained in the interior of B and $\mu(B)<\varepsilon$.*

Then the following theorem holds.

Theorem 10.2. *Let μ be a finite measure on Γ. Then there exists a countable subset D_μ of E_1 such that if A is an interval of E_k and $[a_1, b_1; \ldots; a_k. b_k]$ is its closure and none of the a_i, b_i ($i=1, 2, \ldots, k$) is in D_μ, then A is an interval of continuity of μ.*

Proof. Let An be the cell $(-n, n; \ldots; -n, n]$ $(n \in N)$ and let $g_{i,n}(y)=\mu(A_n \cap \{x : x \in E_k \text{ and } x_i \leqslant y\})$, $i=1, 2, \ldots, k$ and $n \in N$. Then $g_{i,n}(y)$ ($i=1, 2, \ldots, k$ and $n \in N$) is an increasing function of y, and hence has at most a countable number of discontinuities. Let D_μ be the countable set of all the discontinuities of the functions $g_{i,n}(y)$ ($i=1, 2, \ldots, k$ and $n \in N$). To prove the theorem, let A be an interval satisfying the conditions of the theorem, and assume that A is non-degenerate. For sufficiently large n, the closure of A is contained in A_n. Then, for every $\varepsilon>0$, there exist different numbers a_i', a_i'', b_i', b_i'' ($i=1, 2, \ldots, k$) such that for all $i=1, 2, \ldots, k$, $a_i'<a_i<a_i''$, $b_i'<b_i<b_i''$ and $g_{i,n}(a_i'')-g_{i,n}(a_i')<\varepsilon/2k$, $g_{i,n}(b_i'')-g_{i,n}(b_i')<\varepsilon/2k$. Let $A_1=(a_1', b_1'; \ldots; a_k', b_k']$ and $A_2=(a_1'', b_1''; \ldots$ $\ldots; a_k'', b_k'']$. Then it is easy to see that A_1 and A_2 have properties (i), (ii) and (iii) of Definition 10.1. If A is degenerate, say $a_i=b_i$, then we take $a_i'=b_i'$ and $a_i''=b_i''$ and $g_{i,n}(a_i'')-g_{i,n}(a_i')<\varepsilon$. Then A lies in the strip $A_n \cap \{x : a_i<x_i\leqslant a_i''\}$ and hence, if $B=(a_1', b_1'; \ldots; a_i', a_i''; \ldots; a_k' b_k']$, then the closure of A is contained in the interior of B and $\mu(B)<\varepsilon$.

Remark. If ν is a signed measure on Γ, then the same result holds provided it is formulated for $|\nu|$.

Let ν be a signed measure defined on Γ. A point $x \in E_k$ is called a *ν-continuous point* whenever the degenerate interval consisting of x only is a $|\nu|$-continuous interval.

Let X be an element of Γ. We shall denote by Γ_X the semiring of all cells which are contained in X. Furthermore, if π is a partition of X, then by $|\pi|$ we denote the largest diameter of the elements of π, and $|\pi|$ is called the *norm* of π.

Theorem 10.3. *Let Ω be a collection of increasing sequences of partitions of X satisfying the following conditions:* (i) *for every $\omega \in \Omega$, $\lim_{n\to\infty} |\pi_{\omega,n}|=0$,* (ii) *for every $\omega \in \Omega$, the set of all vertices of the cells in $\pi_{\omega,n}$, $n \in N$, contains all $|\nu|$-discontinuous points. Then $R_\Omega(X, \Gamma_X, \nu)=$ $=R(X, \Gamma_X, \nu)$.*

Proof. It follows from Theorem 8.2 that it is only necessary to give the proof for ν^+. Hence we may assume that $\nu=\mu$, where μ is a finite measure on Γ, By Theorem 7.1, $R_{\Omega_\mu} \subset R(X, \Gamma_X,\mu)$, hence we have only to show that $R(X, \Gamma_X,\mu) \subset R_{\Omega_\mu}$. For this purpose, let $f \in R(X, \Gamma_X,\mu)$. Then, for every $\varepsilon>0$, there exists a partition π_ε of X such that $I(u_f(\pi)-l_f(\pi))<\varepsilon/2$ for all $\pi \geqslant \pi_\varepsilon$ (Theorem 4.3(b)). Let $\omega \in \Omega$ and let n_0 be the smallest index such that (i) the vertices of the elements of π_{ω,n_0} contain all the μ-discontinuous vertices of the elements of π_ε, (ii) $|\pi_{\omega, n_0}|$ is less than the length of the smallest edge occurring in the cells of π_ε, (iii) if x is a vertex of an element of π_ε which is a μ-continuous point and if $x \in A \in \pi_{\omega, n_0}$, then $\mu(A) < \varepsilon/4Mm$, where $M = \sup(|f(x)| : x \in X)$ and m is the total number of vertices of all elements of π_ε. Consider now the partition $\pi = \max(\pi_\varepsilon, \pi_{\omega, n})$ and assume that $n \geqslant n_0$. We wish to compare $I(u_f(\pi) - l_f(\pi))$ with $I(u_f(\pi_{\omega, n}) - l_f(\pi_{\omega, n}))$. Observe that if $\pi_{\omega, n}$ and π have an element in common, then its contribution in the two integrals is the same. Now observe that if $A \in \pi_{\omega, n}$ but does not occur in π, then it contains precisely one vertex of some element B of π_ε (follows from property (ii)). Since this vertex is a point of continuity of μ, we have $\mu(A) < \varepsilon/4Mm$. Hence, the difference of the contribution of $A \cap B \in \pi$ and $A \in \pi_{\omega, n}$ in the above integrals is at most $2M\mu(A) \leqslant \varepsilon/2m$. We conclude that

$$|I(u_f(\pi) - l_f(\pi)) - I(u_f(\pi_{\omega, n}) - l_f(\pi_{\omega, n}))| < \varepsilon/2.$$

Since $\pi \geqslant \pi_\varepsilon$, we have $I(u_f(\pi_{\omega, n}) - l_f(\pi_{\omega, n})) < \varepsilon$ for all $n \geqslant n_0$. Thus $f \in R_{\Omega_\mu}$ (Theorem 7.2), which completes the proof of the theorem.

Remark. This theorem shows that the theorem of Fichtenholz is applicable in $E_k (k \geqslant 1)$ for Riemann integration with respect to the semiring of all cells in E_k and the signed measures defined on it. Of course one may take larger semirings in E_k, and ask whether Theorem 10.1 remains valid. If $k > 1$, and if there are $|\nu|$-discontinuous points, then we are not able to give any satisfactory answer to this question. It is well-known that if Γ is the semiring of all intervals and if there are no $|\nu|$-discontinuous points one may take for Ω all increasing sequences of partitions with norm tending to zero, and Theorem 10.1 continues to hold. In this case it is also true that if one restricts ν to the semiring of all cells, this restriction defines the same space of Riemann integrable functions and the same Riemann integral. This can be proved very quickly, since the characteristic function of every finite interval becomes Riemann integrable with respect to this restricted measure space (apply Theorem 10.2).

(b) *Riemann–Stieltjes integration in* E_1. Let Γ be the semiring of all finite intervals in E_1, and let ν be a signed measure defined on Γ. Then there exists a function g on E_1, of *finite variation* over every interval of Γ and *right continuous*, such that $\nu(A) = g(b) - g(a)$ for all left-open intervals (= cells) $a < x \leqslant b$, $\nu(A) = g(b) - g(a-)$ for all closed intervals $a \leqslant x \leqslant b$, $\nu(A) = g(b-) - g(a)$ for all open intervals $a < x < b$, and $\nu(A) = g(b-) - g(a-)$ for all right-open intervals $a \leqslant x < b$ (this follows from Theorem 10.1). We denote by $\{x_n : n \in N\}$ the set of all points of discontinuity of g. Let g be written as the sum of its continuous part g_c and its discrete part g_d. Denote the signed measure defined by g_c by ν_c (observe that $\nu_c(A) = g_c(b) - g_c(a)$ for every interval of Γ with end points a and b, $a \leqslant b$), and denote the signed measure defined by g_d by ν_d. Then $\nu = \nu_c + \nu_d$. Since Γ denotes the semiring of all finite intervals of E_1, we shall denote the semiring of all cells (= left-open intervals) by Γ_1. Then $\Gamma_1 \subset \Gamma$, but $(\Gamma_1)_s \neq \Gamma_s$.

Then the following result holds.

Theorem 10.4. *Let f be a real function defined on E_1. Then:* (i) *$f \in R\,(E_1, \Gamma_1, \nu)$ if and only if $f \in R\,(E_1, \Gamma_1, \nu_c)$ and f is left continuous at the points $x_n (n \in N)$; in this case $\int f d\nu = \int f d\nu_c + \sum_{n=1}^{\infty} f(x_n)\, \nu(\{x_n\})$,* (ii) *$f \in R\,(E_1, \Gamma, \nu)$ if and only if $f \in R\,(E_1, \Gamma, \nu_c)$, and in this case $\int f d\nu = \int f d\nu_c + \sum_{n=1}^{\infty} f(x_n)\, \nu(\{x_n\})$.*

Proof. The proof is straight forward and is left to the reader.

This theorem shows that in general $R(E_1, \Gamma_1, \nu)$ is a proper subspace of $R(E_1, \Gamma, \nu)$. If ν has no discrete part, i.e. g is continuous, then the two Riemann integration theories are equivalent.

Theorem 10.3 shows that the theorem of Fichtenholz applies to Riemann integration with respect to Γ_1. We shall now prove that this is also the case for Riemann integration with respect to Γ, (always with the restriction $k=1$).

Theorem 10.5. *Let X be a finite interval in E_1, let Γ_X be the semiring of all intervals contained in X and let ν be a signed measure defined on Γ_X. If Ω is a non-empty collection of increasing sequences of partitions of X (with respect to Γ_X) such that* (i) *for every $\omega \in \Omega$, $\lim_{n\to\infty} |\pi_{\omega, n}| = 0$,* (ii) *for every $\omega \in \Omega$, if $x \in X$ is a $|\nu|$-discontinuous point, then $\{x\}$ is an element of some $\pi_{\omega, n}$. Then $f \in R(X, \Gamma_{X;}\, \nu)$ if and only if $f \in R_\Omega\,(X, \Gamma_X, \nu)$.*

Proof. Without loss of generality we may assume that ν is a measure μ (Theorem 8.2). Furthermore, we have only to show that $f \in R(X, \Gamma_x, \mu)$ implies $f \in R_{\Omega_\mu}$ (Theorem 8.1). To this end, let $\varepsilon > 0$ be given. Then there exists a partition π_ε of X such that $I(u_f(\pi) - l_f(\pi)) < \varepsilon/2$ for all $\pi \geqslant \pi_\varepsilon$. Let $\omega \in \Omega$ and let n_0 be the first index with the following properties: (i) $|\pi_{\omega, n_0}|$ is less than the length of the smallest non-degenerate element of π_ε, (ii) if $x \in X$ is a μ-discontinuous point and if $\{x\}$ occurs as an element in π_ε or is an end point of one of the elements of π_ε, then $\{x\}$ occurs in π_{ω, n_0}, (iii) if x is an end point of some element of π_ε or if $\{x\}$ occurs in π_ε and is a μ-continuous point and if $x \in A \in \pi_{\omega, n}$, then $\mu(A) \leqslant \varepsilon/4Mm$, where $M = \sup\,(|f(x)| : x \in X)$ and m is the total number of endpoints of all intervals of π_ε. In the same way as in the proof of Theorem 10.3 it follows then that $I(u_f(\pi_{\omega, n}) - l_f(\pi_{\omega, n})) < \varepsilon$ for all $n \geqslant n_0$, which shows that f is Ω_μ-integrable over X (Definition 7.1).

Remark. This theorem also shows that in $E_k (k \geqslant 1)$ the theorem of Fichtenholz is applicable to the theory of Riemann integration with respect to the semiring of all intervals in $E_k(k \geqslant 1)$ and a product measure, i.e. a measure defined by a function $g(x) = g(x_1, \ldots, x_k) = g_1(x_1)g_2(x_2) \cdot \ldots \cdot g_k(x_k)$, where g_i $(i = 1, 2, \ldots, k)$ is a right continuous function which is of finite variation over every finite interval.

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